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EQUIVALENCE OF SOME MATRIX INEQUALITIES

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Abstract. In the present article, some Kantorovich type and Wielandt type matrix inequalities and their equivalent forms are discussed respectively, and the equivalence of these Kantorovich type inequalities with the corresponding Wielandt type inequalities are established too.

1. Introduction

It is well known that considerable progress have been made in the study of the matrix versions of the Cauchy-Schwarz inequality, the Kantorovich inequality, and the Wielandt inequality in the Löwner partial ordering sense. Marshall and Olkin [4], Baksalary and Puntanen[1], and Mond and Pečarić [5, 6] presented some matrix versions of the Kantorovich inequality involving a positive definite or semidefinite matrix. Pečarić et al. [7] derived some general matrix Cauchy-Schwarz type inequalities and Kantorovich type inequalities. Liu and Neudecker [3] also gave some matrix Kantorovich type inequalities. A matrix version of the Wielandt inequality is presented by Wang and Ip [8], and the equivalence of this matrix Wielandt inequality and corresponding matrix Kantotovich inequality was proved by Zhang [9] and Drury et al. [2]. The purpose of this paper is to show some of the matrix Kantorovich type inequalities are equivalent each other, some of the matrix Wielandt type inequalities are equivalent each other, and these matrix Kantorovich type inequalities and the corresponding matrix Wielandt type inequalities are equivalent.

In Section 2, we will discuss some matrix Kantorovich type inequalities and their equivalent forms. In Section 3, we shall give a matrix inequality first, and then we will introduce some matrix versions of Wielandt inequality and their equivalent forms, and show the equivalence of these Wielandt type inequalities and the corresponding Kantorovich type inequalities.

All matrices in this paper is assumed to be complex. For $n\times n$ Hermitian matrices A and B, $A\leqslant B$ will mean that $B-A$ is positive semidefinite. Let A^* and A^+ denote the conjugate transpose and the Moore-Penrose inverse of A respectively, $R(A)$ be the column space of A, and $P_A=A(A^*A)^+A^*=AA^+$ be the orthogonal projector on $R(A)$.

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2. Kantorovich type inequalities and their equivalent forms

In the following theorem, we collect together some of the Kantotovich type inequalities that appeared in the literature [1, 3, 4, 5, 6, 7] except the inequalities (2.3) and (2.5). For the completeness, we will give a unified proof.

THEOREM 2.1. (Kantorovich type inequalities) *Let A be an $n\times n$ positive definite matrix, λ_1 and λ_n be its largest and smallest eigenvalues. If H is an $n\times r$ matrix such that $H^*H=I_r$, then*

$$H^*A^2H \leqslant \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}(H^*AH)^2, \tag{2.1}$$

$$H^*A^2H-(H^*AH)^2 \leqslant \frac{1}{4}(\lambda_1-\lambda_n)^2 I_r, \tag{2.2}$$

$$H^*A^2H-(H^*AH)^2 \leqslant (\sqrt{\lambda_1}-\sqrt{\lambda_n})^2 H^*AH. \tag{2.3}$$

Another equivalent form of each of above inequalities is

$$H^*AH \leqslant \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}(H^*A^{-1}H)^{-1}, \tag{2.4}$$

$$H^*AH-(H^*A^{-1}H)^{-1} \leqslant \frac{1}{4}(\lambda_1-\lambda_n)^2 H^*A^{-1}H, \tag{2.5}$$

$$H^*AH-(H^*A^{-1}H)^{-1} \leqslant (\sqrt{\lambda_1}-\sqrt{\lambda_n})^2 I_r \tag{2.6}$$

respectively.

Proof. Since $0\leqslant(\lambda_1 I_n - A)(A-\lambda_n I_n)$, we have $A^2\leqslant(\lambda_1+\lambda_n)A-\lambda_1\lambda_n I_n$ and then,

$$H^*A^2H \leqslant (\lambda_1+\lambda_n)H^*AH-\lambda_1\lambda_n I_r.$$

The right hand side of this inequality can be expressed as

$$\begin{aligned}
&(\lambda_1+\lambda_n)H^*AH-\lambda_1\lambda_n I_r\\
=&\frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}(H^*AH)^2-\left(\sqrt{\lambda_1\lambda_n}I_r-\frac{\lambda_1+\lambda_n}{2\sqrt{\lambda_1\lambda_n}}H^*AH\right)^2\\
=&(H^*AH)^2+\frac{1}{4}(\lambda_1-\lambda_n)^2I_r-\left(H^*AH-\frac{\lambda_1+\lambda_n}{2}I_r\right)^2\\
=&(H^*AH)^2+(\sqrt{\lambda_1}-\sqrt{\lambda_n})^2H^*AH-(\sqrt{\lambda_1\lambda_n}I_r-H^*AH)^2.
\end{aligned}$$

Hence we obtain (2.1), (2.2) and (2.3).

Replacing H by $A^{-\frac{1}{2}}H(H^*A^{-1}H)^{-\frac{1}{2}}$ in (2.1), (2.2) and (2.3) respectively, and then pre- and post-multiplying them by $(H^*A^{-1}H)^{\frac{1}{2}}$, we obtain (2.4), (2.5) and (2.6) respectively; Conversely, replacing H by $A^{\frac{1}{2}}H(H^*AH)^{-\frac{1}{2}}$ in (2.4), (2.5) and (2.6) respectively, and then pre- and post-multiplying them by $(H^*AH)^{\frac{1}{2}}$, we get (2.1), (2.2) and (2.3) respectively. $\square$

Next we generalized the above theorem to positive semidefinite matrices. The inequality (2.10) below was introduced by Drury et al. [2], The special case of (2.7), (2.8), (2.10) and (2.12) with X^*P_AX being idempotent were considered by Pečarić et al. [7]. And the special case of (2.8) and (2.12) under the condition that $X = AA^+X$ were discussed by Liu and Neudecker [3].

THEOREM 2.2. (Generalized Kantorovich type inequalities) *Let A be an $n\times n$ positive semidefinite matrix, λ_1 and λ_k be its largest and smallest nonzero eigenvalues, where $k = rank(A)$. Then for any $n\times p$ matrix X,*

$$X^*A^2X \leqslant \frac{(\lambda_1+\lambda_k)^2}{4\lambda_1\lambda_k} X^*AX(X^*P_AX)^+X^*AX, \tag{2.7}$$

$$X^*A^2X - X^*AX(X^*P_AX)^+X^*AX \leqslant \frac{1}{4}(\lambda_1-\lambda_k)^2 X^*P_AX, \tag{2.8}$$

$$X^*A^2X - X^*AX(X^*P_AX)^+X^*AX \leqslant (\sqrt{\lambda_1}-\sqrt{\lambda_k})^2 X^*AX. \tag{2.9}$$

Another equivalent form of each of above inequalities is

$$X^*AX \leqslant \frac{(\lambda_1+\lambda_k)^2}{4\lambda_1\lambda_k} X^*P_AX(X^*A^+X)^+X^*P_AX, \tag{2.10}$$

$$X^*AX - X^*P_AX(X^*A^+X)^+X^*P_AX \leqslant \frac{1}{4}(\lambda_1-\lambda_k)^2 X^*A^+X, \tag{2.11}$$

$$X^*AX - X^*P_AX(X^*A^+X)^+X^*P_AX \leqslant (\sqrt{\lambda_1}-\sqrt{\lambda_k})^2 X^*P_AX \tag{2.12}$$

respectively.

Proof. Let $A = U\Lambda U^*$ be the singular value decomposition of A, where Λ is an $k\times k$ diagonal matrix with positive diagonal elements and $k = rank(A)$, U is an $n\times k$ matrix such that $U^*U = I_k$. Then consider the singular value decomposition of U^*X, $U^*X = H\Delta G^*$ where $H^*H = G^*G = I_r$ and $r = rank(U^*X)$, we have $P_A = UU^*$ and $X^*P_AX = G\Delta^2G^*$. Taking $A = \Lambda$ in (2.1), (2.2) and (2.3) respectively, then pre- and post-multiplying them by $G\Delta$ and ΔG^* , noticing that

$$\begin{aligned}
G\Delta(H^*\Lambda^2H)\Delta G^* &= X^*A^2X,\\
G\Delta(H^*\Lambda H)\Delta G^* &= X^*AX,\\
G\Delta(H^*\Lambda H)^2\Delta G^* &= G\Delta(H^*\Lambda H)\Delta G^*(G\Delta^2G^*)^+G\Delta(H^*\Lambda H)\Delta G^*\\
&= X^*AX(X^*P_AX)^+X^*AX,
\end{aligned}$$

we obtain (2.7), (2.8) and (2.9).

Replacing X by $A^{+\frac{1}{2}}X$ in (2.7), (2.8) and (2.9), we get (2.10), (2.11) and (2.12) respectively. Conversely, replacing X by $A^{\frac{1}{2}}X$ in (2.10), (2.11) and (2.12), we obtain (2.7), (2.8) and (2.9). $\square$

3. Wielandt type inequalities and their equivalent forms

First we present an interesting matrix inequality, which is equivalent to the inequality (2.15) in [2, pp. 462], that is, the inequality (3.3) here, and plays a pivotal role in deriving our following results, here we prove it in a different way and give the condition in which the equality holds.

THEOREM 3.1. *Let A be an $n\times n$ positive semidefinite matrix. If X and Y are $n\times p$ and $n\times q$ matrices respectively such that $X^*P_AY=0$, then*

$$X^*AY(Y^*P_AY)^+Y^*AX \leqslant X^*A^2X - X^*AX(X^*P_AX)^+X^*AX. \tag{3.1}$$

Equality holds in (3.1) if and only if

$$R(AX)\subseteq R(P_A(X,Y)), \tag{3.2}$$

where (X,Y) is an $n\times(p+q)$ partitioned matrix.

Another equivalent form of (3.1) is

$$X^*AY(Y^*AY)^+Y^*AX \leqslant X^*AX - X^*P_AX(X^*A^+X)^+X^*P_AX. \tag{3.3}$$

Equality holds in (3.3) if and only if (3.2) holds.

Proof. Since the condition $X^*P_AY=0$ implies $P_{P_AX}P_{P_AY}=P_{P_AY}P_{P_AX}=0$, and since P_{P_AX}, P_{P_AY} are Hermitian idempotent matrices, so $P_{P_AX}+P_{P_AY}$ is a Hermitian idempotent matrix also. By the simple fact that the eigenvalues of a Hermitian idempotent matrix are $0,1$, we know that $I_n\geqslant P_{P_AX}+P_{P_AY}$. Pre- and post-multiplying this inequality by X^*A and AX, we have

$$\begin{aligned} X^*A^2X \geqslant & X^*A(P_{P_AX}+P_{P_AY})AX \\ =& X^*AX(X^*P_AX)^+X^*AX + X^*AY(Y^*P_AY)^+Y^*AY. \end{aligned} \tag{3.4}$$

Thus we obtain (3.1). The equality holds in (3.1) if and only if the equality holds in (3.4), if and only if

$$AX = (P_{P_AX}+P_{P_AY})AX.$$

Noticing that

$$P_{P_AX}+P_{P_AY} = (P_{P_AX},P_{P_AY})(P_{P_AX},P_{P_AY})^* = (P_{P_AX},P_{P_AY})(P_{P_AX},P_{P_AY})^+,$$

and

$$R((P_{P_AX},P_{P_AY})) = R(P_{P_AX})+R(P_{P_AY}) = R(P_AX)+R(P_AY) = R(P_A(X,Y)),$$

we know that the equality holds in(3.1) if and only if $R(AX)\subseteq R(P_A(X,Y))$, that is, (3.2) holds.

Replacing X by $A^{+\frac{1}{2}}X$ and Y by $A^{\frac{1}{2}}Y$ in (3.1), we get (3.3). Conversely, replacing X by $A^{\frac{1}{2}}X$ and Y by $A^{+\frac{1}{2}}Y$ in (3.3), we obtain (3.1). The equality holds in (3.3) if and only if the equality holds in (3.1) and if and only if (3.2) holds. $\square$

Based on the matrix inequality (3.1), we can derive some matrix Wielandt type inequalities easily from the matrix Kantorovich type inequalities appeared in Theorem 2.2. The inequality (3.8) below was established by Drury et al. [2, pp. 464, (2.29)].

THEOREM 3.2. (Generalized Wielandt type inequalities) *Let A be an $n \times n$ positive semidefinite matrix, λ_1 and λ_k be its largest and smallest nonzero eigenvalues where $k = rank(A)$. If X and Y are $n \times p$ matrix and $n \times q$ matrix respectively such that $X^*P_AY = 0$, then*

$$X^*AY(Y^*P_AY)^+Y^*AX \leqslant \left(\frac{\lambda_1 - \lambda_k}{\lambda_1 + \lambda_k}\right)^2 X^*A^2X, \tag{3.5}$$

$$X^*AY(Y^*P_AY)^+Y^*AX \leqslant \frac{1}{4}(\lambda_1 - \lambda_k)^2 X^*P_AX, \tag{3.6}$$

$$X^*AY(Y^*P_AY)^+Y^*AX \leqslant (\sqrt{\lambda_1} - \sqrt{\lambda_k})^2 X^*AX. \tag{3.7}$$

Another equivalent form of each of above inequalities is

$$X^*AY(Y^*AY)^+Y^*AX \leqslant \left(\frac{\lambda_1 - \lambda_k}{\lambda_1 + \lambda_k}\right)^2 X^*AX, \tag{3.8}$$

$$X^*AY(Y^*AY)^+Y^*AX \leqslant \frac{1}{4}(\lambda_1 - \lambda_k)^2 X^*A^+X, \tag{3.9}$$

$$X^*AY(Y^*AY)^+Y^*AX \leqslant (\sqrt{\lambda_1} - \sqrt{\lambda_k})^2 X^*P_AX \tag{3.10}$$

respectively.

Proof. The inequalities (3.5) to (3.10) can be reduced directly from Theorem 2.2 and Theorem 3.1. By the seme way used in the proof of Theorem 3.1, we know that (3.5), (3.6) and (3.7) imply (3.8), (3.9) and (3.10) respectively and vice versa. □

In [9], Zhang proved the equivalence of the Kantorovich inequality (2.4) and the Wielandt inequality (3.8) with A is positive definite by means of submatrices and the eigenvalue interlacing theorem. In [2] Drury et al. presented the equivalence of the Kantorovich inequality (2.10) and the Wielandt inequality (3.8) by making use of the matrix inequality (3.3). In above theorem, we have driven the Wielandt type inequalities (3.5) to (3.10) from the Kantorovich type inequalities (2.7) to (2.12) in Theorem 2.2. Now we will show that the Kantorovich type inequalities (2.7) to (2.12) can be driven easily from the Wielandt type inequalities (3.5) to (3.10) by using (3.1) and (3.3). Hence each of the Kantorovich type inequalities in Theorem 2.2 is equivalent to the corresponding one of the Wielandt type inequalities in Theorem 3.2.

THEOREM 3.3. *Let A be an $n \times n$ positive semidefinite matrix, λ_1 and λ_k be its largest and smallest nonzero eigenvalues where $k = rank(A)$. Then each of the Kantorovich type inequalities in Theorem 2.2 is equivalent to the corresponding one of the Wielandt type inequalities in Theorem 3.2 with $X^*P_AY = 0$.*

Proof. We only need to prove that the Wielandt type inequalities imply the Kantorovich type inequalities. Let $A = U\Lambda U^*$ and $U^*X = H\Delta G^*$ be the singular value decompositions of A and U^*X respectively, just as we do in the proof of Theorem 2.2. Since $H^*H = I_r$, there exists an $k\times(k-r)$ matrix V such that $Q = (H,V)$ is an $k\times k$ unitary matrix. Take $Y = UVV^*U^*$, then $X^*P_AY = X^*UU^*Y = G\Delta H^*VV^*U^* = 0$ since $H^*V = 0$. From the fact that

$$P_A(X,Y)\begin{pmatrix} G & 0 \\ 0 & UV \end{pmatrix}\begin{pmatrix} \Delta^{-1} & 0 \\ 0 & I_{k-r} \end{pmatrix} Q^* = U,$$

we have

$$R(AX) \subseteq R(U) \subseteq R(P_A(X,Y)).$$

Hence, for these X and Y, equalities hold in (3.1) and (3.3), and then, by Theorem 3.2, the inequalities in Theorem 2.2 hold at once. $\square$

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