Journal of
Mathematical
Inequalities
Volume 8, Number 1 (2014), 171–183 doi:10.7153/jmi-08-11

ON SOME MEANS DERIVED FROM THE SCHWAB–BORCHARDT MEAN

EDWARD NEUMAN

(*Communicated by J. Pečarić*)

Abstract. Bivariate means defined as the Schwab-Borchardt mean of two bivariate means are investigated. Explicit formulas for those means are obtained. It is demonstrated that they interpolate inequalities connecting the well known bivariate means. Optimal bounds for the means under discussion are also obtained. The bounding quantities are convex combinations of the generating means.

1. Introduction

There is a renewed interest in the research which deals with inequalities for the bivariate means. Among means which attracted attention of researchers is the Schwab-Borchardt mean. This little known mean has been investigated in [2], [3], and [26, 27]. In [26] the authors have pointed out that the logarithmic mean, two Seiffert means (see [29, 30]), and what some researchers called recently as the Neuman-Sándor mean (see, e.g., [16, 32, 35]), are particular cases of the Schwab-Borchardt mean. They are obtained by forming the Schwab-Borchardt mean of two means from the set containing the geometric mean, the arithmetic mean, and the square-root mean. Lower and upper bounds, in the form of convex combinations of the generating means, for two Seiffert means, logarithmic mean and the Neuman-Sándor means have been obtained by several researches. For more details, the interested reader is referred to [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 20, 23, 24, 25, 28, 31, 33, 34].

This paper is organized as follows. Bivariate means used in the subsequent parts of this work are introduced in Section 2. Two particular means derived from the Schwab-Borchardt mean, denoted by S_{AH} and S_{HA}, are defined and discussed in Section 3. In particular, explicit formulas, interpolation property, and best bounds in the form convex combinations of the generating means H and A are obtained. Similar results for means denoted in this paper by S_{CA} and S_{AC} are established in Section 4. In Section 5 we shall establish four inequalities involving products of two means.

Mathematics subject classification (2010): 26E60, 26D05, 26D07.
Keywords and phrases: Schwab-Borchardt mean, Seiffert means, logarithmic mean, convex combinations, total positivity, inequalities.

Paper JMI-08-11

2. Bivariate means used in this paper

In this section we provide definitions of several bivariate means used in the subsequent sections of this paper.

Let a and b be positive numbers. In order to avoid trivialities we will always assume that $a \neq b$. The unweighted arithmetic mean A of a and b is defined as

$$A = \frac{a+b}{2}.$$

For the reader's convenience let us recall definitions of the first and the second Seiffert means, denoted by P and T, respectively, the Neuman-Sándor mean M, and the logarithmic mean L. Recall that

$$\begin{aligned} P = A\frac{v}{\sin^{-1} v}, \quad & T = A\frac{v}{\tan^{-1} v}, \\ M = A\frac{v}{\sinh^{-1} v}, \quad & L = A\frac{v}{\tanh^{-1} v}, \end{aligned} \tag{1}$$

where

$$v = \frac{a-b}{a+b}. \tag{2}$$

(see [29], [30], [26]). Clearly $0 < |v| < 1$.

Other unweighted bivariate means used in this paper are the harmonic mean H, the geometric mean G, the root-square mean Q and the contra-harmonic mean C which are defined in usual way

$$H = \frac{2ab}{a+b}, \quad G = \sqrt{ab}, \quad Q = \sqrt{\frac{a^2+b^2}{2}}, \quad C = \frac{a^2+b^2}{a+b}. \tag{3}$$

One can easily verify that the means listed in (3) all can be expressed in terms of A and v. We have

$$\begin{aligned} H = A(1-v^2), \quad & G = A\sqrt{1-v^2}, \\ Q = A\sqrt{1+v^2}, \quad & C = A(1+v^2). \end{aligned} \tag{4}$$

All the means mentioned above are comparable. It is known that

$$H < G < L < P < A < M < T < Q < C \tag{5}$$

(see, e.g., [26]).

The four means included in (1) are special cases of the Schwab-Borchardt mean SB which is defined as follows

$$SB(x,y) \equiv SB = \begin{cases} \dfrac{\sqrt{y^2-x^2}}{\cos^{-1}(x/y)} & \text{if } x < y, \\ \dfrac{\sqrt{x^2-y^2}}{\cosh^{-1}(x/y)} & \text{if } y < x \end{cases} \tag{6}$$

(see, e.g., [2], [3]), where both x and y are positive numbers. This mean has been studied extensively in [26], [27], and in [19]. It is well known that the mean SB is strict, nonsymmetric and homogeneous of degree one in its variables.

It has been pointed out in [26] that

$$\begin{aligned} P &= SB(G,A), \quad T = SB(A,Q), \\ M &= SB(Q,A), \quad L = SB(A,G). \end{aligned} \tag{7}$$

Other bivariate means used in this paper are derived from the Schwab-Borchardt mean. They are defined as follows

$$\begin{aligned} S_{AH} &= SB(A,H), \quad S_{HA} = SB(H,A), \quad S_{CA} = SB(C,A), \\ S_{AC} &= SB(A,C), \quad S_{CH} = SB(C,H), \quad S_{HC} = SB(H,C). \end{aligned} \tag{8}$$

In the next sections we will deal with the pairs of means $\{S_{AH}, S_{HA}\}$, $\{S_{CA}, S_{AC}\}$, and $\{S_{CH}, S_{HC}\}$.

3. Means S_{AH} and S_{HA}

For the later use let us record some elementary formulas

$$\cos^{-1}(x/y) = \sin^{-1}\left(\frac{\sqrt{y^2-x^2}}{y}\right) = \tan^{-1}\left(\frac{\sqrt{y^2-x^2}}{x}\right) \tag{9}$$

$(0 < x < y)$ and

$$\cosh^{-1}(x/y) = \sinh^{-1}\left(\frac{\sqrt{x^2-y^2}}{y}\right) = \tanh^{-1}\left(\frac{\sqrt{x^2-y^2}}{x}\right) \tag{10}$$

$(x > y > 0)$. They will be used in the sequel.

We shall establish now formulas for means under discussion. They bear some resemblance of formulas (1).

THEOREM 1. *Let p be defined implicitly as* $\operatorname{sech} p = 1 - v^2$. *Then*

$$S_{AH} = A\frac{\tanh p}{p}. \tag{11}$$

Similarly, if $\cos q = 1 - v^2$, *then*

$$S_{HA} = A\frac{\sin q}{q}. \tag{12}$$

Proof. We shall establish formula (11) using (6) with $x = A$ and $y = H$. Using the first formula of (4) we get

$$\sqrt{A^2 - H^2} = A\sqrt{1-(1-v^2)^2} = Av\sqrt{2-v^2} = A\lambda,$$

where

$$\lambda = v\sqrt{2-v^2}. \tag{13}$$

With $\operatorname{sech} p = 1 - v^2$ we obtain

$$\lambda = \sqrt{1-\operatorname{sech} p}\sqrt{1+\operatorname{sech} p} = \sqrt{1-\operatorname{sech}^2 p} = \tanh p.$$

This in conjunction with (8), (6), and (10) yields

$$S_{AH} = A\frac{\lambda}{\tanh^{-1}\lambda} = A\frac{\tanh p}{p}. \tag{14}$$

For the proof of (12) we use (6) with $x = H$ and $y = A$. Then $\sqrt{A^2 - H^2} = A\lambda$, where λ is the same as in (13). Using $\cos q = 1 - v^2$ we obtain

$$\lambda = \sqrt{1-\cos q}\sqrt{1+\cos q} = \sin q.$$

Then the second formula of (8), the first part of (6), and (9) give

$$S_{HA} = A\frac{\lambda}{\sin^{-1}\lambda} = A\frac{\sin q}{q} \tag{15}$$

which completes the proof of the theorem. □

COROLLARY 1. *Let $\rho = H/A$. Then the following formula*

$$\frac{S_{HA}}{S_{AH}} = \frac{\operatorname{sech}^{-1}(\rho)}{\cos^{-1}(\rho)}$$

is valid.

Proof. Let $\rho = 1 - v^2$. Using the first formula of (4) we have $\rho = H/A$. Using (14) and (15) we obtain

$$\frac{S_{HA}}{S_{AH}} = \frac{\tanh^{-1}\lambda}{\sin^{-1}\lambda} = \frac{p}{q},$$

where λ is defined in (13) and also p and q are defined implicitly in Theorem 1. This in turn gives $p = \operatorname{sech}^{-1}(1-v^2) = \operatorname{sech}^{-1}(\rho)$ and $q = \cos^{-1}(1-v^2) = \cos^{-1}(\rho)$. □

To the end of this section we will deal with inequalities involving means S_{AH} and S_{HA}. For the latter use we recall the following result [21, 22]:

$$(xy^2)^{1/3} < (ySB(y,x))^{1/2} < SB(x,y) < \frac{y + SB(y,x)}{2} < \frac{x+2y}{3}. \tag{16}$$

Letting above $x = A$ and $y = H$ we obtain inequalities which connect S_{AH} with S_{HA}. We omit further details.

We shall prove now that two means under discussion interpolate some of the inequalities which appear in the chain of inequalities (5).

THEOREM 2. *The inequalities*

$$H < S_{AH} < L < S_{HA} < P \tag{17}$$

hold true.

Proof. The first inequality in (17) is obvious because S_{AH} is the mean value of A and H therefore it satisfies $H < S_{AH} < A$. For the proof of the second inequality we shall use formulas (14) and the last part of (1). Then the inequality in question is equivalent to

$$\frac{\lambda}{\tanh^{-1}\lambda} < \frac{v}{\tanh^{-1} v}.$$

Also, it follows from (13) that $\lambda > v$. Since the function $x/\tanh^{-1}x$ is strictly decreasing on the interval $(0,1)$, the desired inequality follows. We shall demonstrate now that the third inequality in (17) is valid. To this aim we shall prove that the second inequality in

$$L < \frac{A+2G}{3} < S_{HA}$$

holds true. The first one is well known (see, e.g., [26]) and can be obtained using the third and fifth members of (16) with $x = A$ and $y = G$. It follows from (4) and (14) that the inequality we have to prove is equivalent to

$$\frac{1+2\sqrt{1-v^2}}{3} < \frac{\lambda}{\sin^{-1}\lambda}.$$

Since $1 - v^2 = (1-\lambda^2)^{1/2}$ (see (5)),

$$\frac{1+2(1-\lambda^2)^{1/4}}{3} < \frac{\lambda}{\sin^{-1}\lambda}.$$

Letting $\lambda = \sin t$ $(0 < t < \pi/2)$ we obtain

$$\frac{1+2\sqrt{\cos t}}{3} < \frac{\sin t}{t}. \tag{18}$$

For the proof of (18) we will utilize the following result [21, 22]:

$$\left(\frac{1+\cos t}{2}\right)^{2/3} < \frac{\sin t}{t} \tag{19}$$

$(0 < t < \pi/2)$. First we shall show that

$$\frac{1+2\sqrt{\cos t}}{3} < \left(\frac{1+\cos t}{2}\right)^{2/3}. \tag{20}$$

Then (18) will follow from (20) and (19). In (20) we use the substitution $\cos t = u^2$. The result is

$$\frac{1+2u}{3} < \left(\frac{1+u^2}{2}\right)^{2/3}. \tag{21}$$

In order to prove validity of (21) it suffices to show that $f(u) > 0$, where $f(u) = 27(1+u^2)^2 - 4(1+2u)^3$ $(0 \leqslant u \leqslant 1)$. Differentiation yields $f'(u) = 12(u-1)(9u^2+u+2)$. Thus $f'(u) < 0$ on $[0,1)$. Since $f(0) = 23$ and $f(1) = 0$, we conclude that the function $f(u)$ is strictly decreasing and positive on the stated domain. This in turn implies that the inequality (18) holds true. The proof of the third inequality in (17) is complete. It follows from (15) and the first part of (1) that the last inequality in (17) is equivalent to

$$\frac{\lambda}{\sin^{-1}\lambda} < \frac{v}{\sin^{-1} v}.$$

Since $\lambda > v$ and the function $x/\sin^{-1} x$ is strictly decreasing on $(0,1)$, the assertion follows. The proof is complete. $\square$

In the proof of the next theorem we will utilize the following result-which is often called L'Hospital's rule for monotonicity (see, e.g., [1]):

THEOREM A. *Let the functions f and g be continuous on $[c,d]$, differentiable on (c,d) and such that $g'(t) \neq 0$ on (c,d). If $\frac{f'(t)}{g'(t)}$ is (strictly) increasing (decreasing) on (c,d), then the functions $\frac{f(t)-f(d)}{g(t)-g(d)}$ and $\frac{f(t)-f(c)}{g(t)-g(c)}$ are also (strictly) increasing (decreasing) on (c,d).*

THEOREM 3. *The inequality*

$$\alpha_1 A + (1-\alpha_1)H < S_{AH} < \beta_1 A + (1-\beta_1)H \tag{22}$$

holds true if $\alpha_1 = 0$ and $\frac{1}{3} \leqslant \beta_1 < 1$. Also, the inequality

$$\alpha_2 A + (1-\alpha_2)H < S_{HA} < \beta_2 A + (1-\beta_2)H \tag{23}$$

is satisfied if $0 < \alpha_2 \leqslant 2/\pi$ and $2/3 \leqslant \beta_2 < 1$.

Proof. For the proof of the first part of the thesis let us write (22) in the equivalent form

$$\alpha_1 < \frac{S_{AH} - H}{A - H} < \beta_1.$$

Using (11), (4), and the fact that $1 - v^2 = \operatorname{sech} p$ we can write the last two-sided inequality in the form

$$\alpha_1 < \frac{\dfrac{\tanh p}{p} - \operatorname{sech} p}{1 - \operatorname{sech} p} < \beta_1,$$

where $p > 0$. Let $\varphi_1(p)$ stand for the middle term in the last inequality. Using elementary identities for the hyperbolic functions we obtain

$$\varphi_1(p) = \frac{\sinh p - p}{p\cosh p - p}.$$

It has been proven in [20, Theorem 3.1] that the function $\varphi_1(p)$ is strictly decreasing on its domain. Elementary computations yield $\varphi_1(0^+) = 1/3$ and $\lim_{p->\infty} \varphi_1(p) = 0$. The second part of the thesis can be established in a similar fashion. First we write (23) as

$$\alpha_2 < \frac{S_{HA} - H}{A - H} < \beta_2.$$

Using (12) and the first formula in (4) in the form $H = A\cos q$ we can write the last two-sided inequality as follows

$$\alpha_2 < \frac{\frac{\sin q}{q} - \cos q}{1 - \cos q} < \beta_2,$$

where $0 < q < \pi/2$. Let $\varphi_2(q)$ denote the second term in the last simultaneous inequality. Then

$$\varphi_2(q) = \frac{\sin q - q\cos q}{q - q\cos q}.$$

Let $f(q) = \sin q - q\cos q$ and $g(q) = q - q\cos q$. Differentiation yields

$$\frac{f'(q)}{g'(q)} = \frac{q\sin q}{1 - \cos q + q\sin q} =: h(q).$$

Also,

$$(1 - \cos q + q\sin q)^2 h'(q) = 2\sin^2(\frac{q}{2})(\sin q - q).$$

Since $\sin q < q$, function $h(q)$ is strictly decreasing on its domain and so is the function $f'(q)/g'(q)$. Using Theorem A we conclude that the function $\varphi_2(q)$ is strictly decreasing on the interval $(0, \pi/2)$. It is easy to demonstrate that $\varphi_2(0^+) = 2/3$ and $\varphi_2(\pi/2) = 2/\pi$. This gives the bounds for α_2 and β_2. The proof is complete. □

4. Means S_{CA} and S_{AC}

In this section we shall establish results for the pair of means $\{S_{CA}, S_{AC}\}$. They can be regarded as the counterparts of the results derived in Section 3 for the pair $\{S_{AH}, S_{HA}\}$. For the sake of brevity let $\gamma = \cosh^{-1}(2) = 1.317...)$. The representation formulas for the means under discussion read as follow.

THEOREM 4. *Let* $\mu = v\sqrt{2 + v^2}$. *If* $\cosh r = 1 + v^2$ $(0 < r < \gamma)$, *then*

$$S_{CA} = A\frac{\mu}{\sinh^{-1}\mu} = A\frac{\sinh r}{r}. \tag{24}$$

Also, if $\sec s = 1 + v^2$, *then*

$$S_{AC} = A\frac{\mu}{\tan^{-1}\mu} = A\frac{\tan s}{s} \tag{25}$$

where $0 < s < \pi/3$.

Proof. Use of (4) gives $\sqrt{C^2-A^2} = Av\sqrt{2+v^2} = A\mu$. Making use of (6) with $x = C$ and $y = A$ followed by application of (10) yields

$$S_{CA} = A\frac{\mu}{\sinh^{-1}\mu}.$$

On the other hand,

$$\mu = \sqrt{\cosh r - 1}\sqrt{\cosh r + 1} = \sinh r.$$

The assertion (24) now follows. For the proof of (25) we follow the lines introduced above with $x = A$ and $y = C$ to obtain

$$S_{AC} = A\frac{\mu}{\tan^{-1}\mu}.$$

Using definition of μ and the assumption that $1+v^2 = \sec s$ we obtain

$$\mu = \sqrt{\sec s - 1}\sqrt{\sec s + 1} = \tan s.$$

The third term of (25) is now obtained and the proof is complete. $\square$

COROLLARY 2. *Let $\sigma = C/A$. Then*

$$\frac{S_{AC}}{S_{CA}} = \frac{\cosh^{-1}(\sigma)}{\sec^{-1}(\sigma)}.$$

Proof. Let $\sigma = 1+v^2$. Making use of (4) we obtain $\sigma = C/A$. Using (24) and (25) we obtain

$$\frac{S_{AC}}{S_{CA}} = \frac{\sinh^{-1}\mu}{\tan^{-1}\mu} = \frac{r}{s},$$

where r and s are defined implicitly in Theorem 4. This in turn gives $r = \cosh^{-1}(1+v^2) = \cosh^{-1}(\sigma)$ and $s = \sec^{-1}(1+v^2) = \sec^{-1}(\sigma)$. The proof is complete. $\square$

Our next result reads as follows.

THEOREM 5. *The following inequalities*

$$T < S_{CA} < Q < S_{AC} < C \tag{26}$$

hold true.

Proof. Let us begin proving the first inequality in (26). To this aim we need to verify the second inequality in

$$T < \frac{A+2Q}{3} < S_{CA}.$$

The first one is well known (see, e.g., [26]) and can be obtained using (16) with $x = A$ and $y = Q$ followed by application of the second formula in (7). It follows from (4) and (24) that the inequality to be proven is equivalent to

$$\frac{1+2\sqrt{1+v^2}}{3} < \frac{\mu}{\sinh^{-1}\mu}.$$

The last inequality can be written in terms of variable r. It follows from the proof of Theorem 4 that $1+v^2 = \cosh r$ and $\mu = \sinh r$, where $0 < r < \gamma$. Thus we have to show that

$$\frac{1+2\sqrt{\cosh r}}{3} < \frac{\sinh r}{r}. \tag{27}$$

To this aim we will utilize the inequality [22, 21]:

$$\left(\frac{1+\cosh r}{2}\right)^{2/3} < \frac{\sinh r}{r}. \tag{28}$$

The inequality (27) will follow if we shall demonstrate that

$$\frac{1+2\sqrt{\cosh r}}{3} < \left(\frac{1+\cosh r}{2}\right)^{2/3}. \tag{29}$$

Letting in (29) $\cosh r = u^2$ we see that the last inequality is equivalent to $f(u) > 0$ $(u > 1)$, where

$$f(u) = 27(1+u^2)^2 - 4(1+2u)^3$$

$(u \geqslant 1)$. We have already encountered this function in the proof of Theorem 3. Recall that $f'(u) = 12(u-1)(9u^2+u+2)$ and also that $f(1) = 0$. Thus the function $f(u)$ is strictly increasing on the stated domain. This in turn implies that the inequalities (29) and (28) are satisfied. Therefore the inequality (27) holds true. For the proof of the second inequality in (26) we use formulas (24) and (4). Then the inequality in question is equivalent to

$$\frac{\mu}{\sinh^{-1}\mu} < \sqrt{1+v^2}.$$

With $\mu = \sinh r$ and $1+v^2 = \cosh r$ $(0 < r < \gamma)$ the last inequality can be written as

$$\frac{\sinh r}{r} < (\cosh r)^{1/2}$$

which holds true for $0 < r < \pi/2$ (see [15]). The proof of the second inequality is complete because $\gamma < \pi/2$. We shall establish now the third inequality in (26). Using (4) and (25) we see that the inequality in question is equivalent to

$$\sqrt{1+v^2} < \frac{\mu}{\tan^{-1}\mu}.$$

It follows from the proof of Theorem 4 that $1+v^2 = \sec s$ and $\mu = \tan s$ $(0 < s < \pi/3)$. Thus the last inequality can be written as

$$(\sec s)^{1/2} < \frac{\tan s}{s}.$$

Multiplying both sides by $\cos s$ we obtain

$$(\cos s)^{1/2} < \frac{\sin s}{s}.$$

Taking into account that

$$(\cos s)^{1/2} < (\cos s)^{1/3}$$

holds true for $0 < s < \pi/2$ and also that

$$(\cos s)^{1/3} < \frac{\sin s}{s}$$

$(0 < s < \pi/2)$ (see, e.g., [18]) we see that the inequality to be proven is valid. This completes the proof of the third inequality in (26). Finally, the last inequality in the chain (26) is obvious because S_{AC} as the mean value of A and C is smaller than $\max\{A,C\} = C$. $\square$

A counterpart of Theorem 3 for the means under discussion reads as follows.

THEOREM 6. *The two-sided inequality*

$$\alpha_1 C + (1-\alpha_1)A < S_{CA} < \beta_1 C + (1-\beta_1)A \tag{30}$$

holds true if $0 < \alpha_1 = (\sqrt{3}-\gamma)/\gamma = 0.315...$ *and* $1/3 \leqslant \beta_1 < 1$. *Also, the inequality*

$$\alpha_2 C + (1-\alpha_2)A < S_{AC} < \beta_2 C + (1-\beta_2)A \tag{31}$$

is satisfied if $0 < \alpha_2 \leqslant (3\sqrt{3}-\pi)/\pi = 0.653...$ *and* $2/3 \leqslant \beta_2 < 1$.

Proof. For the proof of the first part of the thesis let us write inequality (30) as follows

$$\alpha_1 < \frac{S_{CA}-A}{C-A} < \beta_1.$$

Using (26), (4) and the fact that $1+v^2 = \cosh r$ we can rewrite the last two-sided inequality, after elementary transformations, as follows

$$\alpha_1 < \frac{\sinh r - r}{r\cosh r - r} < \beta_1,$$

where $0 < r < \gamma$. We have already encountered the middle term

$$\frac{\sinh r - r}{r\cosh r - r} = \varphi_1(r)$$

in the proof of Theorem 3. Therefore the function $\varphi_1(r)$ is strictly decreasing on the stated domain. Since $\varphi_1(0^+) = 1/3$ and $\varphi_1(\gamma^-) = (\sqrt{3}-\gamma)/\gamma$, the thesis of the first part of the theorem follows. The second part of the thesis can be established in a similar manner. First we write (31) as

$$\alpha_2 < \frac{S_{AC}-A}{C-A} < \beta_2$$

and next use (25), (4), and the formula $1+v^2 = \sec s$, where $0 < s < \pi/3$. The result is

$$\alpha_2 < \varphi_2(s) < \beta_2,$$

where

$$\varphi_2(s) = \frac{\sin s - s\cos s}{s - s\cos s}$$

was used in the proof of Theorem 3. We have demonstrated that the function is strictly decreasing on the interval $(0,\pi/2)$. Easy computations give $\varphi_2(0^+) = 2/3$ and $\varphi_2(\pi/3) = (3\sqrt{3}-\pi)/\pi$. The proof is complete. □

5. Inequalities involving products of two means

Numerous inequalities involving particular means derived from the Schwab-Borchardt mean have been obtained in [27]. Here is the sample of some results

$$PM < AQ^2 \textit{ and } LT < A^2$$

derived in the above cited paper.

In this section we shall establish four inequalities which involve products of two means on each side of an inequality. First we shall prove the following

PROPOSITION 1. *Formulas*

$$S_{HC} = T \textit{ and } S_{CH} = L \tag{32}$$

are valid.

Proof. In the proof of (32) we utilize the invariance property

$$SB(x,y) = SB\left(\frac{x+y}{2}, \sqrt{\frac{x+y}{2}y}\right) \tag{33}$$

(see,e.g., [2, 3]). For the proof of the first formula in (32) we let $x = H$ and $y = C$. Taking into account that

$$H = G^2A^{-1} \textit{ and } C = Q^2A^{-1}$$

and employing homogeneity of SB, the identity $(G^2+Q^2)/2 = A^2$, and (33) we obtain

$$\begin{aligned} S_{HC} &= SB(H,C) = SB(G^2A^{-1}, Q^2A^{-1}) = A^{-1}SB(G^2,Q^2) \\ &= A^{-1}SB(A^2,AQ) = (A^{-1}A)SB(A,Q) = SB(A,Q) = T, \end{aligned}$$

where the last equality follows from (7). The second formula in (32) can be established in a similar way. We omit further details. □

In [4] the authors have proven that the function $SB(x,y)^{-1}$ $(x,y>0)$ is strictly totally positive. This in particular implies that the following inequality

$$SB(x_1,y_1)SB(x_2,y_2) < SB(x_1,y_2)SB(x_2,y_1) \tag{34}$$

holds true provided $x_1 < x_2$ and $y_1 < y_2$.

We are in a position to prove the following

THEOREM 7. *The following inequalities*

$$S_{HA}S_{AC} < AT, \quad S_{AH}S_{CA} < AL \tag{35}$$

and

$$S_{HA}C < S_{CA}T, \quad S_{AH}C < S_{AC}L \tag{36}$$

are valid.

Proof. For the later use let us note that $SB(A,A) = A$. For the proof of the first inequality in (35) we use (34) with $\{x_1,x_2\} = \{H,A\}$, $\{y_1,y_2\} = \{A,C\}$ followed by application of (8) and (32). Similarly, the second part of (35) follows from (34) with $\{x_1,x_2\} = \{A,C\}$ and $\{y_1,y_2\} = \{H,A\}$. Inequalities (36) are established in the same way. The first one follows from (34) by letting $\{x_1,x_2\} = \{H,C\}$ and $\{y_1,y_2\} = \{A,C\}$ while the second inequality in (36) is obtained using (34) with $\{x_1,x_2\} = \{A,C\}$ and $\{y_1,y_2\} = \{H,C\}$. The proof is complete. $\square$

Acknowledgements.

The author is indebted to an anonymous referee for constructive remarks on the first draft of this paper and also for calling his attention to related papers which have been added to the list of References.

REFERENCES

[1] H. ALZER, S.-L. QIU, *Monotonicity theorems and inequalities for complete elliptic integrals*, J. Comput. Appl. Math. **172** (2004), 289–312.

[2] J. M. BORWEIN, P. B. BORWEIN, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, John Wiley and Sons, New York, 1987.

[3] B. C. CARLSON, *Algorithms involving arithmetic and geometric means*, Amer. Math. Monthly **78** (1971), 496–505.

[4] B. C. CARLSON, J. L. GUSTAFSON, *Total positivity of mean values and hypergeometric functions*, SIAM J. Math. Anal. **14** (1983), 389–395.

[5] Y.-M. CHU, S.-W. HOU, *Sharp bounds for Seiffert mean in terms of contra-harmonic mean*, Abstr. Appl. Anal. **2012** (2012), Article ID 425175.

[6] Y.-M. CHU, S.-W. HOU, Z.-H. SHEN, *Sharp bounds for Seiffert mean in terms of root mean square*, J. Inequal. Appl. **2012**, 11 (2012), 15 pages.

[7] Y.-M. CHU AND B.-Y. LONG, *Bounds of the Neuman-Sándor mean using power and identric means*, Abstr. Appl. Anal. **2013**, Article ID 832591, 6 pages.

[8] Y.-M. CHU, B.-Y. LONG, W.-M. GONG AND Y.-Q. SONG, *Sharp bounds for Seiffert and Neuman-Sándor means in terms of generalized logarithmic means*, J. Inequal. Appl. **2013**, 10 (2013), 13 pages.

[9] Y.-M. CHU, Y.-F. QIU, M.-K. WANG, G.-D. WANG, *The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert mean*, J. Inequal. Appl. **2010** (2010), Article ID 436457.

[10] Y.-M. CHU, M.-K. WANG, W.-M. GONG, *Two sharp double inequalities for Seiffert mean*, J. Inequal. Appl. **2011**, 44 (2011), 7 pages.
[11] Y.-M. CHU, M.-K. WANG, Z.-K. WANG, *A best possible double inequality between Seiffert and harmonic means*, J. Inequal. Appl. **2011**, 44 (2011), 7 pages.
[12] Y.-M. CHU, C. ZONG, G.-D. WANG, *Optimal convex combination bounds of Seiffert and geometric means for the arithmetic mean*, J. Math. Inequal. **5** (2011), 429–434.
[13] S.-Q. GAO, H.-Y. GAO, W.-Y. SHI, *Optimal convex combination bounds of the centroidal and harmonic means for the Seiffert mean*, Int. J. Pure Appl. Math. **70** (2011), 701–709.
[14] Z.-Y. HE, W.-M. QIAN, Y.-L. JIANG, Y.-Q. SONG AND Y.-M. CHU, *Bounds for the combination of Neuman-Sándor, arithmetic, and second Seiffert means in terms of contraharmonic mean*, Abstr. Appl. Anal., 2013, Article ID 903982, 5 pages.
[15] R. KLEŃ, M. VISURI, M. VUORINEN, *On Jordan type inequalities for hyperbolic functions*, J. Inequal. Appl., vol. **2010**, Article ID 362548, 14 pages.
[16] Y.-M. LI, B.-Y. LONG, Y.-M. CHU, *Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean*, J. Math. Inequal. **4**, 4 (2012), 567–577.
[17] H. LIU, X.-J. MENG, *The optimal convex combination bounds for Seiffert's mean*, J. Inequal. Appl. **2011** (2011), Article ID 686834.
[18] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.
[19] E. NEUMAN, *Inequalities for the Schwab-Borchardt mean and their applications*, J. Math. Inequal. **5**, 4 (2011), 601–609.
[20] E. NEUMAN, *A note on a certain bivariate mean*, J. Math. Inequal. **6**, 4 (2012), 637–643.
[21] E. NEUMAN, *Refinements and generalizations of certain inequalities involving trigonometric and hyperbolic functions*, Adv. Inequal. Appl. **1**, 1 (2012), 1–11.
[22] E. NEUMAN, *Inequalities involving Jacobian elliptic, trigonometric and hyperbolic functions*, J. Inequal. Spec. Funct. **3**, 2 (2012), 16–21.
[23] E. NEUMAN, *A one-parameter family of bivariate means*, J. Math. Inequal., **7**, 3 (2013), 399–412.
[24] E. NEUMAN, *Sharp inequalities involving Neuman-Sándor and logarithmic means*, J. Math. Inequal., **7**, 3 (2013), 413–419.
[25] E. NEUMAN, *Inequalities involving certain bivariate means II*, J. Inequal. Spec. Funct., in press.
[26] E. NEUMAN, J. SÁNDOR, *On the Schwab-Borchardt mean*, Math. Pannon. **14**, 2 (2003), 253–266.
[27] E. NEUMAN, J. SÁNDOR, *On the Schwab-Borchardt mean II*, Math. Pannon. **17**, 1 (2006), 49–59.
[28] W.-M. QIAN AND Y.-M. CHU, *On certain inequalities for Neuman-Sándor mean*, Abstr. Appl. Anal., 2013, Article ID 790783, 6 pages.
[29] H.-J. SEIFFERT, *Problem 887*, Nieuw. Arch. Wisk. **11** (1993), 176.
[30] H.-J. SEIFFERT, *Aufgabe 16*, Würzel **29** (1995), 87.
[31] Y.-Q. SONG, W.-M. QIAN, Y.-L. JIANG AND Y.-M. CHU, *Optimal lower generalized logarithmic mean bound for the Seiffert mean*, J. Appl. Math., 2013, Article ID 273653, 5 pages.
[32] M.-K. WANG, Y.-M. CHU, B.-Y. LIU, *Sharp inequalities for the Neuman-Sándor mean in terms of arithmetic and contra-harmonic means*, preperint, arXiv: 1209.5825 vol 1 [math CA] 26 Sept. 2012.
[33] A. WITKOWSKI, *Interpolations of Schwab-Borchardt means*, Math. Inequal. Appl. **16**, 1 (2013), 193–206.
[34] T.-H. ZHAO, Y.-M. CHU, Y.-L. JIANG AND Y.-M. LI, *Best possible bounds for Neuman-Sándor mean by identric, quadratic and contraharmonic means*, Abstr. Appl. Anal., 2013, Article ID 348326, 12 pages.
[35] T.-H. ZHAO, Y.-M. CHU, B.-Y. LIU, *Optimal bounds for the Neuman-Sándor mean in terms of the convex combinations of harmonic, geometric, quadratic and contra-harmonic means*, Abstract Appl. Anal. Volume 2012, Article ID 302635, 9 pages.

(Received March 9, 2013)

Edward Neuman
Department of Mathematics, Mailcode 4408
Southern Illinois University
1245 Lincoln Drive, Carbondale
IL 62901, USA
e-mail: edneuman@siu.edu