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OPTIMAL BOUNDS FOR NEUMAN–SÁNDOR MEAN IN TERMS OF THE CONVEX COMBINATION OF LOGARITHMIC AND QUADRATIC OR CONTRA–HARMONIC MEANS

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Abstract. In this article, we present the least values α_1, α_2, and the greatest values β_1, β_2 such that the double inequalities

$$\alpha_1 L(a,b)+(1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 L(a,b)+(1-\beta_1)Q(a,b)$$

$$\alpha_2 L(a,b)+(1-\alpha_2)C(a,b) < M(a,b) < \beta_2 L(a,b)+(1-\beta_2)C(a,b)$$

hold for all $a,b>0$ with $a\neq b$, where $L(a,b)$, $M(a,b)$, $Q(a,b)$ and $C(a,b)$ are respectively the logarithmic, Neuman-Sándor, quadratic and contra-harmonic means of a and b.

1. Introduction

For $a,b>0$ with $a\neq b$ the Neuman-Sándor mean $M(a,b)$ [1] is defined by

$$M(a,b)=\frac{a-b}{2\sinh^{-1}\left(\frac{a-b}{a+b}\right)}, \tag{1.1}$$

where $\sinh^{-1}(x)=\log(x+\sqrt{x^2+1})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean $M(a,b)$ can be found in the literature [1, 2].

Let $H(a,b)=2ab/(a+b)$, $G(a,b)=\sqrt{ab}$, $L(a,b)=(b-a)/(\log b-\log a)$, $P(a,b)=(a-b)/(4\arctan\sqrt{a/b}-\pi)$, $A(a,b)=(a+b)/2$, $T(a,b)=(a-b)/[2\arctan((a-b)/(a+b))]$, $Q(a,b)=\sqrt{(a^2+b^2)/2}$ and $C(a,b)=(a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic and contra-harmonic means of a and b, respectively. Then it is well-known that the inequalities

$$H(a,b)<G(a,b)<L(a,b)<P(a,b)<A(a,b)<M(a,b)<T(a,b)<Q(a,b)<C(a,b)$$

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hold for $a,b>0$ with $a\neq b$.

Neuman and Sándor [1, 2] proved that the inequalities

$$\frac{\pi}{4\log(1+\sqrt{2})}T(a,b)<M(a,b)<\frac{A(a,b)}{\log(1+\sqrt{2})},$$

$$\sqrt{2T^2(a,b)-Q^2(a,b)}<M(a,b)<\frac{T^2(a,b)}{Q(a,b)},$$

$$H(T(a,b),A(a,b))<M(a,b)<L(A(a,b),Q(a,b)),\quad T(a,b)>H(M(a,b),Q(a,b)),$$

$$M(a,b)<\frac{A^2(a,b)}{P(a,b)},\quad A^{2/3}(a,b)Q^{1/3}(a,b)<M(a,b)<\frac{2A(a,b)+Q(a,b)}{3},$$

$$\sqrt{A(a,b)T(a,b)}<M(a,b)<\sqrt{A^2(a,b)+T^2(a,b)},$$

$$\frac{G(x,y)}{G(1-x,1-y)}<\frac{L(x,y)}{L(1-x,1-y)}<\frac{P(x,y)}{P(1-x,1-y)}$$

$$<\frac{A(x,y)}{A(1-x,1-y)}<\frac{M(x,y)}{M(1-x,1-y)}<\frac{T(x,y)}{T(1-x,1-y)},$$

$$\frac{1}{A(1-x,1-y)}-\frac{1}{A(x,y)}<\frac{1}{M(1-x,1-y)}-\frac{1}{M(x,y)}<\frac{1}{T(1-x,1-y)}-\frac{1}{T(x,y)},$$

$$A(x,y)A(1-x,1-y)<M(x,y)M(1-x,1-y)<T(x,y)T(1-x,1-y)$$

hold for all $a,b>0$ and $x,y\in(0,1/2]$ with $a\neq b$ and $x\neq y$.

Li et al. [3] showed that the double inequality $L_{p_0}(a,b)<M(a,b)<L_2(a,b)$ holds for all $a,b>0$ with $a\neq b$, where $L_p(a,b)=[(b^{p+1}-a^{p+1})/((p+1)(b-a))]^{1/p}(p\neq -1,0)$, $L_0(a,b)=1/e(b^b/a^a)^{1/(b-a)}$ and $L_{-1}(a,b)=(b-a)/(\log b-\log a)$ is the $p-$th generalized logarithmic mean of a and b, and $p_0=1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p}=2\log(1+\sqrt{2})$.

In [4], Neuman proved that the double inequalities

$$\alpha Q(a,b)+(1-\alpha)A(a,b)<M(a,b)<\beta Q(a,b)+(1-\beta)A(a,b)$$

and

$$\lambda C(a,b)+(1-\lambda)A(a,b)<M(a,b)<\mu C(a,b)+(1-\mu)A(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leqslant[1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})]=0.3249\cdots$, $\beta\geqslant 1/3$, $\lambda\leqslant[1-\log(1+\sqrt{2})]/\log(1+\sqrt{2})=0.1345\cdots$ and $\mu\geqslant 1/6$.

Very recently, inequalities for quotients involving the Neuman-Sándor mean $M(a,b)$ were obtained in [5].

The main purpose of this paper is to find the least values α_1, α_2 and the greatest values β_1, β_2 such that the double inequalities

$$\alpha_1 L(a,b)+(1-\alpha_1)Q(a,b)<M(a,b)<\beta_1 L(a,b)+(1-\beta_1)Q(a,b),$$

$$\alpha_2 L(a,b)+(1-\alpha_2)C(a,b)<M(a,b)<\beta_2 L(a,b)+(1-\beta_2)C(a,b)$$

hold for all $a,b>0$ with $a\neq b$. All numerical computations are carried out using MATHEMATICAL software.

2. Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

LEMMA 2.1. *Let $F(x) = x/\sqrt{1+x^2}$. Then the double inequality*

$$x - \frac{x^3}{2} < F(x) < x - \frac{x^3}{2} + \frac{2}{5}x^5 < x - \frac{x^3}{2} + \frac{x^5}{2} \tag{2.1}$$

holds for all $x \in (0,1)$.

Proof. Inequality (2.1) follows easily from the inequalities

$$F^2(x) - \left(x - \frac{x^3}{2}\right)^2 = \frac{x^6}{4(1+x^2)}(3-x^2) > 0$$

and

$$\left(x - \frac{x^3}{2} + \frac{2}{5}x^5\right)^2 - F^2(x) = \frac{x^6}{100(1+x^2)}(5 + 65x^2 - 24x^4 + 16x^6) > 0$$

together with $x - x^3/2 > 0$ for all $x \in (0,1)$. □

LEMMA 2.2. *The inequality* $\sinh^{-1}(x) > x - x^3/6$ *holds for all* $x \in (0,1)$.

Proof. Let

$$w(x) = \sinh^{-1}(x) - x + \frac{x^3}{6}. \tag{2.2}$$

Then simple computations lead to

$$w(0) = 0, \tag{2.3}$$

$$w'(x) = \frac{w_1(x)}{\sqrt{1+x^2}}, \tag{2.4}$$

where $w_1(x) = 1 - \sqrt{1+x^2} + x^2\sqrt{1+x^2}/2$. Note that

$$w_1(0) = 0, \quad w_1'(x) = \frac{3x^3}{2\sqrt{1+x^2}} > 0 \tag{2.5}$$

for $x \in (0,1)$.

Therefore, Lemma 2.2 follows easily from (2.2)–(2.5). □

LEMMA 2.3. *The inequality* $(30 + 9x^2 - 68x^4)\sinh^{-1}(x) < 30x + 4x^3$ *holds for all* $x \in (0,1)$.

Proof. Let

$$\eta(x) = (30+9x^2-68x^4)\sinh^{-1}(x) - 30x - 4x^3. \tag{2.6}$$

Then simple computations lead to

$$\eta(0) = 0, \tag{2.7}$$

$$\eta'(x) = \eta_1(x) + 2x(9-136x^2)\sinh^{-1}(x), \tag{2.8}$$

where

$$\eta_1(x) = \frac{30+9x^2-68x^4}{\sqrt{1+x^2}} - 30 - 12x^2, \tag{2.9}$$

$$\eta_1(0) = 0, \tag{2.10}$$

$$\eta_1{}'(x) = -\frac{x[12+263x^2+204x^4+24(1+x^2)^{3/2}]}{(1+x^2)^{3/2}} < 0 \tag{2.11}$$

for $x \in (0,1)$.

We claim that $\eta'(x) < 0$ for all $x \in (0,1)$. Therefore, Lemma 2.3 follows easily from (2.6) and (2.7).

Indeed, if $x \in (3/(2\sqrt{34}),1)$, then from (2.8) and (2.10) together with (2.11) we clearly see that $\eta'(x) \leqslant \eta_1(x) < \eta_1(0) = 0$ for $x \in (0,1)$.

If $x \in (0, 3/(2\sqrt{34}))$, then Lemma 2.1, (2.8) and (2.9) together with the fact that $30+9x^2-68x^4 > 0$ and $\sinh^{-1}(x) < x$ lead to the conclusion that

$$\begin{aligned}\eta'(x) &< (30+9x^2-68x^4)\left(1-\frac{x^2}{2}+\frac{x^4}{2}\right) - 30 - 12x^2 + 2x^2(9-136x^2)\\ &= -\frac{x^4}{2}(659-77x^2+68x^4) < 0\end{aligned}$$

for $x \in (0,1)$. □

LEMMA 2.4. *Let*

$$G(x) = \frac{x}{\sqrt{1+x^2}(\sinh^{-1}(x))^2} - \frac{1}{\sinh^{-1}(x)}.$$

Then $G(x)$ is strictly decreasing on $(0,1)$. Moreover, the double inequality

$$-\frac{x}{3} < G(x) < -\frac{x}{3} + \frac{17x^3}{90} \tag{2.12}$$

holds for all $x \in (0,1)$.

Proof. Differentiating $G(x)$ yields

$$G'(x) = -\frac{2}{(1+x^2)^{3/2}(\sinh^{-1}(x))^3}G_1(x), \tag{2.13}$$

where

$$G_1(x) = x\sqrt{1+x^2} - (1+\frac{x^2}{2})\sinh^{-1}(x). \tag{2.14}$$

It follows from (2.14) that

$$G_1(0) = 0, \tag{2.15}$$

$$G_1'(x) = x\left[\frac{3x}{2\sqrt{1+x^2}} - \sinh^{-1}(x)\right] > x[x - \sinh^{-1}(x)] > 0 \tag{2.16}$$

for $x \in (0,1)$.

The first statement follows easily from (2.13) and (2.15) together with (2.16).

To prove inequality (2.12), it suffices to show that inequalities

$$g_1(x) := x - \sqrt{1+x^2}\sinh^{-1}(x) + \frac{1}{3}x\sqrt{1+x^2}(\sinh^{-1}(x))^2 > 0 \tag{2.17}$$

and

$$g_2(x) := x - \sqrt{1+x^2}\sinh^{-1}(x) + \left(\frac{1}{3}x - \frac{17}{90}x^3\right)\sqrt{1+x^2}(\sinh^{-1}(x))^2 < 0 \tag{2.18}$$

hold for $x \in (0,1)$.

We first prove inequality (2.17). From the expression of $g_1(x)$ we get

$$g_1(0) = 0, \tag{2.19}$$

$$g_1'(x) = \frac{\sinh^{-1}(x)}{3\sqrt{1+x^2}} g_1^*(x), \tag{2.20}$$

where

$$g_1^*(x) = -3x + 2x\sqrt{1+x^2} + (1+2x^2)\sinh^{-1}(x). \tag{2.21}$$

It follows from Lemma 2.2 and (2.21) one has

$$g_1^*(x) > -3x + 2x + (1+2x^2)\left(x - \frac{1}{6}x^3\right) = \frac{1}{6}x^3(11-2x^2) > 0 \tag{2.22}$$

for $x \in (0,1)$.

Therefore, inequality (2.17) follows from (2.19) and (2.20) together with (2.22).

Next, we prove inequality (2.18). From the expression of $g_2(x)$ we have

$$g_2(0) = 0, \tag{2.23}$$

$$g_2'(x) = \frac{\sinh^{-1}(x)}{90\sqrt{1+x^2}} g_2^*(x), \tag{2.24}$$

where

$$g_2^*(x) = 2x[(30-17x^2)\sqrt{1+x^2} - 45] + (30+9x^2-68x^4)\sinh^{-1}(x). \tag{2.25}$$

It follows from Lemma 2.3 and (2.25) together with $\sqrt{1+x^2}<1+x^2/2$ that

$$g_2^*(x)<2x\left[(30-17x^2)\left(1+\frac{x^2}{2}\right)-45\right]+30x+4x^3=-17x^5<0 \tag{2.26}$$

for $x\in(0,1)$.

Therefore, inequality (2.18) follows from (2.23) and (2.24) together with (2.26). □

LEMMA 2.5. *Let*

$$H(x)=\frac{1}{\log[(1+x)/(1-x)]}-\frac{2x}{(1-x^2)\log^2[(1+x)/(1-x)]}.$$

Then $H(x)$ is strictly decreasing on $(0,1)$. Moreover, the inequality

$$H(x)>-\frac{x}{3}-\frac{x^3}{2} \tag{2.27}$$

holds for $0<x<3/4$, and the inequality

$$H(x)<-\frac{x}{3}-\frac{x^3}{6} \tag{2.28}$$

holds for all $0<x<1$.

Proof. Differentiating $H(x)$ gives

$$H'(x)=\frac{4H_1(x)}{(1-x^2)^2\log^3[(1+x)/(1-x)]}, \tag{2.29}$$

where

$$H_1(x)=2x-\log\left(\frac{1+x}{1-x}\right). \tag{2.30}$$

From (2.30) one has

$$H_1(0)=0,\quad H_1'(x)=-\frac{2x^2}{1-x^2}<0 \tag{2.31}$$

for $x\in(0,1)$.

Therefore, the first conclusion follows easily from (2.29) and (2.31).

To show inequalities (2.27) and (2.28), it suffices to prove that inequality

$$\begin{aligned} h_1(x):=&(1-x^2)\log^2\left(\frac{1+x}{1-x}\right)\left[H(x)+\frac{x}{3}+\frac{x^3}{2}\right]\\ =&(1-x^2)\left(\frac{x}{3}+\frac{x^3}{2}\right)\log^2\left(\frac{1+x}{1-x}\right)+(1-x^2)\log\left(\frac{1+x}{1-x}\right)-2x>0 \end{aligned} \tag{2.32}$$

holds for $0 < x < 3/4$, and inequality

$$\begin{aligned} h_2(x) :=& (1-x^2)\log^2\left(\frac{1+x}{1-x}\right)\left[H(x)+\frac{x}{3}+\frac{x^3}{6}\right] \\ =& (1-x^2)\left(\frac{x}{3}+\frac{x^3}{6}\right)\log^2\left(\frac{1+x}{1-x}\right)+(1-x^2)\log\left(\frac{1+x}{1-x}\right)-2x<0 \end{aligned} \tag{2.33}$$

holds for $0 < x < 1$.

We first prove inequality (2.32). From the expression of $h_1(x)$ one has

$$h_1(0)=0, \tag{2.34}$$

$$h_1'(x)=\frac{1}{6}\log\left(\frac{1+x}{1-x}\right)h_1^*(x), \tag{2.35}$$

where

$$h_1^*(x)=4x(3x^2-1)+(2+3x^2-15x^4)\log\left(\frac{1+x}{1-x}\right). \tag{2.36}$$

Equation (2.36) leads to

$$h_1^*(0)=0, \quad h_1^*(3/4)=0.0025\cdots>0, \tag{2.37}$$

$$h_1^{*\prime}(x)=\frac{2x}{1-x^2}h_1^{**}(x), \tag{2.38}$$

where

$$\begin{aligned} h_1^{**}(x) =& 23x-33x^3+3(1-11x^2+10x^4)\log\left(\frac{1+x}{1-x}\right) \\ =& 23x-33x^3+30\left(\frac{\sqrt{10}}{10}-x\right)\left(\frac{\sqrt{10}}{10}+x\right)(1-x^2)\log\left(\frac{1+x}{1-x}\right). \end{aligned} \tag{2.39}$$

It is not difficulty to verify that

$$\begin{cases} \log\left(\frac{1+x}{1-x}\right)>2x, & x\in(0,1), \\ \log\left(\frac{1+x}{1-x}\right)<2x+x^3, & x\in(0,\sqrt{3}/3). \end{cases} \tag{2.40}$$

We assert that

$$h_1^{**}(x)>0 \tag{2.41}$$

for all $0 < x \leqslant 1/2$. In fact, if $0 < x \leqslant \sqrt{10}/10$, then (2.39) and (2.40) lead to $h_1^{**}(x) \geqslant 23x-33x^3+6x(1-11x^2+10x^4)=x(29-99x^2+60x^4)>0$. If $\sqrt{10}/10 < x \leqslant 1/2$, then (2.39) and (2.40) imply that $h_1^{**}(x)>23x-33x^3+3(1-11x^2+10x^4)(2x+x^3)=x(29-96x^2+27x^4+30x^6)>0$.

From (2.39) and the monotonicity of $\log[(1+x)/(1-x)]$ we get

$$h_1^{**}(1/2)=3.667\cdots>0, \quad h_1^{**}(3/4)=-8.48\cdots<0, \tag{2.42}$$

$$h_1^{**\prime}(x) = 29-159x^2+6x(20x^2-11)\log\left(\frac{1+x}{1-x}\right) < 29-\frac{159}{4}+\frac{9}{2}\left(\frac{45}{4}-11\right)\log 7$$
$$= \frac{-86+9\log 7}{8} < 0 \tag{2.43}$$

for $1/2 < x < 3/4$.

From (2.41)–(2.43) we clearly see that there exists $x_0 \in (1/2, 3/4)$ such that $h_1^{**}(x) > 0$ for $x \in (0, x_0)$ and $h_1^{**}(x) < 0$ for $x \in (x_0, 3/4)$. Then (2.37) and (2.38) lead to the conclusion that

$$h_1^*(x) > 0 \tag{2.44}$$

for $x \in [0, 3/4]$.

Therefore, inequality (2.32) follows from (2.34) and (2.35) together with (2.44).

Next, we prove inequality (2.33). From the expression of $h_2(x)$ we obtain

$$h_2(0) = 0, \tag{2.45}$$

$$h_2'(x) = \frac{1}{6}\log\left(\frac{1+x}{1-x}\right)h_2^*(x), \tag{2.46}$$

where

$$h_2^*(x) = 4x(x^2-1)+(2-3x^2-5x^4)\log\left(\frac{1+x}{1-x}\right)$$
$$= 4x(x^2-1)+5(x^2+1)(x+\sqrt{\frac{2}{5}})(\sqrt{\frac{2}{5}}-x)\log\left(\frac{1+x}{1-x}\right). \tag{2.47}$$

We announce that

$$h_2^*(x) < 0 \tag{2.48}$$

for all $x \in (0,1)$. Indeed, if $0 < x < \sqrt{3}/3$, then (2.40) and (2.47) lead to $h_2^*(x) < 4x(x^2-1)+(2-3x^2-5x^4)(2x+x^3) = -x^5(13+5x^2) < 0$; if $\sqrt{3}/3 \leqslant x \leqslant \sqrt{2/5}$, then from (2.40) and (2.47) together with the monotonicity of $\log[(1+x)/(1-x)]$ we get $h_2^*(x) \leqslant 4x(x^2-1)+(2-3x^2-5x^4)\log[(\sqrt{5}+\sqrt{2})/(\sqrt{5}-\sqrt{2})] = 4x^3+2\log[(7+2\sqrt{10})/3]-4x-3\log[(7+2\sqrt{10})/3]x^2-5\log[(7+2\sqrt{10})/3]x^4 \leqslant 4(\sqrt{2/5})^3+2\log[(7+2\sqrt{10})/3]-4\sqrt{3}/3-3\log[(7+2\sqrt{10})/3]\times(\sqrt{3}/3)^2-5\log[(7+2\sqrt{10})/3]\times(\sqrt{3}/3)^4 = 8\sqrt{10}/25+4\log[(7+2\sqrt{10})/3]/9-4\sqrt{3}/3 = -0.6348\cdots < 0$; if $\sqrt{2/5} < x < 1$, then (2.40) and (2.47) lead to $h_2^*(x) < 4x(x^2-1)+2x(2-3x^2-5x^4) = -2x^3(1+5x^2) < 0$.

Therefore, inequality (2.33) follows from (2.45) and (2.46) together with (2.48). □

LEMMA 2.6. *Let* $\lambda_1 = 1-1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\cdots$, *then the function* $\varphi(x) = (1-\lambda_1)F(x)+2\lambda_1 H(x)$ *is strictly decreasing on* $(3/4,1)$, *where* $F(x)$ *and* $H(x)$ *are defined as in Lemmas 2.1 and 2.5, respectively.*

Proof. Differentiating (2.29) gives

$$H''(x) = \frac{8\zeta(x)}{(1-x^2)^3\log^4[(1+x)/(1-x)]}, \tag{2.49}$$

where

$$\zeta(x) = -6x + 3(1+x^2)\log\left(\frac{1+x}{1-x}\right) - 2x\log^2\left(\frac{1+x}{1-x}\right). \tag{2.50}$$

From (2.50) and $\log[(1+x)/(1-x)] > \log 7 = 1.945\cdots > 3/2$ we get

$$\zeta(3/4) = -\frac{9}{2} + \frac{75}{16}\log 7 - \frac{3}{2}\log^2 7 = -1.0583\cdots < 0, \tag{2.51}$$

$$\zeta'(x) = \frac{2}{1-x^2}\left[6x^2 - (x+3x^3)\log\left(\frac{1+x}{1-x}\right) - (1-x^2)\log^2\left(\frac{1+x}{1-x}\right)\right] \tag{2.52}$$

$$< \frac{2}{1-x^2}\left[6x^2 - \frac{3}{2}(x+3x^3) - \frac{9}{4}(1-x^2)\right] \tag{2.53}$$

$$= -\frac{3[3-x^2+5x(1-x)]}{2(1+x)} < 0 \tag{2.54}$$

for $x \in (3/4, 1)$.

It follows from (2.49) and (2.51) together with (2.52) that $H'(x)$ is strictly decreasing on $(3/4,1)$. Then from (2.29) and (2.30) we have

$$\begin{aligned}\varphi'(x) =& \frac{1-\lambda_1}{(1+x^2)^{3/2}} + 2\lambda_1 H'(x) < \frac{64}{125}(1-\lambda_1) + 2\lambda_1 H'(3/4)\\ =& \frac{64}{125} - \left[\frac{64}{125} + \frac{1024(2\log 7 - 3)}{49\log^3 7}\right]\lambda_1 = -0.0893\cdots < 0\end{aligned}$$

for $x \in (3/4, 1)$. This completes the result. □

LEMMA 2.7. *Let $H(x)$ be defined as in Lemma 2.5 and $\lambda_2 = 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\cdots$, then the function $\phi(x) = 2(1-\lambda_2)x + 2\lambda_2 H(x)$ is strictly decreasing on $[4/5, 1)$. Moreover, $\phi(x) > 1/2$ for $x \in [3/4, 4/5]$.*

Proof. From the proof of Lemma 2.6 we know that $H'(x)$ is strictly decreasing on $(3/4,1)$. Then from (2.29) and (2.30) we get

$$\begin{aligned}\phi'(x) =& 2(1-\lambda_2) + 2\lambda_2 H'(x) \leqslant 2(1-\lambda_2) + 2\lambda_2 H'(4/5)\\ =& 2 - \left[2 + \frac{250(5\log 3 - 4)}{81\log^3 3}\right]\lambda_2 = -0.369\cdots < 0\end{aligned}$$

for $x \in [4/5, 1)$. This in turn implies that $\phi(x)$ is strictly decreasing on $[4/5, 1)$.

Moreover, it follows from $\phi'(3/4) = 2 - [1024(2\log 7 - 3)/(49\log^3 7) + 2]\lambda_2 = 0.04012\cdots > 0$ and $\phi'(4/5) = 2 - [250(5\log 3 - 4)/(81\log^3 3) + 2]\lambda_2 = -0.369\cdots < 0$ together with the monotonicity of $H'(x)$ on $(3/4,1)$ that there exists $x_1 \in (3/4, 4/5)$, such that $\phi'(x)>0$ for $x\in[3/4,x_1)$ and $\phi'(x)<0$ for $x\in(x_1,4/5]$. This implies that

$$\begin{aligned}\phi(x)&\geqslant \min\{\phi(3/4),\phi(4/5)\}\\ &=\min\left\{\frac{3}{2}-\left(\frac{3}{2}+\frac{48-14\log 7}{7\log^2 7}\right)\lambda_2,\frac{8}{5}-\left(\frac{8}{5}+\frac{20-9\log 3}{9\log^2 3}\right)\lambda_2\right\}\\ &=\min\{0.5120\cdots,0.5048\cdots\}>\frac{1}{2}\end{aligned}$$

for $x\in[3/4,4/5]$. □

3. Main Results

THEOREM 3.1. *The double inequality*

$$\alpha_1 L(a,b)+(1-\alpha_1)Q(a,b)<M(a,b)<\beta_1 L(a,b)+(1-\beta_1)Q(a,b)$$

holds true for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_1\geqslant 2/5$ *and* $\beta_1\leqslant 1-1/[\sqrt{2}\log(1+\sqrt{2})]=0.1977\cdots$.

Proof. Since $L(a,b)$, $M(a,b)$ and $Q(a,b)$ are symmetric and homogenous of degree 1. Without loss generality, we assume that $a>b$. Let $x=(a-b)/(a+b)$, $\lambda_1=1-1/[\sqrt{2}\log(1+\sqrt{2})]=0.1977\cdots$ and $p\in(0,1)$. Then $x\in(0,1)$,

$$\frac{L(a,b)}{A(a,b)}=\frac{2x}{\log[(1+x)/(1-x)]},\quad \frac{M(a,b)}{A(a,b)}=\frac{x}{\sinh^{-1}(x)},\quad \frac{Q(a,b)}{A(a,b)}=\sqrt{1+x^2}. \tag{3.1}$$

$$\frac{Q(a,b)-M(a,b)}{Q(a,b)-L(a,b)}=\frac{\sqrt{1+x^2}-x/\sinh^{-1}(x)}{\sqrt{1+x^2}-2x/\log[(1+x)/(1-x)]}, \tag{3.2}$$

$$\lim_{x\to 0^+}\frac{\sqrt{1+x^2}-x/\sinh^{-1}(x)}{\sqrt{1+x^2}-2x/\log[(1+x)/(1-x)]}=\frac{2}{5}, \tag{3.3}$$

$$\lim_{x\to 1^-}\frac{\sqrt{1+x^2}-x/\sinh^{-1}(x)}{\sqrt{1+x^2}-2x/\log[(1+x)/(1-x)]}=\lambda_1, \tag{3.4}$$

$$\begin{aligned}&\frac{pL(a,b)+(1-p)Q(a,b)-M(a,b)}{A(a,b)}\\ =&p\frac{2x}{\log[(1+x)/(1-x)]}+(1-p)\sqrt{1+x^2}-\frac{x}{\sinh^{-1}(x)}:=D_p(x).\end{aligned} \tag{3.5}$$

Equation (3.5) leads to

$$D_p(0^+)=0,\quad D_p(1^-)=\sqrt{2}(1-p)-\frac{1}{\log(1+\sqrt{2})},\quad D_{\lambda_1}(1^-)=0, \tag{3.6}$$

$$\begin{aligned}D'_p(x) =&\frac{(1-p)x}{\sqrt{1+x^2}}+\frac{x}{\sqrt{1+x^2}[\sinh^{-1}(x)]^2}-\frac{1}{\sinh^{-1}(x)}\\&+2p\left[\frac{1}{\log[(1+x)/(1-x)]}-\frac{2x}{(1-x^2)\log^2[(1+x)/(1-x)]}\right]\\=&(1-p)F(x)+G(x)+2pH(x),\end{aligned} \tag{3.7}$$

where $F(x)$, $G(x)$ and $H(x)$ are defined as in Lemmas 2.1, 2.4 and 2.5, respectively.

From Lemmas 2.1, 2.4 and 2.5 together with (3.7) one has

$$\begin{aligned}D'_{2/5}(x) &= \frac{3}{5}F(x)+G(x)+\frac{4}{5}H(x)\\&<\frac{1}{5}\left[3\Big(x-\frac{x^3}{2}+\frac{2}{5}x^5\Big)+5\Big(-\frac{1}{3}x+\frac{17}{90}x^3\Big)-4\Big(\frac{1}{3}x+\frac{1}{6}x^3\Big)\right]\\&=-\frac{x^3}{225}(55-54x^2)<0\end{aligned} \tag{3.8}$$

for all $x\in(0,1)$.

It follows from (3.5) and (3.6) together with (3.8) we clearly see that

$$\frac{2}{5}L(a,b)+\frac{3}{5}Q(a,b)<M(a,b). \tag{3.9}$$

Next, we prove that

$$\lambda_1 L(a,b)+(1-\lambda_1)Q(a,b)>M(a,b). \tag{3.10}$$

It follows from Lemmas 2.1, 2.4 and 2.5 together with (3.7) we clearly see that

$$\begin{aligned}D'_{\lambda_1}(x) >&(1-\lambda_1)\left(x-\frac{1}{2}x^3\right)-\frac{1}{3}x-2\lambda_1\left(\frac{1}{3}x+\frac{1}{2}x^3\right)=\frac{(1+\lambda_1)x}{2}\left[\frac{2(2-5\lambda_1)}{3(1+\lambda_1)}-x^2\right]\\>&\frac{(1+\lambda_1)x}{2}\left[\frac{2(2-5\lambda_1)}{3(1+\lambda_1)}-\left(\frac{3}{4}\right)^2\right]>\frac{(1+\lambda_1)x}{2}\times 0.00045>0\end{aligned} \tag{3.11}$$

for $x\in(0,3/4)$.

From Lemmas 2.4 and 2.6 together with (3.7) we clearly see that $D'_{\lambda_1}(x)$ is strictly decreasing on $x\in(3/4,1)$, $D'_{\lambda_1}(1^-)=-\infty$ and $D'_{\lambda_1}(3/4)=(1-\lambda_1)F(3/4)+G(3/4)+2\lambda_1 H(3/4)=3(1-\lambda_1)/5+(3-5\log 2)/(5\log^2 2)+(14\log 7-48)\lambda_1/(7\log^2 7)=0.1326\cdots>0$. Hence, we know that there exists $x_2\in(3/4,1)$ such that $D'_{\lambda_1}(x)>0$ for $x\in[3/4,x_2)$ and $D'_{\lambda_1}(x)<0$ for $x\in(x_2,1)$. This in conjunction with (3.11) leads to that $D_{\lambda_1}(x)$ is strictly increasing on $(0,x_2]$ and strictly decreasing on $[x_2,1)$.

Therefore, inequality (3.10) follows from (3.5) and (3.6) together with the piecewise monotonicity of $D_{\lambda_1}(x)$, and Theorem 3.1 follows from (3.9) and (3.10) in conjunction with the following statements.

- If $\alpha_1<2/5$, then equations (3.2) and (3.3) lead to the conclusion that there exists $0<\delta_1<1$ such that $M(a,b)<\alpha_1 L(a,b)+(1-\alpha_1)Q(a,b)$ for all $a,b>0$ with $(a-b)/(a+b)\in(0,\delta_1)$.

- If $\beta_1 > \lambda_1$, then equations (3.2) and (3.4) lead to the conclusion that there exists $0 < \delta_2 < 1$ such that $M(a,b) > \beta_1 L(a,b) + (1-\beta_1)Q(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (1-\delta_2, 1)$. $\square$

THEOREM 3.2. *The double inequality*

$$\alpha_2 L(a,b) + (1-\alpha_2)C(a,b) < M(a,b) < \beta_2 L(a,b) + (1-\beta_2)C(a,b)$$

holds true for all $a,b > 0$ *with* $a \neq b$ *if and only if* $\alpha_2 \geqslant 5/8$ *and* $\beta_2 \leqslant 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\cdots$.

Proof. We will follow, to some extent, lines in the proof of Theorem 3.1. Since $L(a,b)$, $M(a,b)$ and $C(a,b)$ are symmetric and homogenous of degree 1. Without loss generality, we assume that $a > b$. Let $x = (a-b)/(a+b)$, $\lambda_2 = 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\cdots$ and $q \in (0,1)$. Then $x \in (0,1)$, making use of (3.1) and $C(a,b)/A(a,b) = 1 + x^2$ we get

$$\frac{C(a,b)-M(a,b)}{C(a,b)-L(a,b)} = \frac{1+x^2 - x/\sinh^{-1}(x)}{1+x^2-2x/\log[(1+x)/(1-x)]}, \tag{3.12}$$

$$\lim_{x\to 0^+} \frac{1+x^2 - x/\sinh^{-1}(x)}{1+x^2-2x/\log[(1+x)/(1-x)]} = \frac{5}{8}, \tag{3.13}$$

$$\lim_{x\to 1^-} \frac{1+x^2 - x/\sinh^{-1}(x)}{1+x^2-2x/\log[(1+x)/(1-x)]} = \lambda_2, \tag{3.14}$$

$$\begin{aligned}&\frac{qL(a,b)+(1-q)C(a,b)-M(a,b)}{A(a,b)}\\ =&q\frac{2x}{\log[(1+x)/(1-x)]} + (1-q)(1+x^2) - \frac{x}{\sinh^{-1}(x)} := E_q(x).\end{aligned} \tag{3.15}$$

Equation (3.15) leads to

$$E_q(0^+) = 0, \quad E_q(1^-) = \sqrt{2}(1-q) - \frac{1}{\log(1+\sqrt{2})}, \quad E_{\lambda_2}(1^-) = 0, \tag{3.16}$$

$$\begin{aligned}E_q'(x) =&2(1-q)x + \frac{x}{\sqrt{1+x^2}[\sinh^{-1}(x)]^2} - \frac{1}{\sinh^{-1}(x)}\\ &+2q\left[\frac{1}{\log[(1+x)/(1-x)]} - \frac{2x}{(1-x^2)\log^2[(1+x)/(1-x)]}\right]\\ =&2(1-q)x + G(x) + 2qH(x),\end{aligned} \tag{3.17}$$

where $G(x)$ and $H(x)$ are defined as in Lemmas 2.4 and 2.5, respectively.

From Lemmas 2.4 and 2.5 together with (3.17) one has

$$\begin{aligned} E'_{5/8}(x) &= \frac{3}{4}x + G(x) + \frac{5}{4}H(x) \\ &< \frac{3}{4}x - \frac{x}{3} + \frac{17x^3}{90} - \frac{5}{4}\left(\frac{x}{3} + \frac{x^3}{6}\right) = -\frac{7x^3}{360} < 0 \end{aligned} \tag{3.18}$$

for all $x \in (0,1)$.

Equations (3.15) and (3.16) together with (3.18) lead to the conclusion that

$$\frac{5}{8}L(a,b) + \frac{3}{8}C(a,b) < M(a,b). \tag{3.19}$$

Next, we prove that

$$\lambda_2 L(a,b) + (1-\lambda_2)C(a,b) > M(a,b). \tag{3.20}$$

From Lemmas 2.4, 2.5 and 2.7 together with (3.17) we have

$$\begin{aligned} E'_{\lambda_2}(x) &> 2(1-\lambda_2)x - \frac{x}{3} - 2\lambda_2\left(\frac{x}{3} + \frac{x^3}{2}\right) \\ &= \lambda_2 x\left(\frac{5-8\lambda_2}{3\lambda_2} - x^2\right) = \lambda_2 x(1.1850\cdots - x^2) > 0 \end{aligned} \tag{3.21}$$

for $x \in (0,3/4)$,

$$E'_{\lambda_2}(x) = \phi(x) + G(x) > \frac{1}{2} + G\left(\frac{4}{5}\right) > \frac{1}{2} - \frac{1}{3} \times \frac{4}{5} = \frac{7}{30} > 0 \tag{3.22}$$

for $x \in [3/4, 4/5)$,

$$\begin{aligned} E'_{\lambda_2}(4/5) =& \frac{8}{5}(1-\lambda_2) + G\left(\frac{4}{5}\right) + 2\lambda_2 H\left(\frac{4}{5}\right) \\ =& \frac{8}{5} + \frac{4 - \sqrt{41}\log^2(\frac{4+\sqrt{41}}{5})}{\sqrt{41}\log^2(\frac{4+\sqrt{41}}{5})} + \left(\frac{18\log 9 - 80}{9\log^2 9} - \frac{8}{5}\right)\lambda_2 \\ =& 0.3037\cdots > 0, \end{aligned} \tag{3.23}$$

$$E'_{\lambda_2}(1^-) = -\infty, \tag{3.24}$$

and $E'_{\lambda_2}(x)$ is strictly decreasing on $[4/5, 1)$.

From (3.21)–(3.24) and the monotonicity of $E'_{\lambda_2}(x)$ we know that there exists $x_3 \in (4/5, 1)$ such that $E_{\lambda_2}(x)$ is strictly increasing on $(0, x_3]$ and strictly decreasing on $[x_3, 1)$.

Therefore, inequality (3.20) follows from (3.15) and (3.16) together with the piecewise monotonicity of $E'_{\lambda_2}(x)$, and Theorem 3.2 follows from (3.19) and (3.20) in conjunction with the following statements.

- If $\alpha_2 < 5/8$, then equations (3.12) and (3.13) lead to the conclusion that there exists $0 < \delta_3 < 1$ such that $M(a,b) < \alpha_2 L(a,b) + (1-\alpha_2)C(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (0,\delta_3)$.
- If $\beta_2 > \lambda_2$, then equations (3.12) and (3.14) lead to the conclusion that there exists $0 < \delta_4 < 1$ such that $M(a,b) > \beta_2 L(a,b) + (1-\beta_2)C(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (1-\delta_4, 1)$. □

REMARK 3.3. The bounds in (3.9) and (3.19) are not comparable to each other. In fact, if we let

$$J(a,b) = \left[\frac{5}{8}L(a,b) + \frac{3}{8}C(a,b)\right] - \left[\frac{2}{5}L(a,b) + \frac{3}{5}Q(a,b)\right], \tag{3.25}$$

then numerical computations show that

$$\begin{aligned}
&J(1,10) = 0.0588\cdots > 0,\ \ J(1,20) = 0.0918\cdots > 0,\ \ J(1,30) = 0.0826\cdots > 0,\\
&J(1,40) = 0.0461\cdots > 0,\ \ J(1,50) = -0.0095\cdots < 0,\ \ J(1,60) = -0.0798\cdots < 0,\\
&J(1,70) = -0.1617\cdots < 0,\ \ J(1,80) = -0.2531\cdots < 0,\ \ J(1,90) = -0.3526\cdots < 0.
\end{aligned}$$

More precisely, let $a > b$ and $x = a/b > 1$, then

$$J(a,b) = \frac{3}{40(x+1)\log x}\left[3(x^2-1) + \left(5(x^2+1) - 4\sqrt{2}(x+1)\sqrt{x^2+1}\right)\log x\right] \tag{3.26}$$

$$\begin{aligned}
&:= \frac{3}{40(x+1)\log x} f(x),\\
&f(1) = f'(1) = f''(1) = f'''(1) = f^{(4)}(1) = 0, \quad f^{(5)}(1) = 44\sqrt{2}, && (3.27)\\
&\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left\{3\left(\frac{1}{\log x} - \frac{1}{x^2\log x}\right)\right.\\
&\qquad \left. + \left[5\left(1+\frac{1}{x^2}\right) - 4\sqrt{2}\left(1+\frac{1}{x}\right)\sqrt{1+\frac{1}{x^2}}\right]\right\} x^2\log x && (3.28)\\
&= -\infty.
\end{aligned}$$

Equations (3.25)–(3.28) imply that there exist small enough $\delta > 0$ and large enough $X > 1$ such that

$$\frac{5}{8}L(a,b) + \frac{3}{8}C(a,b) > \frac{2}{5}L(a,b) + \frac{3}{5}Q(a,b)$$

for all $a/b \in (1, 1+\delta)$ and

$$\frac{5}{8}L(a,b) + \frac{3}{8}C(a,b) < \frac{2}{5}L(a,b) + \frac{3}{5}Q(a,b)$$

for all $a/b \in (X, +\infty)$.

REMARK 3.4. The bound in (3.10) is better than that in (3.20).

Proof. Let $a>b$ and $x=(a-b)/(a+b)\in(0,1)$, then

$$\begin{aligned}&[\lambda_2 L(a,b)+(1-\lambda_2)C(a,b)]-[\lambda_1 L(a,b)+(1-\lambda_1)Q(a,b)]\\&=\left[(\lambda_2-\lambda_1)\frac{2x}{\log(1+x)-\log(1-x)}+(1-\lambda_2)(1+x^2)-(1-\lambda_1)\sqrt{1+x^2}\right]A(a,b)\end{aligned}\tag{3.29}$$

Let

$$g(x)=(\lambda_2-\lambda_1)\frac{2x}{\log(1+x)-\log(1-x)}+(1-\lambda_2)(1+x^2)-(1-\lambda_1)\sqrt{1+x^2},\tag{3.30}$$

$$g_1(x)=(\lambda_2-\lambda_1)\frac{2x}{\log(1+x)-\log(1-x)},\tag{3.31}$$

$$g_2(x)=(1-\lambda_2)(1+x^2)-(1-\lambda_1)\sqrt{1+x^2}.\tag{3.32}$$

We claim that

$$\frac{2x}{\log(1+x)-\log(1-x)}>1-\frac{x^2}{3}-\frac{x^4}{4}\tag{3.33}$$

for all $x\in(0,0.96)$. Indeed, let

$$\varphi(x)=2x-\left(1-\frac{x^2}{3}-\frac{x^4}{4}\right)\log\frac{1+x}{1-x}.\tag{3.34}$$

Then

$$\varphi(0)=0,\quad \varphi(0.96)=0.0501\cdots>0,\tag{3.35}$$

$$\varphi'(x)=\frac{x\varphi_1(x)}{6(1-x^2)},\tag{3.36}$$

where

$$\varphi_1(x)=-8x+3x^3+(4+2x^2-6x^4)\log\left(\frac{1+x}{1-x}\right).\tag{3.37}$$

Equation (3.37) leads to

$$\begin{aligned}\varphi_1(x)&>-8x+3x^3+(4+2x^2-6x^4)\left(2x+\frac{2x^3}{3}+\frac{2x^5}{5}\right)\\&=\frac{x^3}{15}(145-136x^2-48x^4-36x^6)\\&\geqslant\frac{x^3}{15}(145-136\times0.8^2-48\times0.8^4-36\times0.8^6)\\&=\frac{x^3}{15}\times28.862\cdots>0\end{aligned}\tag{3.38}$$

for $x \in (0, 0.8]$, and

$$\varphi_1(0.8) = 1.3374\cdots > 0, \quad \varphi_1(0.96) = -2.11813\cdots < 0, \tag{3.39}$$

$$\varphi_1'(x) = x\varphi_2(x), \tag{3.40}$$

where

$$\begin{aligned} \varphi_2(x) &= 21x + (4 - 24x^2)\log\left(\frac{1+x}{1-x}\right), \\ \varphi_2(0.8) &= -8.16047\cdots < 0, \\ \varphi_2'(x) &= \frac{1}{1-x^2}\left[29 - 69x^2 - 48x(1-x^2)\log\left(\frac{1+x}{1-x}\right)\right] \\ &< \frac{29 - 69 \times 0.8^2}{1-x^2} = -\frac{15.16}{1-x^2} < 0 \end{aligned} \tag{3.41, 3.42}$$

for $x \in (0.8, 0.96)$.

From (3.38)–(3.42) we clearly see that there exists $x_1 \in (0.8, 0.96)$ such that $\varphi_1(x) > 0$ for $x \in (0, x_1)$ and $\varphi_1(x) < 0$ for $x \in (x_1, 0.96)$. Then (3.36) implies that $\varphi(x)$ is strictly increasing on $(0, x_1]$ and strictly decreasing on $[x_1, 0.96)$.

Therefore, inequality (3.33) follows from (3.34) and (3.35) together with the piecewise monotonicity of $\varphi(x)$.

It follows from (3.30)–(3.33) and $\sqrt{1+x^2} < 1 + x^2/2$ that

$$\begin{aligned} g(x) &> (\lambda_2 - \lambda_1)\left(1 - \frac{x^2}{3} - \frac{x^4}{4}\right) + (1-\lambda_2)(1+x^2) - (1-\lambda_1)\left(1 + \frac{x^2}{2}\right) \\ &= \frac{x^2}{12}[6 + 10\lambda_1 - 16\lambda_2 - 3(\lambda_2 - \lambda_1)x^2] \\ &> \frac{x^2}{12}[6 + 10\lambda_1 - 16\lambda_2 - 3(\lambda_2 - \lambda_1) \times 0.96^2] \\ &= \frac{x^2}{12} \times 0.404282\cdots > 0 \end{aligned} \tag{3.43}$$

for $x \in (0, 0.96)$, and

$$g_1'(x) = \frac{(\lambda_2 - \lambda_1)[2(1-x^2)\log(\frac{1+x}{1-x}) - 4x]}{(1-x^2)[\log(\frac{1+x}{1-x})]^2} \tag{3.44}$$

$$g_2'(x) = \frac{x}{2\log(1+\sqrt{2})}\left(2 - \frac{\sqrt{2}}{\sqrt{1+x^2}}\right), \tag{3.45}$$

$$g_1''(x) = \frac{8(\lambda_2 - \lambda_1)[2x - \log(\frac{1+x}{1-x})]}{(1-x^2)^2[\log(\frac{1+x}{1-x})]^3} < 0 \tag{3.46}$$

for $x \in [0.96, 1)$.

From (3.30)–(3.32) and (3.44), (3.45) together with (3.46) we clearly see that

$$g'(x) = g_1'(x) + g_2'(x) < g_1'(0.96) + g_2'(1) = -0.0718\cdots < 0 \tag{3.47}$$

for $x \in [0.96, 1)$.

Note that

$$\lim_{x \to 1} g(x) = 0. \tag{3.48}$$

Therefore, Remark 3.4 follows easily from (3.29), (3.30), (3.43), (3.47) and (3.48). □

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