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PROPERTIES OF THE SYMBOL OF MULTIDIMENSIONAL SINGULAR INTEGRALS IN THE WEIGHTED SPACES AND OSCILLATING MULTIPLIERS

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Abstract. Differential properties of symbols of multidimensional singular integrals in the weighted space of Bessel potentials on the sphere with the weighted functions, having singularities on a sphere are studied. The main results are applied to obtaining theorems on Fourier multipliers of spherical harmonic expansions.

1. Introduction

We intend to investigate symbols of multidimensional singular integrals within the frameworks of oscillatory multipliers of spheral harmonic expansions. The principle reason is in the fact that symbol has representation as an operator with oscillatory multipliers. Firstly, we develop some results for the weighted space of Bessel potentials on the sphere.

Let S^{n-1} be the unit sphere in R^n. We will denote the points of this sphere by x', y' such that $x' = (\theta, \theta')$, $y' = (\omega, \omega')$, where $\theta', \omega' \in S^{n-2}$ and $0 \leqslant \theta, \omega < \pi$. Let $Y_m(x')$ be n-dimensional spherical harmonics of order m. Then for each function $f \in C^\infty(S^{n-1})$ we can write the expansion $f(x') = \sum_{m=0}^{\infty} f_m Y_m(x')$, where f_m are Fourier coefficients of the function f. Any operator Λ, acting on a function f by the formula

$$\Lambda f(x') = \sum_{m=0}^{\infty} \lambda_m f_m Y_m(x')$$

is called the operator with multiplier λ_m. The numbers λ_m are called (p,q)- multiplier on the sphere S^{n-1} if

$$\|\Lambda f\|_q \leqslant C \|f\|_p,$$

where $\|\cdot\|_r$ is L_r- norm on the sphere S^{n-1}.

The theory of singular integrals and the theory of Riesz potentials give us examples of so-called oscillating multipliers $i^m a(m)$, where $a(m)$ is a real-valued multiplier (see, for example, the monographs [1–3] and the article [4]).

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Oscillating multipliers in the Sobolev space on sphere with $p = 2$ were studied in Mikhlin's monograph [1]. The properties of oscillating multipliers in the space of Bessel potentials were studied in our papers [5] and [6].

We recall that the space of Bessel potentials on the sphere S^{n-1} is the completion of the space $C^{\infty}\left(S^{n-1}\right)$ by the norm $\left\|(E+\delta)^{\frac{l}{2}} f\right\|_p$, where $l > 0$, E is a unit operator and δ is the spherical part of Laplace operator Δ. The operator δ is also called Beltrami operator on the sphere. In [7] and [8][1] we denote this space by $H_p^l\left(S^{n-1}\right)$.

The purpose of this paper is to study oscillating multipliers in the weighted spaces of Bessel potentials on the sphere, with a weight function, having singularities at the pole or at the equator of the sphere.

Now we introduce these weighted spaces: $C_\beta H_p^l\left(S^{n-1}\right)$ is the space of functions with the finite norm

$$\|g\|_{p,l,C_\beta} = \|g\|_p + \left\|(\cos\theta)^\beta \delta^{l/2} g\right\|_p$$

and $S_\beta H_p^l\left(S^{n-1}\right)$ is the space of the function g with the norm

$$\|g\|_{p,l,S_\beta} = \|g\|_p + \left\|(\sin\theta)^\beta \delta^{l/2} g\right\|_p,$$

where l and β are real numbers, $p > 1$, and δ, as above, is the Beltrami operator on the sphere. Note that this operator is a non-negative-definite self-adjoint differential operator of second order in L_2. The spectrum of the operator δ consists of the eigenvalues $\lambda_m = m(m+n-2)$ and corresponding eigenfunctions are the spherical functions $Y_m(x')$.

We also set $C_\beta H_p^0 = C_\beta L_p$ and $S_\beta H_p^0 = S_\beta L_p$. It is clear that the letters C and S in these notations indicate the weighted functions are cosine and sine, respectively.

In the special case $\beta = 0$ the norms $\|g\|_{p,l,C_0}$ and $\|g\|_{p,l,S_0}$ are equivalent to the norm of the space $H_p^l\left(S^{n-1}\right)$ of Bessel potentials on the sphere

$$\|g\|_{p,l} = \left\|(E+\delta)^{l/2} g\right\|_p.$$

Differential properties of the symbol and characteristics of n-dimensional singular integral were investigated in our works [7] and [8]. Some other results were proved in the book [9], in the papers [14], [19] and in the *PhD* dissertations [15]–[17]. Very interesting connections are shown in the articles [10]–[13].

We present here the main theorem of the papers [7], [8], which will be used in this study. Denoting, as in [8], operator "characteristic $\to$symbol" by A, we can write

$$(Af)\left(x'\right) = \int\limits_{S^{n-1}} \left\{ \ln \frac{1}{\left|\left(x',y'\right)\right|} - \frac{i\pi}{2} sgn\left(x',y'\right) \right\} f\left(y'\right) dy'. \tag{1}$$

In [8] we prove the inverse formula for this operator, which has the form

[1] Gadziev = Gadjiev.

$A^{-1} = \sum_{k=1}^{n/2} C_k \delta^k A\tau$, if n is an even number,

$A^{-1} = \sum_{k=1}^{(n-1)/2} C_k (-\Delta)^{1/2} \delta^k A\tau$, if n is odd,

where C_k are real numbers, and $\tau f(x) = f(-x)$. Note that the other expression of this formula was proposed in [4].

THEOREM A. (see [7], [8]) *Let* $1 < p < \infty$ *and* $l = \frac{n}{2} - \left|\frac{1}{p} - \frac{1}{2}\right|(n-2)$. *Then the operator A is bounded from* $H_p^l\left(S^{n-1}\right)$ *to* $L_p\left(S^{n-1}\right)$.

THEOREM B. (see [7], [8]) *Let* $1 < p < \infty$ *and* $\gamma = \frac{n}{2} + \left|\frac{1}{p} - \frac{1}{2}\right|(n-2)$. *Then the operator* A^{-1} *is bounded from* $H_p^\gamma\left(S^{n-1}\right)$ *to* $L_p\left(S^{n-1}\right)$.

Using the well known result that the operator A acts on each function $f \in C^\infty\left(S^{n-1}\right)$, $f(x') = \sum_{m=0}^{\infty} f_m Y_m(x')$ by the formula [1]

$$(Af)(x') = \sum_{m=0}^{\infty} i^m \pi^{n/2} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+n}{2}\right)} f_m Y_m(x'),$$

we can rephrase the theorems A and B for oscillating mutipliers

THEOREM C. (see [5], [6]) *Let* p *and* l *be as in Theorem A. Then* $i^m \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+n}{2}\right)}$ *is a multiplier from* $H_p^l\left(S^{n-1}\right)$ *to* $L_p\left(S^{n-1}\right)$.

THEOREM D. (see [5], [6]) *Let* p *and* γ *be as in Theorem B. Then* $(-i)^m \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}$ *is a multiplier from* $L_p\left(S^{n-1}\right)$ *to* $H_p^\gamma\left(S^{n-1}\right)$.

Theorem D follows from Theorem B and the following obvious multiplier representation of the operator A^{-1}

$$\left(A^{-1}f\right)(\theta) = \sum_{m=1}^{\infty} (-i)^m \pi^{-n/2} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} f_m Y_m(x').$$

2. Boundedness of the operator A

We shall prove the theorems of type A, B, C and D in weighted spaces $S_\beta H_p^l$ and $C_\beta H_p^l$. In connection with these spaces, we note the following. In [8] it was shown that if the function f in (1) has a singularity at the equator of the sphere S^{n-1}, then Af has a singularity at the pole, and conversely, if f has a singularity at the pole, then Af will have a singularity at the equator. Thus, in contrast to the singular integrals and pseudodiferential operators, the operator A does not preserve the singular support and this circumstance dictates the consideration of the weighted spaces $C_\beta H_p^l$ and $S_\beta H_p^l$.

For brevity we use the notation

$$\int_{d_1<\theta<d_2} g\left(x'\right) dx' = \int_{d_1}^{d_2} \int_{S^{n-2}} g\left(\theta,\theta'\right) (\sin\theta)^{n-2}\, d\theta d\theta'.$$

THEOREM 1. *Let* $p>1$, $l=\frac{n}{2}-\left|\frac{1}{p}-\frac{1}{2}\right|(n-2)$ *be an integer and* $-\frac{1}{p}<\beta<l-\frac{1}{p}$. *Then the operator* A *defined in (1), is a bounded operator from* $S_\beta L_p$ *to* $C_\beta H_p^l$.

Proof. Consider the norm $\|Af\|_{C_\beta H_p^l}$. Since the operator A acts from L_p to L_p for any $p\in(1,\infty)$ it is sufficient to estimate only the term with the weighted function $(\cos\theta)^\beta$. Moreover, $(\cos\theta)^\beta$ is separated from zero outside of the neighborhood of $\pi/2$. Therefore, we must estimate only the part where θ is near to $\pi/2$. So, consider the integral

$$\int_{0<\left|\frac{\pi}{2}-\theta\right|<\frac{1}{2}} (\cos\theta_1)^{p\beta} \left|\delta^{1/2}(Af)(x')\right|^p dx', \quad x'=\left(\theta,\theta'\right).$$

We note that the integrals with $0<\frac{\pi}{2}-\theta<\frac{1}{2}$ and $0<\theta-\frac{\pi}{2}<\frac{1}{2}$ are estimated in the same way, and therefore we present only estimates in the case $0<\frac{\pi}{2}-\theta<\frac{1}{2}$.

We denote the kernel of the operator A in (1) by $K(x',y')$, that is let $K(x'\cdot y')=\ln\frac{1}{|(x'\cdot y')|}-\frac{i\pi}{2}sgn(x'\cdot y')$.

Then we have

$$\begin{aligned}&\int_{0<\frac{\pi}{2}-\theta<\frac{1}{2}} (\cos\theta)^{p\beta} \left|\delta^{l/2}\int_{S^{n-1}} f\left(y'\right) K\left(x'\cdot y'\right) dy'\right|^p dx' \\ &\leqslant C\sum_{k=1}^{\infty} \int_{2^{-k-1}<\frac{\pi}{2}-\theta<2^{-k}} (\cos\theta)^{p\beta} \left|\delta^{l/2}\left(\int_{0<\omega<2^{-k-2}} + \int_{2^{-k-2}<\omega<2^{-k+1}}\right.\right. \\ &\quad \left.\left. + \int_{2^{-k+1}<\omega<\pi}\right) f\left(y'\right) K\left(x'\cdot y'\right) dy'\right|^{p} dx' = I_1+I_2+I_3. \qquad (2)\end{aligned}$$

Denoting by $\chi_k(\omega)$ the characteristic function of the set $-2^{-k-2}<\omega<2^{-k+1}$, we obtain by using the Theorem A

$$\begin{aligned}
I_2 &\leqslant C\sum_{k=1}^{\infty}\int\limits_{2^{-k-1}<\frac{\pi}{2}-\theta<2^{-k}}\left(\frac{\pi}{2}-\theta\right)^{p\beta}\left|\delta^{l/2}\int\limits_{S^{n-1}}K\left(x'\cdot y'\right)\chi_k(\omega)\,dy'\right|^p dx' \\
&\leqslant C\sum_{k=1}^{\infty}2^{-kp\beta}\int\limits_{S^{n-1}}\left|\delta^{l/2}A\left(\chi_k f\right)\right|^p dx' = C\sum_{k=1}^{\infty}2^{-kp\beta}\left\|\delta^{l/2}A\left(\chi_k f\right)\right\|_p^p \\
&\leqslant C\sum_{k=1}^{\infty}2^{-kp\beta}\left\|\chi_k f\right\|_p^p = C\sum_{k=1}^{\infty}\int\limits_{S^{n-1}}\left|\sin\omega\right|^{p\beta}\chi_k(\omega)\left|f\left(y'\right)\right|^p dy' \\
&= C\int\limits_{S^{n-1}}\left|\sin\omega\right|^{p\beta}\left|f\left(y'\right)\right|^p dy'.
\end{aligned}$$

That is

$$I_2 \leqslant \|f\|_{p.l,S_\beta}^p. \tag{3}$$

Consider I_1. Since 1 is a natural number

$$\left|\delta^{l/2}K\left(x'\cdot y'\right)\right| \leqslant \frac{c}{\left|\cos\left(x'\cdot y'\right)\right|^l}$$

and we can write

$$\begin{aligned}
I_1+I_3 \leqslant \sum_{k=1}^{\infty}\int\limits_{2^{-k-1}<\frac{\pi}{2}-\theta<2^{-k}}\left(\frac{\pi}{2}-\theta\right)^{p\beta}&\left\{\left(\int\limits_{0<\omega<2^{-k-2}}\frac{|f(y')|}{\left|\cos\left(x'\cdot y'\right)\right|^l}dy'\right)^p dx'\right. \\
&\left.+\left(\int\limits_{2^{-k+1}<\omega<\pi}\frac{|f(y')|}{\left|\cos\left(x'\cdot y'\right)\right|^l}dy'\right)^p dx'\right\}.
\end{aligned} \tag{4}$$

Consider only the integrals with respect to the variable ω. We have

$$\begin{aligned}
&\int\limits_{\omega>2^{-k+1}}\frac{|f(y')|}{\left|\cos\left(x'\cdot y'\right)\right|^l}dy' \leqslant C\int\limits_{\omega>2^{-k+1}}\frac{|f(y')|}{\left|\cos\left(x'\cdot y'\right)\right|^l}\omega^{n-2}d\omega d\omega' \\
&\leqslant \int\limits_{2^{-k+1}}^{\pi}\omega^{n-2}d\omega\left(\int\limits_{S^{n-2}}\frac{dy'}{\left|\cos\left(x'\cdot y'\right)\right|^l}\right)^{1/p'}\left(\int\limits_{S^{n-2}}\left|f\left(y'\right)\right|^p dy'\right)^{1/p}.
\end{aligned}$$

Moreover

$$\left(\int\limits_{S^{n-2}}\frac{dy'}{\left|\cos\left(x'\cdot y'\right)\right|^l}\right)^{1/p'} = C\left(\int\limits_0^{\pi}\frac{(\sin\varphi)^{n-2}\,d\varphi}{\left|\cos\theta\cos\omega+\sin\theta\sin\omega\cos\varphi\right|^{lp}}\right)^{1/p'}$$

$$\leqslant C\left(-\frac{1}{\sin\theta\sin\omega}\int\limits_0^{\pi}\frac{d\left(\sin\theta\sin\omega\cos\varphi\right)}{|\cos\theta\cos\omega+\sin\theta\sin\omega\cos\varphi|^{lp}}\right)^{1/p'}$$
$$=C\left(\frac{1}{\sin\theta\sin\omega}\left[\frac{1}{|\cos(\theta+\omega|^{lp'-1}}-\frac{1}{|\cos(\theta-\omega|^{lp'-1}}\right]\right)^{1/p'}.$$

The inequalities $\frac{\pi}{2}-\theta<2^{-k},\ \omega>2^{-k+1}$ imply $\theta+\omega>\frac{\pi}{2}$. Therefore, if $\omega<\frac{\pi}{2}$, then

$$|\cos(\theta+\omega)|=\sin\theta\sin\omega-\cos\theta\cos\omega<\cos(\theta-\omega)$$
$$\sin\theta\sin\omega>|\cos(\theta+\omega)|$$

and we obtain

$$\left(\int\limits_{S^{n-1}}\frac{dy'}{|\cos(x'\cdot y')|^{lp'}}\right)^{1/p'}\leqslant\frac{C}{|\cos(\theta+\omega)|^{l}}. \tag{5}$$

If $\omega>\frac{\pi}{2}$, then

$$\cos(\theta+\omega)=-\cos\theta\,|\cos\omega|+\sin\theta\sin\omega$$
$$|\cos(\theta+\omega)|=\cos\theta\,|\cos\omega|+\sin\theta\sin\omega$$
$$\cos(\omega-\theta)=\cos\theta\cos\omega+\sin\theta\sin\omega=-\cos\theta\,|\cos\omega|+\sin\theta\sin\omega.$$

Consider two cases. If $\omega>\frac{\pi}{2}$ and $\omega-\theta<\frac{\pi}{2}$ then

$$|\cos(\theta+\omega)|>\cos(\omega-\theta)$$

$$\sin\theta\sin\omega>\cos(\omega-\theta)$$

and so

$$\left(\int\limits_{S^{n-2}}\frac{dy'}{|\cos(x'\cdot y')|^{lp'}}\right)^{1/p'}\leqslant\frac{C}{|\cos(\omega-\theta)|^{l}}. \tag{6}$$

In the case $\omega>\frac{\pi}{2}$ and $\omega-\theta>\frac{\pi}{2}$

$$\begin{aligned}|\cos\theta\cos\omega+\sin\theta\sin\omega\cos\varphi|&=|\cos\theta\cos\omega+\sin\theta\sin\omega-\sin\theta\sin\omega(1-\cos\varphi)|\\&=\left|\cos(\theta-\omega)-2\sin\theta\sin\omega\sin^2\frac{\varphi}{4}\right|\\&=|\cos(\theta-\omega)|+2\sin\theta\sin\omega\sin^2\frac{\varphi}{2}>|\cos(\theta-\omega)|.\end{aligned}$$

Therefore, in this case

$$\left(\int\limits_{S^{n-2}} \frac{dy'}{|\cos(x'\cdot y')|^{1p'}}\right)^{1/p'} = C\left(\int\limits_0^\pi \frac{(\sin\varphi)^{n-2}d\varphi}{|\cos\theta\cos\omega+\sin\theta\sin\omega\cos\varphi|^{lp'}}\right)^{1/p'} \leqslant \frac{C}{|\cos(\theta-\omega)|^l}. \tag{7}$$

Inequalities (5), (6) and (7) cover all possible cases, when $0<\theta<\dfrac{\pi}{2}$ and $0<\omega<\pi$.

Now consider the second integral with respect to ω in (4). Since $\dfrac{\pi}{2}-\theta>2^{-k-1}$ and $\omega<2^{-k-2}$, then $\theta+\omega<\dfrac{\pi}{2}$. Hence, as above

$$\int\limits_{\omega<2^{-k-2}} \frac{|f(y')|}{|\cos(x'\cdot y')|^l}dy' \leqslant \int\limits_0^{2^{k-2}} \omega^{n-2}d\omega\left(\int\limits_{S^{n-2}} \frac{dy'}{|\cos(x'\cdot y')|^{lp'}}\right)^{1/p'}\left(\int\limits_{S^{n-2}} |f(y')|^p\,dy'\right)^{1/p}$$

and we have the estimate

$$\left(\int\limits_{S^{n-2}} \frac{dy'}{|\cos(x'\cdot y')|^{1p'}}\right)^{1/p'} = C\left(\int\limits_0^\pi \frac{(\sin\varphi)^{n-2}d\varphi}{|\cos\theta\cos\omega+\sin\theta\sin\omega\cos\varphi|^{lp'}}\right)^{1/p'}$$

$$= C\left(\int\limits_0^\pi \frac{(\sin\varphi)^{n-2}d\varphi}{\left|\cos(\theta+\omega)+2\sin\theta\cos\omega\cos^2\dfrac{\varphi}{2}\right|^{lp'}}\right)^{1/p'} \leqslant \frac{C}{|\cos(\theta+\omega)|^l}.$$

Therefore, we have the estimates

$$\left(\int\limits_{S^{n-2}} \frac{dy'}{|\cos(x'\cdot y')|^{lp'}}\right)^{l/p'} \leqslant \begin{cases} \dfrac{C}{|\cos(\theta+\omega)|^l} & if\ \ \theta,\omega<\dfrac{\pi}{2},\ \ \theta+\omega>\dfrac{\pi}{2}\ \ or\ \ \theta+\omega<\dfrac{\pi}{2} \\ \dfrac{C}{|\cos(\theta-\omega)|^l} & if\ \ \omega>\dfrac{\pi}{2},\quad \theta+\omega>\dfrac{\pi}{2},\ \left(\theta<\dfrac{\pi}{2}\right). \end{cases}$$

Denoting

$$F(\omega) = \left(\int\limits_{S^{n-2}} |f(y)|^p\,dy'\right)^{1/p},$$

we can write

$$\int\limits_{\omega>2^{-k+1}} \frac{|f(y')|}{|\cos(x'\cdot y')|^l}dy' \leqslant \int\limits_{2^{-k+1}}^\pi \omega^{n-2}F(\omega)\left\{\frac{1}{|\cos(\theta+\omega)|^l}+\frac{1}{|\cos(\theta-\omega)|^l}\right\}d\omega.$$

Since $\omega > 2^{-k+1}, \frac{\pi}{2} - \theta < 2^{-k}$, then $\frac{\omega}{2} > \frac{\pi}{2} - \theta$ or $\omega > \frac{\omega}{2} + \frac{\pi}{2} - \theta$, $\omega - \frac{\pi}{2} + \theta > \frac{\omega}{2}$. In addition $\frac{\pi}{2} - \theta + \omega > \omega$ and therefore

$$\int\limits_{\omega > 2^{-k+1}} \frac{|f(y)|}{|\cos(y' \cdot x')|^l} dy' \leqslant \int\limits_{2^{-k+1}}^{\pi} F(\omega)\, \omega^{n-2-l} d\omega.$$

So for the integral I_3 in (4) we can write

$$\begin{aligned} I_3 &\leqslant \sum_{k=1}^{\infty} \int\limits_{2^{-k-1} < \frac{\pi}{2} - \theta < 2^{-k}} \left(\frac{\pi}{2} - \theta\right)^{p\beta} \left(\int\limits_{2^{-k+1}}^{\pi} F(\omega)\, \omega^{n-2-l} d\omega \right)^p d\theta \\ &\leqslant C \sum_{k=1}^{\infty} \int\limits_{2^{-k-1}}^{2^{-k}} \theta^{p\beta} \left(\int\limits_{\theta}^{\pi} F(\omega)\, \omega^{n-2-l} d\omega \right)^p d\theta. \end{aligned}$$

Given the condition $\beta > -\frac{1}{p}$, we can apply the Hardy inequality and get

$$I_3 \leqslant \sum_{k=1}^{\infty} \int\limits_{2^{-k-1}}^{2^{-k}} \theta^{p\beta} \left(\theta^{n-1-l} F(\theta)\right)^p d\theta = \sum_{k=1}^{\infty} \int\limits_{2^{-k-1}}^{2^{-k}} \theta^{p\beta} \theta^{(n-1-l)p} F(\theta) d\theta.$$

In the case $p \leqslant 2$ we have $l = \frac{n-2}{p'} + 1$ where $p' = \frac{p}{p-1}$ and then $(n-1-l)p = n-2$.

If $p \geqslant 2$, then $l = \frac{n-2}{p} + 1$ and $(n-1-l)p = n = 2$.

Hence $\theta^{(n-2)(p-1)} = \theta^{(n-2)(p-2)} \theta^{n-2} < \frac{\pi}{2}^{(n-2)(p-2)} \cdot \theta^{n-2}$. Therefore, since $x' = (\theta, \theta')$,

$$\begin{aligned} I_3 &\leqslant C \sum_{k=1}^{\infty} \int\limits_{2^{-k-1}}^{2^{-k}} (\sin\theta)^{p\beta} (\sin\theta)^{n-2} \int\limits_{S^{n-2}} |f(x')|^p d\theta' d\theta \\ &\leqslant C \int\limits_{S^{n-1}} (\sin x')^{p\beta} |f(\theta)|^p dx'. \end{aligned} \tag{8}$$

For the integral I_1 in (4) we have

$$I_1 \leqslant \sum_{k=1}^{\infty} \int\limits_{2^{-k-1} < \frac{\pi}{2} - \theta < 2^{-k}} \left(\frac{\pi}{2} - \theta\right)^{p\beta} \left(\int\limits_{0}^{2^{-k-2}} F(\omega) \frac{\omega^{n-2}}{\left|\frac{\pi}{2} - \omega - \theta\right|^l} d\omega \right)^p d\theta.$$

Since $\frac{\pi}{2}-\theta>2^{-k-1}$ and $\omega<2^{-k-2}$, then $\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)>\omega$, that is $\frac{\pi}{2}-\theta>\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)+\omega$. So $\frac{\pi}{2}-\theta-\omega>\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)$. Therefore, replacing $\frac{\pi}{2}-\theta$ by θ we have

$$I_1\leqslant\sum_{k=1}^{\infty}\int_{2^{-k-1}}^{2^{-k}}\theta^{p(\beta-l)}\left(\int_0^{\theta}F(\omega)\omega^{n-2}d\omega\right)^p d\theta.$$

Since $\beta<l-\frac{1}{p}$, we can apply the other inequality of Hardy and get

$$\begin{aligned}I_1&\leqslant\sum_{k=1}^{\infty}\int_{2^{-k-1}}^{2^{-k}}\theta^{p(\beta-l)}\left(\theta^{n-1}F(\theta)\right)^p d\theta\\&\leqslant C\sum_{k=1}^{\infty}\int_{2^{-k-1}}^{2^{-k}}(\sin\theta)^{p\beta}(\sin\theta)^{n-2}\int_{S^{n-2}}|f(\theta)|^p d\theta' d\theta\\&=C\int_{S^{n-1}}(\sin\theta)^{p\beta}\left|f\left(x'\right)\right|^p dx'.\end{aligned}\tag{9}$$

Form (8), (9) and (4)

$$I_1+I_3\leqslant\int_{S^{n-1}}|\sin\theta|^{p\beta}\left|f\left(x'\right)\right|^p dx'=\|f\|_{p,l,S_\beta}^p.\tag{10}$$

From (10), (3) and (2) the proof is complete. □

Similarly we prove

THEOREM 2. *Let $p>1$, $l=\frac{n}{2}-\left|\frac{1}{p}-\frac{1}{2}\right|(n-2)$ be a natural number and $-\frac{n-1}{p}<\beta<l-\frac{n-1}{p}$. Then, A is a bounded operator from $C_\beta L_p$ to $S_\beta H_p^l$.*

From Theorems 1-2 it follows

COROLLARY. *Let $p>1$, $l=\frac{n}{2}-\left|\frac{1}{p}-\frac{1}{2}\right|(n-2)$ be an integer and α any positive number. Then the operator A is bounded from $S_\beta H_p^\alpha$ to $C_p H_p^{\alpha+l}$ if $-\frac{1}{p}<\beta<l-\frac{1}{p}$ and is bounded from $C_\beta H_p^\alpha$ to $S_\beta H_p^{\alpha+l}$ if $-\frac{n-1}{p}<\beta<l-\frac{n-1}{p}$.*

Proof. Let $f\in S_\beta H_p^\alpha$ and then $(\sin\theta)^\beta\,\delta^{\alpha/2}f=g(\theta)\in L_p$. Introducing a function $\varphi(\theta)=\frac{g(\theta)}{(\sin\theta)^\beta}$ we have $\varphi\in S_\beta L_p$ and $\|\varphi\|_{p,\beta,0}=\|g\|_p$. By Theorem 1 $A\varphi\in$

$C_\beta H_p^l$ that is

$$\left\|(\cos)^\beta \delta^{l/2} A\varphi\right\|_p \leqslant C\|\varphi\|_{p,\beta,0} = C\|g\|_p.$$

But $\varphi = \delta^{\alpha/2} f$ and $g = (\sin\theta)^\beta \delta^{\alpha/2} f$.

Therefore

$$\left\|(\cos\theta)^\beta \delta^{\frac{l+\alpha}{2}} Af\right\|_p \leqslant \left\|(\sin\theta)^\beta \delta^{\frac{\alpha}{2}} f\right\|_p$$

an the first part is proved.

Similarly we can prove the second assertion. □

3. Boundedness of the operator A^{-1} and concluding remarks

Now we can prove corresponding statements about the boundedness of the inverse operator A^{-1} in the weighted spaces.

THEOREM 3. *Let* $p > 1$, $\gamma = \frac{n}{2} + \left|\frac{1}{p} - \frac{1}{2}\right|(n-2)$ *be an integer and* $-\frac{1}{p} < \beta < \gamma - \frac{1}{p}$. *Then the inverse operator* A^{-1} *is bounded from* $S_\beta H_p^\gamma$ *to* $C_\beta L_p$.

Proof. If $A^{-1}f \in S_\beta H_p^\gamma$, then by Theorem 1 $f \in C_\beta H_p^{\gamma+l}$. Hence, for any $j : 1 \leqslant j \leqslant \frac{n}{2}$

$$\delta^j AA^{-1} f \in C_\beta H_p^{\gamma+l-2j} \subset C_\beta H_p^{\gamma+l-n},$$

where $l = \frac{n}{2} - \left|\frac{1}{p} - \frac{1}{2}\right|(n-2)$. From this we have

$$\gamma - n - l = \frac{n}{2} - \left|\frac{1}{p} - \frac{1}{2}\right|(n-2) - n + \frac{n}{2} - \left|\frac{1}{p} - \frac{1}{2}\right|(n-2) = 0. \quad \square$$

By the same way we can prove

THEOREM 4. *Let* $p > 1$, $\gamma = \frac{n}{2} + \left|\frac{1}{p} - \frac{1}{2}\right|(n-2)$ *be an integer and* $-\frac{n-1}{p} < \beta < \gamma - \frac{n-1}{p}$. *Then the inverse operator* A^{-1} *is bounded from* $C_\beta H_p^\gamma$ *to* $S_\beta L_p$.

Theorems 1, 2, 3, 4 and corollary to Theorems 1-2 allow us to formulate the following results on oscillating multipliers.

THEOREM 5. *Let* $p > 1$, $l = \frac{n}{2} - \left|\frac{1}{p} - \frac{1}{2}\right|(n-2)$ *be an integer and* $-\frac{n-1}{p} < \beta < l - \frac{n-1}{p}$. *Then* $i^m \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n+m}{2}\right)}$ *is a multiplier from* $C_\beta H_p^\alpha$ *to* $S_\beta H_p^{l+\alpha}$ *for any* $\alpha > 0$.

THEOREM 6. *Let* $p > 1$, $l = \frac{n}{2} - \left|\frac{1}{p} - \frac{1}{2}\right|(n-2)$ *be an integer and* $-\frac{n-1}{p} < \beta < l - \frac{n-1}{p}$.

Then $i^m \frac{\Gamma(m/2)}{\Gamma\left(\frac{n+m}{2}\right)}$ is a multiplier from $C_\beta H_p^\alpha$ to $S_\beta H_p^{l+\alpha}$ for any $\alpha > 0$.

As a final remark we note that it would be interesting to prove theorems 1, 2, 3 and 4 for a fractional values l. Lemma 4 of the paper [18] and Lemma 2.1 of the paper [8] may be probably useful for this. The other direction is the obtaining the weighted $L_p \to L_q$ estimates by using the corresponding generalizations of the paper [8] given in [19].

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REFERENCES

[1] S. G. MIKHLIN, *Multidimensional singular integrals and integral equations*, Fizmatgiz, Moscow, 1962, English transl., Pergamon Press, 1965.

[2] ELIAS M. STEIN, *Singular integral operators and differentiability properties of functions*, Princeton Univ. Press, Princeton, State N.J., 1970.

[3] S. G. MIKHLIN, S. PRESSDORF, *Singular integral operators*, Springer Verlag, Berlin Heidelberg, New York-Tokyo, 1986.

[4] S. G. SAMKO, *Singular integral on the sphere and the construction characteristics by symbol*, Izv. VUZ ov Mathematics, No. 4 (251) (1983), 28–42. (Russian)

[5] A. D. GADJIEV, *Exact theorems on multipliers of spherical expansions and some applications*, in "Special problems of function theory", issue 4, Baku, Elm, 1989, 73–100. (Russian)

[6] A. D. GADJIEV, *Multipliers of Fourier Series in spherical functions and properties of the symbol of the multidimensional singular operators*, Dokl. Akad. Nauk USSR, **266**, 2 (1982); English transl. Soviet Math. Dokl. **26**, 2 (1982).

[7] A. D. GADZIEV, *On a connection between the properties of the symbol and the characteristic of a multidimensional singular integral operator in L_p-spaces*, Dokl. Akad. Nauk USSR **250** (1980), 531–534. English transl. in Soviet Math. Dokl. **21** (1980).

[8] A. D. GADZIEV, *On differentiability properties of the symbol of a multidimensional singular integral operator*, Matem. Sbornik **114(156)**, 4 (198); English transl. Math. USSR Sbornik **42**, 4 (1982), 427–450.

[9] F. G. MAKSUDOV, A. D. GADJIEV, *Spherical convolutions and oscillating multipliers*, Baku, Elm, 2000, 1–159. (Russian)

[10] YU. N. SKIBA, *Instability of the Rossby – Haurwitz wave in the invariant sets of perturbations*, J. Math. Anal. Appl. **290** (2004), 686–701.

[11] YU. N. SKIBA, *On the spectral problem in the linear stability study of flows on a sphere*, J. Math. Anal. Appl. **270** (2002), 165–180.

[12] BORIS RUBIN, *Inversion and characterization of the hemispherical transform*, J. D'Anal. Math. **77** (1999), 105–128.

[13] BORIS RUBIN, *Notes on Radon Transforms in integral geometry*, Fract. Calc. Appl. Anal. **6**, 1 (2003), 25–72.

[14] KH. P. RUSTAMOV, *On the smoothness of the symbol of multidimensional singular integral operators*, Analysis Math. **13** (1987), 93–199.

[15] V. S. KRYUCHKOV, *Study of differential properties of symbol of Kalderon – Zygmund singular integral*, PhD Dissertation, Moscow, 1983 (Russian).

[16] B. G. VAKULOV, *Spherical operators of potential type in generalized Holder spaces on the sphere*, PhD Dissertation, Rostov on Don, 1986 (Russian).

[17] P. M. PAVLOV, *Spherical hypersingular integrals and its applications*, PhD Dissertation, Rostov on Don, 1987 (Russian).
[18] ROBERT S. STRICHARTZ, *Convolution with kernels having singularities on a sphere*, Trans. American Math. Soc. **148** (1970), 461–471.
[19] V. S. KRYUCHKOV, *Differential properties of the symbol of a Calderon–Zygmund singular integral operator*, Proc. of the Steklov Inst. of Math. **170**, 1 (1987), 169–183 (Russian).

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