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LOCAL HÖLDER ESTIMATES FOR GENERAL ELLIPTIC $p(x)$–LAPLACIAN EQUATIONS

FENGPING YAO

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Abstract. In this paper we obtain the interior Hölder regularity of the gradients of weak solutions for general elliptic $p(x)$-Laplacian equations

$$\operatorname{div}\left(a\left(x,\nabla u\right)\right)=\operatorname{div}\left(|\mathbf{f}|^{p(x)-2}\mathbf{f}\right),$$

under some proper assumptions on a and the Hölder continuous functions $p,\mathbf{f}$.

1. Introduction

In this paper we mainly study the interior Hölder regularity of the gradients of weak solutions for the following general elliptic $p(x)$-Laplacian equation

$$\operatorname{div}\left(a\left(x,\nabla u\right)\right)=\operatorname{div}\left(|\mathbf{f}|^{p(x)-2}\mathbf{f}\right) \quad \text{in} \quad \Omega, \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ and $\mathbf{f}=(f^1,\cdots,f^n)$ is a given vector field satisfying

$$1<\gamma_1=\inf_{\Omega}p(x)\leqslant\sup_{\Omega}p(x)=\gamma_2<\infty, \tag{1.2}$$

$$p(x)\in C^{0,\alpha_1}_{loc}(\Omega) \quad \text{and} \quad f^i(x)\in C^{0,\alpha_2}_{loc}(\Omega) \tag{1.3}$$

for $1\leqslant i\leqslant n$, where the constants $\alpha_1,\alpha_2\in(0,1)$. Moreover, the structural conditions on the function $a(x,\xi)$ are given as follows

$$\left[a\left(x,\xi\right)-a\left(x,\eta\right)\right]\cdot\left(\xi-\eta\right)\geqslant\begin{cases} C_1|\xi-\eta|^{p(x)}, & p(x)\geqslant 2\\ C_1\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p(x)-2}{2}}|\xi-\eta|^2, & 1<p(x)<2,\end{cases} \tag{1.4}$$

$$\left|a\left(x,\xi\right)\right|\leqslant C_2\left(1+|\xi|^2\right)^{\frac{p(x)-1}{2}}, \tag{1.5}$$

$$a\left(x,\xi\right)\cdot\xi\geqslant C_3\left(\mu^2+|\xi|^2\right)^{\frac{p(x)}{2}}-C_4, \tag{1.6}$$

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$$|a(x,\xi)-a(y,\xi)| \leqslant C_5|x-y|^{\alpha_3}\left|\log\left(\mu^2+|\xi|^2\right)\right|\left(\mu^2+|\xi|^2\right)^{\frac{p(x)-1}{2}} \tag{1.7}$$

for all $\xi,\eta \in \mathbb{R}^n$, $x,y \in \Omega$ and some positive constants μ,α_3,C_i, $i=1,2,3,4,5$. We point out that the above structural condition (1.4) for $p(x)=p$ is the monotonicity condition (see [6]) while the remaining conditions were previously used in [3, 8].

When $p(x)=p$, many authors [4, 5, 9, 13, 14, 19, 20, 23, 26] studied the regularity for weak solutions of quasilinear elliptic equations

$$\operatorname{div}(a(x,\nabla u)) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) \quad \text{in} \quad \Omega \tag{1.8}$$

under some proper assumptions on $a,\mathbf{f}$.

When $p(x)$ is not a constant, such elliptic problems (1.1) appear in mathematical models of various physical phenomena, such as the electro-rheological fluids (see, e.g., [2, 24, 25]). Especially when $a(x,\nabla u)=|\nabla u|^{p(x)-2}\nabla u$ and $\mathbf{f}=0$, (1.1) is reduced to the $p(x)$-Laplacian elliptic equation

$$\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)=0 \quad \text{in } \Omega, \tag{1.9}$$

which can be derived from the variational problem

$$\Phi(u)=\min_{v|_{\partial\Omega}=\varphi}\Phi(v)=:\min_{v|_{\partial\Omega}=\varphi}\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}dx.$$

There have been many investigations [8, 22, 27] on Hölder estimates for the $p(x)$-Laplacian elliptic equation (1.9). Recently, Challal and Lyaghfouri [7] obtained the local L^∞ estimates of $|\nabla u|^{p(x)}$ for the weak solutions of (1.9). Moreover, Acerbi and Mingione [3] have proved that

$$|\mathbf{f}|^{p(x)} \in L^q_{loc}(\Omega) \quad \Longrightarrow \quad |\nabla u|^{p(x)} \in L^q_{loc}(\Omega) \ \text{ for any } \ q>1$$

for weak solutions of (1.1) under some assumptions on $a, p(x), \mathbf{f}$.

Recently, many authors (see for example [10, 11, 12, 15, 16, 18]) have studied the properties of the variable exponent Lebesgue-Sobolev spaces. We denote by $L^{p(x)}(\Omega)$ the variable exponent Lebesgue space

$$L^{p(x)}(\Omega)=\left\{f:\Omega\to\mathbb{R} \mid f \text{ is measurable and} \int_\Omega |f|^{p(x)}\,dx<\infty\right\} \tag{1.10}$$

equipped with the Luxemburg type norm

$$\|f\|_{L^{p(x)}(\Omega)}=\inf\left\{\lambda>0 \ : \ \int_\Omega\left|\frac{f}{\lambda}\right|^{p(x)}dx\leqslant 1\right\}. \tag{1.11}$$

Furthermore, we define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega) \ : \ |\nabla u|\in L^{p(x)}(\Omega)\right\} \tag{1.12}$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}. \tag{1.13}$$

Moreover, $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Actually, the $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are Banach spaces.

As usual, the solutions of (1.1) are taken in a weak sense. We now state the definition of weak solutions.

DEFINITION 1.1. Assume that $\mathbf{f} \in L^{p(x)}_{loc}(\Omega)$. A function $u \in W^{1,p(x)}_{loc}(\Omega)$ is a local weak solution of (1.1) in Ω if for any $\varphi \in W_0^{1,p(x)}(\Omega)$, we have

$$\int_\Omega a(x,\nabla u)\cdot\nabla\varphi\,dx = \int_\Omega |\mathbf{f}|^{p(x)-2}\mathbf{f}\cdot\nabla\varphi\,dx.$$

Now let us state the main result of this work.

THEOREM 1.2. *If u is a local weak solution of problem (1.1) under the assumptions (1.2)–(1.7), then ∇u is locally Hölder continuous.*

2. Proof of the main result

In this section we provide the proof of Theorem 1.2. We first recall the following reverse Hölder inequality.

LEMMA 2.1. (see [3], Theorem 5) *If u is a local weak solution of problem (1.1) under the assumptions (1.2)–(1.6), then there exist positive constants $\sigma_0, R_0 < 1, C$, depending on $n, \mu, \gamma_1, \gamma_2, C_i > 0,\ i = 1,2,3,4$, such that*

$$⨍_{B_{R/2}} |\nabla u|^{p(x)(1+\sigma)}\,dx \leqslant C\left(⨍_{B_R} |\nabla u|^{p(x)}\,dx\right)^{1+\sigma} + C\left(⨍_{B_R} 1+|\mathbf{f}|^{p(x)(1+\sigma)}\,dx\right)$$

for any $R \leqslant R_0$ and $\sigma \leqslant \sigma_0$.

We denote

$$p_m(R) = \min_{\overline{B_R}} p(x) \quad \text{and} \quad p_M(R) = \max_{\overline{B_R}} p(x).$$

Furthermore, we can obtain the following result.

LEMMA 2.2. *If u is a local weak solution of problem (1.1) under the assumptions (1.2)–(1.6), then there exists a positive constant $R_1 \in (0,R_0)$ such that*

$$⨍_{B_{R/2}} |\nabla u|^{p(x)(1+\sigma_0)}\,dx \leqslant CR^{-n\sigma_0}⨍_{B_R}\left(|\nabla u|^{p_M(R)}+1\right)\,dx$$

for any $R \leqslant R_1$ with $B_{R_0} \subset \Omega$.

Proof. We shall initially take R_1 small enough such that $0 < R_1 < R_0 < 1$ and

$$|p(x) - p(y)| \leqslant C_1|x-y|^{\alpha_1} \leqslant C_1(2R_1)^{\alpha_1} \leqslant \frac{\sigma_0\gamma_1}{\sigma_0+2}$$

for any $x, y \in B_{R_1}$. Assume that $R \leqslant R_1$. Then we find that $p_m(R) \geqslant \gamma_1 > 1$ and

$$p_M(R) \leqslant p_M(R) - p_m(R) + p_m(R) \leqslant p_m(R) + \frac{\sigma_0\gamma_1}{\sigma_0+2},$$

which implies that

$$\begin{aligned} p_M(R) &\leqslant p_M(R)(1+\sigma_0/2) \leqslant \left(p_m(R) + \frac{\sigma_0\gamma_1}{\sigma_0+2}\right)(1+\sigma_0/2) \\ &\leqslant p_m(R)\left(1+\frac{\sigma_0}{\sigma_0+2}\right)(1+\sigma_0/2) = p_m(R)(1+\sigma_0) \\ &\leqslant p(x)(1+\sigma_0). \end{aligned} \tag{2.1}$$

From Lemma 2.1 and the fact that $\{f^i\} \in C^{0,\alpha_2}_{loc}(\Omega)$ we have

$$\fint_{B_{R/2}} |\nabla u|^{p(x)(1+\sigma_0)}\,dx \leqslant C\left\{\left(\fint_{B_R} |\nabla u|^{p(x)}\,dx\right)^{1+\sigma_0} + 1\right\},$$

which implies that

$$\begin{aligned} \fint_{B_{R/2}} |\nabla u|^{p(x)(1+\sigma_0)}\,dx &\leqslant C\left\{\left(\fint_{B_R} |\nabla u|^{p(x)}\,dx\right)^{\sigma_0} \times \left(\fint_{B_R} |\nabla u|^{p(x)}\,dx\right) + 1\right\} \\ &\leqslant C\left\{R^{-n\sigma_0}\fint_{B_R} |\nabla u|^{p(x)}\,dx + 1\right\} \\ &\leqslant CR^{-n\sigma_0}\fint_{B_R}\left(|\nabla u|^{p(x)}+1\right)dx \\ &\leqslant CR^{-n\sigma_0}\fint_{B_R}\left(|\nabla u|^{p_M(R)}+1\right)dx \end{aligned} \tag{2.2}$$

for any $R \leqslant R_1 \leqslant R_0 < 1$ with $B_{R_0} \subset \Omega$, since $u \in W^{1,p(x)}_{loc}(\Omega)$. $\square$

Assume that $R \leqslant R_1/2 \leqslant R_0/2$ with $B_{R_0} \subset \Omega$. We may assume that

$$p(x_M) = p_M(2R) = \max_{\overline{B_{2R}}} p(x).$$

For simplicity we write

$$p_2 =: p(x_M).$$

Let us consider the weak solution of the following reference equation

$$\begin{cases} \operatorname{div} a(x_M, \nabla v) = 0 & \text{in } B_R, \\ v = u & \text{on } \partial B_R, \end{cases}$$

which implies that

$$\begin{cases} \operatorname{div} a(x_M, \nabla v) = \operatorname{div}\left(|\mathbf{f}(x_M)|^{p_2-2}\mathbf{f}(x_M)\right) & \text{in } B_R, \\ v = u & \text{on } \partial B_R, \end{cases} \tag{2.3}$$

since $\operatorname{div}\left(|\mathbf{f}(x_M)|^{p_2-2}\mathbf{f}(x_M)\right) = 0$.

LEMMA 2.3. (see [21], Lemma 5.1) *If v is the weak solution of problem (2.3), then we have*

$$\sup_{B_{R/2}} |\nabla v|^{p_2} \leqslant CR^{-n} \int_{B_R} |\nabla v|^{p_2}\, dx, \tag{2.4}$$

$$⨍_{B_\rho} \left|\nabla v - (\nabla v)_{B_\rho}\right|^{p_2} dx \leqslant C\left(\frac{\rho}{R}\right)^{\beta_1} ⨍_{B_R} \left|\nabla v - (\nabla v)_{B_R}\right|^{p_2} dx, \tag{2.5}$$

$$\int_{B_R} |\nabla v|^{p_2}\, dx \leqslant C \int_{B_R} |\nabla u|^{p_2} + 1 dx, \tag{2.6}$$

for any $\rho < R$, where

$$(\nabla v)_{B_\rho} = ⨍_{B_\rho} \nabla v\, dx,$$

and $\beta_1 \in (0,1)$ and C are two positive constants depending on $p_2, n, \mu, \gamma_1, \gamma_2, C_i$ $(1 \leqslant i \leqslant 5)$.

LEMMA 2.4. *Assume that $R \leqslant R_1/2 \leqslant R_0/2$ with $B_{R_0} \subset \Omega$. If u is a local weak solution of problem (1.1) under the assumptions (1.2)–(1.7) and v is the weak solution of the reference equation (2.3), then there exists a positive constant $\beta \in (0,1)$, depending on $n, \sigma_0, \gamma_1, \gamma_2, \alpha_i$ $(1 \leqslant i \leqslant 3)$, such that*

$$⨍_{B_R} |\nabla u - \nabla v|^{p_2}\, dx \leqslant CR^{\beta} ⨍_{B_{2R}} (|\nabla u|^{p_2} + 1)\, dx,$$

where C depends on $n, \mu, \gamma_1, \gamma_2, \alpha_i$ $(1 \leqslant i \leqslant 3)$, C_j $(1 \leqslant j \leqslant 5)$, and the Hölder norms of $\{f^i\}$, $p(x)$.

Proof. Without loss of generality we may as well select the test function $\varphi = u - v$. From the definitions of weak solutions we have

$$\int_{B_R} a(x, \nabla u) \cdot \nabla(u - v)\, dx = \int_{B_R} |\mathbf{f}(x)|^{p(x)-2} \mathbf{f}(x) \cdot \nabla(u - v)\, dx$$

and

$$\int_{B_R} a(x_M, \nabla v) \cdot \nabla(u - v)\, dx = \int_{B_R} |\mathbf{f}(x_M)|^{p_2-2} \mathbf{f}(x_M) \cdot \nabla(u - v)\, dx.$$

After a direct calculation we show the resulting expression as

$$I_4 = I_1 + I_2 + I_3,$$

where

$$I_1 = -\!\!\!\!\!\!\int_{B_R} (a(x,\nabla u) - a(x_M,\nabla u)) \cdot \nabla(u-v)\, dx,$$
$$I_2 = -\!\!\!\!\!\!\int_{B_R} \left(|\mathbf{f}(x)|^{p(x)-2}\mathbf{f}(x) - |\mathbf{f}(x_M)|^{p(x)-2}\mathbf{f}(x_M)\right) \cdot \nabla(u-v)\, dx,$$
$$I_3 = -\!\!\!\!\!\!\int_{B_R} \left(|\mathbf{f}(x_M)|^{p(x)-2}\mathbf{f}(x_M) - |\mathbf{f}(x_M)|^{p_2-2}\mathbf{f}(x_M)\right) \cdot \nabla(u-v)\, dx,$$
$$I_4 = -\!\!\!\!\!\!\int_{B_R} (a(x_M,\nabla u) - a(x_M,\nabla v)) \cdot \nabla(u-v)\, dx.$$

Estimate of I_1. From (1.7) and Hölder's inequality, we have

$$\begin{aligned} I_1 &\leqslant C -\!\!\!\!\!\!\int_{B_R} |x-x_M|^{\alpha_3} \left|\log\left(\mu^2+|\nabla u|^2\right)\right| \left(\mu^2+|\nabla u|^2\right)^{\frac{p_2-1}{2}} |\nabla(u-v)|\, dx \\ &\leqslant CR^{\alpha_3} -\!\!\!\!\!\!\int_{B_R} \left(1+|\nabla u|^{p_2-1+\delta}\right) |\nabla(u-v)|\, dx \\ &\leqslant CR^{\alpha_3} \left(-\!\!\!\!\!\!\int_{B_R} |\nabla(u-v)|^{p_2}\, dx\right)^{1/p_2} \left(-\!\!\!\!\!\!\int_{B_R} (|\nabla u|+1)^{\frac{(p_2-1+\delta)p_2}{p_2-1}}\, dx\right)^{\frac{p_2-1}{p_2}} \end{aligned}$$

for any $\delta > 0$. Selecting proper $\delta \in (0,(\gamma_1-1)\sigma_0/2)$, we deduce from (2.1) that

$$\frac{p_2(p_2-1+\delta)}{p_2-1} = p_2\left(1+\frac{\delta}{p_2-1}\right) \leqslant p_2\left(1+\frac{\delta}{\gamma_1-1}\right) \leqslant p_2\left(1+\frac{\sigma_0}{2}\right) \leqslant p(x)(1+\sigma_0).$$

Therefore, Lemma 2.2 implies that

$$\begin{aligned} I_1 &\leqslant CR^{\alpha_3} \left(-\!\!\!\!\!\!\int_{B_R} |\nabla(u-v)|^{p_2}\, dx\right)^{1/p_2} \left(-\!\!\!\!\!\!\int_{B_R} (|\nabla u|+1)^{p(x)(1+\sigma_0)}\, dx\right)^{\frac{p_2-1}{p_2}} \\ &\leqslant CR^{\alpha_3-\frac{n\sigma_0(p_2-1)}{p_2}} \left(-\!\!\!\!\!\!\int_{B_R} |\nabla(u-v)|^{p_2}\, dx\right)^{1/p_2} \left(-\!\!\!\!\!\!\int_{B_{2R}} (|\nabla u|^{p_2}+1)\, dx\right)^{\frac{p_2-1}{p_2}} \\ &\leqslant CR^{\alpha_3-n\sigma_0+\frac{n\sigma_0}{\gamma_2}} \left(-\!\!\!\!\!\!\int_{B_R} |\nabla(u-v)|^{p_2}\, dx\right)^{1/p_2} \left(-\!\!\!\!\!\!\int_{B_{2R}} (|\nabla u|^{p_2}+1)\, dx\right)^{\frac{p_2-1}{p_2}}, \end{aligned}$$

since

$$R^{\alpha_3-\frac{n\sigma_0(p_2-1)}{p_2}} = R^{\alpha_3-n\sigma_0+\frac{n\sigma_0}{p_2}} \leqslant R^{\alpha_3-n\sigma_0+\frac{n\sigma_0}{\gamma_2}}.$$

Estimate of I_2. We consider the following two cases.

Case 1. $p(x) \geqslant 2$. Using the elementary inequality

$$\left||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right| \leqslant C(|\xi|+|\eta|)^{p-2}|\xi-\eta|$$

for any $p \geqslant 2$, $\xi, \eta \in \mathbb{R}^n$ with $C = C(p)$, Hölder's inequality and the fact that $\{f^i\} \in C^{0,\alpha_2}_{loc}(\Omega)$, we have

$$\begin{aligned} I_2 &\leqslant \fint_{B_R} \Big| |\mathbf{f}(x)|^{p(x)-2}\mathbf{f}(x) - |\mathbf{f}(x_M)|^{p(x)-2}\mathbf{f}(x_M) \Big| |\nabla(u-v)|\, dx, \\ &\leqslant \fint_{B_R} (|\mathbf{f}(x)| + |\mathbf{f}(x_M)|)^{p(x)-2} |\mathbf{f}(x) - \mathbf{f}(x_M)| |\nabla(u-v)|\, dx, \\ &\leqslant CR^{\alpha_2} \fint_{B_R} |\nabla(u-v)|\, dx \\ &\leqslant CR^{\alpha_2} \left(\fint_{B_R} |\nabla u - \nabla v|^{p_2}\, dx \right)^{\frac{1}{p_2}}. \end{aligned}$$

Case 2. $1 < p(x) < 2$. Using the elementary inequality

$$\big| |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \big| \leqslant C|\xi - \eta|^{p-1}$$

for any $p \in (1,2)$, $\xi, \eta \in \mathbb{R}^n$ with $C = C(p)$, Hölder's inequality and the fact that $\{f^i\} \in C^{0,\alpha_2}_{loc}(\Omega)$, we have

$$\begin{aligned} I_2 &\leqslant \fint_{B_R} |\mathbf{f}(x) - \mathbf{f}(x_M)|^{p(x)-1} |\nabla(u-v)|\, dx \\ &\leqslant C \fint_{B_R} R^{(p(x)-1)\alpha_2} |\nabla(u-v)|\, dx \\ &\leqslant CR^{(\gamma_1-1)\alpha_2} \fint_{B_R} |\nabla(u-v)|\, dx \\ &\leqslant CR^{(\gamma_1-1)\alpha_2} \left(\fint_{B_R} |\nabla u - \nabla v|^{p_2}\, dx \right)^{\frac{1}{p_2}}. \end{aligned}$$

Estimate of I_3. From the mean-value theorem, Hölder's inequality and the facts that $p(x) \in C^{0,\alpha_1}_{loc}(\Omega)$ and $\{f^i\} \in C^{0,\alpha_2}_{loc}(\Omega)$, we have

$$\begin{aligned} I_3 &\leqslant C \fint_{B_R} |p_2 - p(x)| \left(1 + |\mathbf{f}(x_M)|^{p_2-1}\right) |\ln(|\mathbf{f}(x_M)|)| |\nabla(u-v)|\, dx \\ &\leqslant CR^{\alpha_1} \fint_{B_R} |\nabla(u-v)|\, dx \\ &\leqslant CR^{\alpha_1} \left(\fint_{B_R} |\nabla u - \nabla v|^{p_2}\, dx \right)^{\frac{1}{p_2}}. \end{aligned}$$

Combining all the estimates of I_i $(1 \leqslant i \leqslant 3)$, we obtain

$$\begin{aligned} I_4 &= I_1 + I_2 + I_3 \\ &\leqslant CR^{\beta_2} \left(\fint_{B_R} |\nabla u - \nabla v|^{p_2}\, dx \right)^{\frac{1}{p_2}} \left(\fint_{B_{2R}} (|\nabla u|^{p_2} + 1)\, dx \right)^{\frac{p_2-1}{p_2}}, \end{aligned} \tag{2.7}$$

where

$$\beta_2 =: \min\left\{\alpha_1, \alpha_2, (\gamma_1 - 1)\alpha_2, \alpha_3 - n\sigma_0 + \frac{n\sigma_0}{\gamma_2}\right\}.$$

Without loss of generality we may as well assume that $\sigma_0 > 0$ small enough such that $\sigma_0 n(1 - 1/\gamma_2) < \alpha_3$. Then we observe that $\beta_2 > 0$.

Estimate of I_4. We consider the following two cases.

Case 1. $p_2 = p(x_M) \geqslant 2$. Using the following condition

$$[a(x,\xi) - a(x,\eta)] \cdot (\xi - \eta) \geqslant C_1 |\xi - \eta|^{p(x)},$$

we have

$$I_4 \geqslant C \fint_{B_R} |\nabla u - \nabla v|^{p_2} \, dx.$$

From (2.7) we have

$$\begin{aligned} \fint_{B_R} |\nabla u - \nabla v|^{p_2} \, dx &\leqslant C R^{\frac{\beta_2 p_2}{p_2 - 1}} \fint_{B_{2R}} (|\nabla u|^{p_2} + 1) \, dx \\ &\leqslant C R^{\frac{\beta_2 \gamma_2}{\gamma_2 - 1}} \fint_{B_{2R}} (|\nabla u|^{p_2} + 1) \, dx. \end{aligned}$$

Case 2. $1 < p_2 = p(x_M) < 2$. From the following condition

$$[a(x,\xi) - a(x,\eta)] \cdot (\xi - \eta) \geqslant C \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p(x)-2}{2}} |\xi - \eta|^2,$$

we have

$$I_4 \geqslant C \fint_{B_R} \left(\mu^2 + |\nabla u|^2 + |\nabla v|^2\right)^{\frac{p_2 - 2}{2}} |\nabla u - \nabla v|^2 \, dx. \tag{2.8}$$

Therefore, using Hölder's inequality and (2.6), we have

$$\begin{aligned} \fint_{B_R} |\nabla u - \nabla v|^{p_2} \, dx &= \fint_{B_R} \left(\mu^2 + |\nabla u|^2 + |\nabla v|^2\right)^{\frac{p_2-2}{4} + \frac{2-p_2}{4}} |\nabla u - \nabla v| \, |\nabla u - \nabla v|^{p_2 - 1} \, dx \\ &\leqslant \left(\fint_{B_R} \left(\mu^2 + |\nabla u|^2 + |\nabla v|^2\right)^{\frac{p_2-2}{2}} |\nabla u - \nabla v|^2 \, dx\right)^{\frac{1}{2}} \\ &\quad \times \left(\fint_{B_R} \left(\mu^2 + |\nabla u|^2 + |\nabla v|^2\right)^{\frac{2-p_2}{2}} |\nabla u - \nabla v|^{2p_2 - 2} \, dx\right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, it follows from (2.6) and (2.8) that

$$\begin{aligned} \fint_{B_R} |\nabla u - \nabla v|^{p_2} \, dx &\leqslant C I_4^{1/2} \left(\fint_{B_R} \left(1 + |\nabla u|^2 + |\nabla v|^2\right)^{\frac{2-p_2}{2}} \left(1 + |\nabla u|^2 + |\nabla v|^2\right)^{\frac{2p_2-2}{2}} \, dx\right)^{\frac{1}{2}} \\ &\leqslant C I_4^{1/2} \left(\fint_{B_R} (1 + |\nabla u|^{p_2} + |\nabla v|^{p_2}) \, dx\right)^{\frac{1}{2}} \end{aligned}$$

$$\leqslant CI_4^{1/2}\left(\fint_{B_R}(1+|\nabla u|^{p_2})\,dx\right)^{\frac{1}{2}}$$
$$\leqslant CI_4^{1/2}\left(\fint_{B_{2R}}(1+|\nabla u|^{p_2})\,dx\right)^{\frac{1}{2}}.$$

Thus, in view of (2.7) we find that

$$\fint_{B_R}|\nabla u-\nabla v|^{p_2}\,dx\leqslant CR^{\beta_2/2}\left(\fint_{B_R}|\nabla u-\nabla v|^{p_2}\,dx\right)^{\frac{1}{2p_2}}\left(\fint_{B_{2R}}(1+|\nabla u|^{p_2})\,dx\right)^{\frac{2p_2-1}{2p_2}},$$

which implies that

$$\fint_{B_R}|\nabla u-\nabla v|^{p_2}\,dx\leqslant CR^{\frac{\beta_2p_2}{2p_2-1}}\fint_{B_{2R}}(|\nabla u|^{p_2}+1)\,dx$$
$$\leqslant CR^{\frac{\beta_2\gamma_2}{2\gamma_2-1}}\fint_{B_{2R}}(|\nabla u|^{p_2}+1)\,dx.$$

Thus, combining the estimates of I_4 in Case 1 and Case 2, we obtain

$$\fint_{B_R}|\nabla u-\nabla v|^{p_2}\,dx\leqslant CR^{\frac{\beta_2\gamma_2}{2\gamma_2-1}}\fint_{B_{2R}}(|\nabla u|^{p_2}+1)\,dx,$$

which can finish the proof by choosing $\beta=\frac{\beta_2\gamma_2}{2\gamma_2-1}$. □

LEMMA 2.5. *Assume that $B_{8R_0}\subset\Omega_0\subset\subset\Omega$. If u is a local weak solution of problem (1.1) under the assumptions (1.2)–(1.7), then for any $\tau\in(0,n)$ there exist positive constants $R_2\in(0,R_1)$ and C, depending on $n,\mu,\gamma_1,\gamma_2,\alpha_i$ $(1\leqslant i\leqslant 3)$, C_j $(1\leqslant j\leqslant 5)$, dist$\{\Omega_0,\Omega\}$, such that*

$$\fint_{B_\rho}|\nabla u|^{p_2}\,dx\leqslant C\rho^{-\tau} \tag{2.9}$$

for any $0<\rho\leqslant R/16\leqslant R_2/16<R_1/16\leqslant R_0/16$.

Proof. From (2.5) in Lemma 2.3, Lemma 2.4 and Hölder's inequality we have

$$\int_{B_\rho}|\nabla u|^{p_2}\,dx$$
$$\leqslant C\left(\int_{B_\rho}|\nabla u-\nabla v|^{p_2}\,dx+\int_{B_\rho}\left|\nabla v-(\nabla v)_{B_\rho}\right|^{p_2}dx+\rho^n\left|(\nabla v)_{B_\rho}\right|^{p_2}\right)$$
$$\leqslant C\left(R^\beta\int_{B_{R/4}}(|\nabla u|^{p_2}+1)\,dx+\left(\frac{\rho}{R}\right)^{n+\beta_1}\int_{B_R}\left|\nabla v-(\nabla v)_{B_R}\right|^{p_2}dx+\int_{B_\rho}|\nabla v|^{p_2}\,dx\right)$$
$$\leqslant C\left(R^\beta\int_{B_{R/4}}(|\nabla u|^{p_2}+1)\,dx+\left(\frac{\rho}{R}\right)^{n+\beta_1}\int_{B_R}\left|\nabla v-(\nabla v)_{B_R}\right|^{p_2}dx+\rho^n\sup_{B_\rho}|\nabla v|\right),$$

which implies that

$$\begin{aligned}
&\int_{B_\rho} |\nabla u|^{p_2}\,dx \\
&\leqslant C\left(R^\beta \int_{B_{R/4}} (|\nabla u|^{p_2}+1)\,dx + \left(\frac{\rho}{R}\right)^{n+\beta_1} \int_{B_R} |\nabla v|^{p_2}\,dx + \left(\frac{\rho}{R}\right)^{n} \int_{B_R} |\nabla v|^{p_2}\,dx \right) \\
&\leqslant C\left(R^\beta \int_{B_{R/4}} (|\nabla u|^{p_2}+1)\ dx + \left(\frac{\rho}{R}\right)^{n} \int_{B_R} (|\nabla u|^{p_2}+1)\,dx \right)
\end{aligned}$$

for any $\rho \leqslant R/8 \leqslant R_1/8$, in view of (2.4) and (2.6) in Lemma 2.3. So, we have

$$\int_{B_\rho} |\nabla u|^{p_2}\,dx \leqslant C\left[\left(R^\beta + \left(\frac{\rho}{R}\right)^n \right) \int_{B_R} |\nabla u|^{p_2}\ dx + R^n \right].$$

Finally, similarly to the proof of (12) in [8], from a covering and iteration argument (see Lemma 3.2 in [1]), for every $\tau \in (0,n)$ there exist positive constants $R_2, C > 0$ such that for all $0 < \rho \leqslant R/16 \leqslant R_2/16 \leqslant R_1/16 \leqslant R_0/16$ such that (2.9) holds. □

Now we are ready to prove the main result, Theorem 1.2.

Proof. From Young's inequality and Hölder's inequality, we have

$$\begin{aligned}
&\int_{B_\rho} \left|\nabla u - (\nabla u)_{B_\rho}\right|^{p_2} dx \\
&\leqslant C\left[\int_{B_\rho} \left|\nabla u - (\nabla v)_{B_\rho}\right|^{p_2} dx + \rho^n \left|(\nabla u)_{B_\rho} - (\nabla v)_{B_\rho}\right|^{p_2} \right] \\
&\leqslant C \int_{B_\rho} \left|\nabla u - (\nabla v)_{B_\rho}\right|^{p_2} dx \\
&\leqslant C\left[\rho^n \fint_{B_\rho} \left|\nabla v - (\nabla v)_{B_\rho}\right|^{p_2} dx + \int_{B_R} |\nabla u - \nabla v|^{p_2}\,dx \right].
\end{aligned}$$

Therefore, it follows from (2.5)–(2.6) in Lemma 2.3 and Lemma 2.5 that

$$\begin{aligned}
&\int_{B_\rho} \left|\nabla u - (\nabla u)_{B_\rho}\right|^{p_2} dx \\
&\leqslant C\left[\left(\frac{\rho}{R}\right)^{\beta_1} \rho^n \fint_{B_{R/16}} \left|\nabla v - (\nabla v)_{B_{R/16}}\right|^{p_2} dx + R^\beta \int_{B_{2R}} (|\nabla u|^{p_2}+1)\ dx \right] \\
&\leqslant C\left[\left(\frac{\rho}{R}\right)^{\beta_1} \rho^n \fint_{B_{R/16}} |\nabla v|^{p_2}\,dx + R^{n+\beta-\tau} \right] \\
&\leqslant C\left[\left(\frac{\rho}{R}\right)^{\beta_1} \rho^n \fint_{B_R} (|\nabla u|^{p_2}+1)\,dx + R^{n+\beta-\tau} \right] \\
&\leqslant C\left[\left(\frac{\rho}{R}\right)^{\beta_1} \rho^n R^{-\tau} + R^{n+\beta-\tau} \right] = CR^{-\tau-\beta_1}\left[\rho^{n+\beta_1} + R^{n+\beta+\beta_1} \right]
\end{aligned}$$

for any $0 < \rho \leqslant R/32 \leqslant R_2/32 \leqslant R_1/32 \leqslant R_0/32$ with $B_{8R_0} \subset \Omega_0 \subset\subset \Omega$. Choose $\rho = \frac{R^{1+\mu}}{32}$, where $\mu = \frac{\beta}{n+\beta_1} > 0$. Then we obtain

$$\rho^{n+\beta_1} \leqslant CR^{n+\beta+\beta_1},$$

which implies that

$$\int_{B_\rho} \left|\nabla u - (\nabla u)_{B_\rho}\right|^{p_2} dx \leqslant CR^{n+\beta-\tau} \leqslant C\rho^{\frac{n+\beta-\tau}{1+\mu}} \leqslant C\rho^{n+\frac{\beta-\mu n-\tau}{1+\mu}}.$$

Choose $\tau = \frac{\beta\beta_1}{2(n+\beta_1)} > 0$. Then we have

$$\beta - \mu n - \tau = \frac{\beta\beta_1}{n+\beta_1} - \tau = \frac{\beta\beta_1}{2(n+\beta_1)} > 0.$$

Furthermore, from Hölder's inequality we have

$$\begin{aligned} &⨍_{B_\rho} \left|\nabla u - (\nabla u)_{B_\rho}\right|^{\gamma_1} dx \\ &\leqslant \left(⨍_{B_\rho} \left|\nabla u - (\nabla u)_{B_\rho}\right|^{p_2} dx\right)^{\frac{\gamma_1}{p_2}} \leqslant C\rho^{\frac{(\beta-\mu n-\tau)\gamma_1}{(1+\mu)p_2}} \leqslant C\rho^{\frac{(\beta-\mu n-\tau)\gamma_1}{(1+\mu)\gamma_2}} =: C\rho^\alpha \end{aligned}$$

for any $0 < \rho \leqslant R/32 \leqslant R_2/32 \leqslant R_1/32 \leqslant R_0/32$ with $B_{8R_0} \subset \Omega_0 \subset\subset \Omega$. It is easy to see that

$$\alpha = \frac{(\beta - \mu n - \tau)\gamma_1}{(1+\mu)\gamma_2} > 0.$$

Then from Campanato's theorem (see [17], Theorem 1.3 of Chapter 3) we conclude that $u \in C^{1,\alpha}(B_{R_2/32})$. Thus, we can complete the proof of Theorem 1.2 by an elementary covering argument. □

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Fengping Yao
Department of Mathematics
Shanghai University
Shanghai 200444, China
e-mail: yfp@shu.edu.cn