Journal of Mathematical Inequalities
Volume 8, Number 2 (2014), 251–265 doi:10.7153/jmi-08-17

WEIGHTED TURÁN TYPE INEQUALITY FOR RATIONAL FUNCTIONS WITH PRESCRIBED POLES

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(*Communicated by I. Raşa*)

Abstract. Firstly, we introduce a new type of weight functions named as N-doubling weights, which is an essential generalization of the well known doubling weights. Secondly, we establish a weighted Turán type inequality with N-doubling weights and a Nikolskii-Turán type inequality for rational functions with prescribed poles. Our results generalize some known Turán type inequality both for polynomials and rational functions.

1. Introduction

It is well know that Bernstein's inequality for trigonometric polynomials and Markov's inequality for algebraic polynomials play an important role in establishing the converse results in approximation theory. There are many natural and important generalizations and improvements on Bernstein's inequality and Markov's inequality. For example, one of the most important recent progress on this direction can be found in [4], where the authors established some important Bernstein's type inequalities with doubling weights and A^* weights.

In 1939, Turán [7] established an inequality which was later referred as Turán's inequality. Let H_n be the class of real algebraic polynomials of degree n, whose zeros all lie in the interval $[-1,1]$. Then Turán's inequality can be stated as follows:

$$\|f'\|_\infty \geqslant C\sqrt{n}\|f\|_\infty$$

for all $f \in H_n$, where $\|f\|_\infty$ is the usual supremum norm of f.

Turán's inequality have attracted attention of many mathematicians and a lot of generalizations were achieved. For examples, It was generalized to L_p spaces for $0 < p < \infty$ (see [8]–[10], [15]–[18]), the optimal constants were estimated (see [1]–[3], [5], [8]–[10]), the weighted Turán's inequality ([11], [12], [14]) and Nikol'skii type inequalities ([16]–[18]) were considered as well.

It is natural to ask if one can generalize Turán type inequality to the rational system $R_n = \{p/q : \ p,q \in \Pi_n\}$ with restricted zeros and poles. Min [6] in 1999 made such an attempt to establish the following results:

Mathematics subject classification (2010): 41A17, 26D10.
Keywords and phrases: Weighted Turán type inequality, generalized doubling weights, rational functions with prescribed poles.
Research of the first author is supported by NSF of China (11271248/A010603), Research of the third author is supported by NSF of China (10901044) and Program for excellent Young Teachers in HZNU
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THEOREM 1. *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset C\setminus[-1,1]$ *be paired by complex conjugation, and let* $\{a_k\}$ *satisfy that* $|a_k|-1>\rho$ *for some* $\rho>2$ *and all* $k=1,2,\cdots,n$. *Then, for* $f\in\mathscr{P}_n(a_1,a_2,\cdots,a_n)$ *with all zeros in* $[-1,1]$, *we have*

$$\|f'\|_\infty \geqslant \frac{\sqrt{\rho^2-4}}{6\rho}\sqrt{n}\|f\|_\infty$$

for $n\geqslant \max\left\{\frac{\rho+2}{9(\rho-2)},\frac{4\rho^2}{\rho^2-4}\right\}$, *where*

$$\mathscr{P}_n(a_1,a_2,\cdots,a_n)=\left\{\frac{P(x)}{\prod_{k=1}^n(x-a_k)},\ P\in\Pi_n\right\}.$$

THEOREM 2. *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset C\setminus[-1,1]$ *be paired by complex conjugation, and let* $\{a_k\}$ *satisfy that* $|a_k|-1>\rho$ *for some* $\rho>2$ *and all* $k=1,2,\cdots,n$. *Then, for* $f\in\mathscr{P}(a_1,a_2,\cdots,a_n)$ *with all zeros in* $[-1,1]$, *we have*

$$\|f'\|_2^2 \geqslant \frac{1}{2}\int_{-1}^1 \mathscr{B}_n(x)f^2(x)dx,$$

where

$$\mathscr{B}(x)=\sum_{k=1}^n\frac{a_k^2-1}{(x-a_k)^2}>0,\ \ x\in[-1,1],$$

and $\|f\|_2$ *is the* L^2 *norm of* f, *that is,*

$$\|f\|_2:=\left(\int_{-1}^1|f(x)|^2\,dx\right)^{1/2}.$$

Yu and Zhou [15] generalized the Min's results to the general space L_p for $1\leqslant p\leqslant\infty$ while removing the unpleasant restriction $\rho>2$ in Theorem 1.

THEOREM 3. *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset C\setminus[-1,1]$ *be paired by complex conjugation, and let* $\{a_k\}$ *satisfy that* $|a_k|-1>\rho$ *for some* $\rho>0$ *and all* $k=1,2,\cdots,n$. *Then, for* $f\in\mathscr{P}_n(a_1,a_2,\cdots,a_n)$ *with all zeros in* $[-1,1]$, *we have*

$$\|f'\|_p\geqslant C(\rho)\sqrt{n}\|f\|_p,\ \ 1\leqslant p\leqslant\infty. \tag{1.1}$$

In this paper, we will generalize (1.1) to the weighted cases and establish the Nikol'skii type inequality. The paper is organized as follows: In section 2, we introduce a new type of weight functions named as N-doubling weights, which is an essential generalization of the well known doubling weights. We give some properties of N-doubling weights and the main results (Theorem 4 and Theorem 5). In section 3, we give some auxiliary lemmas. The proofs of the results are given in section 4.

2. The main results

We say an integrable nonnegative function $W(x)$ is a doubling weight if it satisfies the so-called doubling condition (see [4], for example)

$$W(2I) \leqslant LW(I)$$

for all intervals I, where L is a constant independent of I, $2I$ is the interval twice the length of I and with midpoint at the midpoint of I (note that parts of $2I$ may lie outside $[-1,1]$, where we set $W(x) = 0$), and

$$W(I) := \int_I W(u)dd$$

for any measurable set $I \subseteq [-1,1]$.

As we know, doubling condition has been applied widely in Fourier analysis and harmonic analysis. Now, we further extend the doubling condition as follows:

DEFINITION 1. An integrable nonnegative function $W(x)$ is said to be an N-doubling weight function if $W(x)$ is defined in the interval $[-1,1]$, and there is a constant $L \geqslant 1$ such that

$$W(2I) \leqslant LW(2I \backslash I)$$

holds for any $2I \subseteq [-1,1]$, where L is independent of $I, 2I \backslash I = \{x | x \in 2I, x \notin I\}$.

The following proposition means that the N-doubling condition is an essential generalization of the doubling condition.

PROPOSITION 1. *A doubling weight function is an N-doubling weight function, but the converse is not true.*

To consider the weighted Turán inequality, Wang and Zhou [11] introduced a class of weight functions named as *Generalized Jacobi Weight* functions (GJW), that is, one says $W(x) \in GJW$, if $W(x) \geqslant 0$, $\int_{-1}^{1} W(x)dx < \infty$ and $W(x_1) \approx W(x_2)$ for any $-1 < x_1 < x_2 \leqslant 0$ and $|x_2 - x_1| < 1 + x_1$ or for any $0 < x_2 < x_1 \leqslant 1$ and $|x_2 - x_1| < 1 - x_1$, where $W(x_1) \approx W(x_2)$ means that there is a constant $M \geqslant 1$ (M depends on $W(x)$) such that $M^{-1}W(x_1) \leqslant W(x_2) \leqslant MW(x_1)$.

Wei and Yu [14] pointed out that it is still an open problem whether any $W(x) \in GJW$ must be a doubling weight. However, we have

PROPOSITION 2. *Any $W(x) \in GJW$ must be an N-doubling weight.*

Proposition 2 shows that N-doubling condition is also a generalization of the condition GJW.

Now, we can state our main results as follows:

THEOREM 4. *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset C\setminus[-1,1]$ *be paired by complex conjugation, and let* $\{a_k\}$ *satisfy that* $|a_k|-1>\rho$ *for some* $\rho>0$ *and all* $k=1,2,\cdots,n$. *Then, for* $f\in\mathscr{P}_n(a_1,a_2,\cdots,a_n)$ *with all zeros in* $[-1,1]$, *we have*

$$\|f'\|_{p,W} \geqslant C_{\rho,L}\sqrt{n}\|f\|_{p,W}, \tag{2.1}$$

where $\|f\|_{p,W}$ *is the weighted* L^p *norm, that is,*

$$\|f\|_{p,W} := \left(\int_{-1}^{1}|f(x)|^pW(x)dx\right)^{1/p},\ \ 0<p<\infty.$$

Letting the poles tend to infinity, we have

COROLLARY 1. *Let* $W(x)$ *be an N-doubling weight. If* $f\in H_n$, *then for any* $0<p<\infty$, *we have*

$$\|f'\|_{p,W} \geqslant C_L\sqrt{n}\|f\|_{p,W}.$$

THEOREM 5. *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset C\setminus[-1,1]$ *be paired by complex conjugation, and let* $\{a_k\}$ *satisfy that* $|a_k|-1>\rho$ *for some* $\rho>0$ *and all* $k=1,2,\cdots,n$. *Suppose that* $0<p\leqslant q\leqslant\infty$, $1-1/p+1/q\geqslant 0$. *Then, for any* $f\in\mathscr{P}_n(a_1,a_2,\cdots,a_n)$ *with all its zeros in* $[-1,1]$, *we have*

$$\|f'\|_p \geqslant C_\rho(\sqrt{n})^{1-1/p+1/q}\|f\|_q. \tag{2.2}$$

In this paper, we always use C_x to indicate a positive constant depending only upon x, and C to indicate an absolute positive constant, which may take different values at different situations. We also point out that the constants $C_{\rho,L}$ in Theorem 4, C_L in Corollary 1, and C_ρ in Theorem 2 may also depend on p when $0<p<1$.

3. Auxiliary lemmas

Denote by $-1\leqslant x_1<x_2<\cdots<x_s\leqslant 1$ all the distinct zeros of $f\in\mathscr{P}_n(a_1,a_2,\cdots,a_n)$ and by l_k the multiplicity of x_k, $1\leqslant k\leqslant s$. Take a positive number σ satisfying $1<\sigma^{-1}<1+\rho$, say, $\sigma=\frac{1}{1+\rho/2}$, and let

$$m_\sigma(x) := \frac{(1-\sigma x^2)f'(x)}{f(x)} = \sum_{k=1}^{s}\frac{l_k(1-\sigma x^2)}{x-x_k} - \sum_{j=1}^{n}\frac{1-\sigma x^2}{x-a_j}.$$

The function $m_\sigma(x)$ will play very important roles in proving our results. By using the compression factor $1-\sigma x^2$, we can prove Turán's inequality for rational functions with prescribed poles as convenient as that for polynomials.

LEMMA 1. *Let* $a_j=\alpha_j+i\gamma_j$, $j=1,2,\cdots,n$. *Then*

$$|m'_\sigma(x)| = \sum_{k=1}^{s}\frac{l_k(1-\sigma x_k^2)}{(x-x_k)^2} + \sum_{j=1}^{n}\frac{\sigma a_j^2-1}{(x-a_j)^2}, \tag{3.1}$$

$$|m'_\sigma(x)| = \sum_{k=1}^{s} \frac{l_k(1-\sigma x_k^2)}{(x-x_k)^2} + \sum_{j=1}^{n} \frac{\sigma|a_j|^2-1}{|x-a_j|^2} + \sum_{j=1}^{n} \frac{2(1-\sigma x^2)\gamma_j^2}{|x-a_j|^4}. \tag{3.2}$$

Proof. Firstly, (3.1) can be proved in the same way as Lemma 2 in [15]. To prove (3.2), we only need to prove the following equality:

$$\sum_{j=1}^{n} \frac{\sigma a_j^2-1}{(x-a_j)^2} = \sum_{j=1}^{n} \frac{\sigma|a_j|^2-1}{|x-a_j|^2} + \sum_{j=1}^{n} \frac{2(1-\sigma x^2)\gamma_j^2}{|x-a_j|^4}.$$

In fact, by noting that any nonreal element in $\{a_j\}_{j=1}^n$ is paired by complex conjugation, we have

$$\begin{aligned}
\sum_{j=1}^{n} \frac{\sigma a_j^2-1}{(x-a_j)^2} &= \frac{1}{2}\sum_{j=1}^{n}\Big(\frac{\sigma a_j^2-1}{(x-a_j)^2} + \frac{\sigma \bar{a}_j^2-1}{(x-\bar{a}_j)^2}\Big) \\
&= \sum_{j=1}^{n} \frac{(\sigma|a_j|^2-1)(x^2-2x\alpha_j+|a_j|^2)+2(1-\sigma x^2)\gamma_j^2}{|x-a_j|^4} \\
&= \sum_{j=1}^{n} \frac{\sigma|a_j|^2-1}{|x-a_j|^2} + \sum_{j=1}^{n} \frac{2(1-\sigma x^2)\gamma_j^2}{|x-a_j|^4}. \quad \square
\end{aligned}$$

LEMMA 2. *There is a unique point $\beta_v \in (x_v, x_{v+1}), v = 1,2,\cdots,s-1$, such that $m_\sigma(\beta_v) = 0$. In other words, $f'(x)$ keeps the same sign as $m_\sigma(x)$ in (x_v,β_v) and (β_v, x_{v+1}), respectively.*

Proof. The proof can be done exactly in the same way as that of Lemma 3 in [15]. We omit the details here. $\square$

For $x \in [-1,1]$, it is easy to show that

$$\sum_{k=1}^{s} \frac{l_k(1-\sigma x_k^2)}{(x-x_k)^2} \geqslant (1-\sigma)\sum_{k=1}^{s} \frac{l_k}{(x-x_k)^2} \geqslant \frac{1-\sigma}{4}n,$$

and for $x \in [-\sigma^{-1}, \sigma^{-1}]$, by Lemma 1, we have

$$\begin{aligned}
\sum_{j=1}^{n} \frac{\sigma a_j{}^2-1}{(x-a_j)^2} \geqslant \sum_{j=1}^{n} \frac{\sigma|a_j|^2-1}{|x-a_j|^2} &\geqslant \sum_{j=1}^{n} \frac{\sigma|a_j|^2-1}{(1+\rho+|a_j|)^2} \\
&\geqslant \sum_{j=1}^{n} \frac{\sigma(1+\rho)^2-1}{4(1+\rho)^2} > \frac{\rho}{4(1+\rho)^2}n. \qquad (3.3)
\end{aligned}$$

Thus, for $x \in [-1,1]$, we have

$$|m'_\sigma(x)| \geqslant \left(\frac{1-\sigma}{4} + \frac{\rho}{4(1+\rho)^2}\right)n,$$

$$|m'_\sigma(x)| \geqslant \frac{1-\sigma}{(x-x_k)^2}, \quad 1 \leqslant k \leqslant s,$$

$$|m'_\sigma(x)| \geqslant \frac{\sigma|a_j|^2-1}{|x-a_j|^2}, \quad 1 \leqslant j \leqslant n.$$

Set

$$d_j = |m'_\sigma(\beta_j)|^{-1}, \quad j=1,2,\cdots,s-1,$$

and take

$$\delta = \min\left\{\sqrt{1-\sigma}, \sqrt{\sigma(1+\rho)^2-1}\right\}.$$

Then, we have

$$\sqrt{d_v} \leqslant \min_{1\leqslant k\leqslant s, 1\leqslant j\leqslant n}\left\{\frac{|\beta_v-x_k|}{\delta}, \frac{|\beta_v-a_j|}{\delta}, C_\rho n^{-1/2}\right\}, \quad v=1,2,\cdots,n. \tag{3.4}$$

In the sequel, we always assume that all inequalities hold for sufficiently large n if not specified.

LEMMA 3. *If* $x \in [x_v,\ \beta_v - \frac{\delta}{8}\sqrt{d_v}] \cup [\beta_v + \frac{\delta}{8}\sqrt{d_v}, x_{v+1}]$, *then*

$$|m_\sigma(x)| \geqslant \frac{16}{625}\frac{\delta}{\sqrt{d_v}}. \tag{3.5}$$

If $x \in [\beta_v - \frac{\delta}{4}\sqrt{d_v},\ \beta_v + \frac{\delta}{4}\sqrt{d_v}]$, *then*

$$|m_\sigma(x)| \leqslant \frac{128}{81}\frac{\delta}{\sqrt{d_v}}. \tag{3.6}$$

Proof. For $x \in [\beta_v - \frac{\delta}{4}\sqrt{d_v},\ \beta_v + \frac{\delta}{4}\sqrt{d_v}]$, $k=1,2,\cdots,s$, by (3.4), we have

$$\frac{3}{4}|x_k-\beta_v| \leqslant |x_k-\beta_v| - \frac{\delta}{4}\sqrt{d_v} \leqslant |x_k-x| \leqslant |\beta_v-x_k| + \frac{\delta}{4}\sqrt{d_v} \leqslant \frac{5}{4}|x_k-\beta_v|,$$

which leads to

$$\frac{16}{25}\sum_{k=1}^{s}\frac{l_k(1-\sigma x_k^2)}{(\beta_v-x_k)^2} \leqslant \sum_{k=1}^{s}\frac{l_k(1-\sigma x_k^2)}{(x-x_k)^2} \leqslant \frac{16}{9}\sum_{k=1}^{s}\frac{l_k(1-\sigma x_k^2)}{(\beta_v-x_k)^2}. \tag{3.7}$$

For $j=1,2,\cdots,n$, we have

$$\frac{3}{4}|\beta_v-a_j| \leqslant |\beta_v-a_j| - \frac{\delta}{4}\sqrt{d_v} \leqslant |x-a_j| \leqslant |\beta_v-a_j| + \frac{\delta}{4}\sqrt{d_v} \leqslant \frac{5}{4}|\beta_v-a_j|.$$

Thus,

$$\frac{16}{25}\sum_{j=1}^{n}\frac{\sigma|a_j|^2-1}{|\beta_v-a_j|^2} \leqslant \sum_{j=1}^{n}\frac{\sigma|a_j|^2-1}{|x-a_j|^2} \leqslant \frac{16}{9}\sum_{j=1}^{n}\frac{\sigma|a_j|^2-1}{|\beta_v-a_j|^2}, \tag{3.8}$$

and

$$\frac{256}{625}\sum_{j=1}^{n}\frac{2(1-\sigma x^2)\gamma_j^2}{|\beta_\nu-a_j|^4}\leqslant\sum_{j=1}^{n}\frac{2(1-\sigma x^2)\gamma_j^2}{|x-a_j|^4}\leqslant\frac{256}{81}\sum_{j=1}^{n}\frac{2(1-\sigma x^2)\gamma_j^2}{|\beta_\nu-a_j|^4}. \tag{3.9}$$

For $x\in[\beta_\nu-\frac{\delta}{4}\sqrt{d_\nu},\ \beta_\nu+\frac{\delta}{4}\sqrt{d_\nu}]$ and sufficiently large n, we have

$$\frac{1}{2}(1-\sigma\beta_\nu^2)\leqslant 1-\sigma x^2\leqslant 2(1-\sigma\beta_\nu^2). \tag{3.10}$$

By (3.9) and (3.10), we deduce that

$$\frac{128}{625}\sum_{j=1}^{n}\frac{2(1-\sigma\beta_\nu{}^2)\gamma_j^2}{|\beta_\nu-a_j|^4}\leqslant\sum_{j=1}^{n}\frac{2(1-\sigma x^2)\gamma_j^2}{|x-a_j|^4}\leqslant\frac{512}{81}\sum_{j=1}^{n}\frac{2(1-\sigma\beta_\nu{}^2)\gamma_j^2}{|\beta_\nu-a_j|^4}. \tag{3.11}$$

Combining (3.7), (3.8), (3.11) and Lemma 1, we have

$$\frac{128}{625}|m'_\sigma(\beta_\nu)|\leqslant|m'_\sigma(x)|\leqslant\frac{512}{81}|m'_\sigma(\beta_\nu)|. \tag{3.12}$$

In view of $m_\sigma(\beta_\nu)=0$, it follows that

$$\begin{aligned}\left|m_\sigma(\beta_\nu\pm\frac{\delta}{4}\sqrt{d_\nu})\right|&=\left|\int_{\beta_\nu\pm\frac{\delta}{4}\sqrt{d_\nu}}^{\beta_\nu}m'_\sigma(x)dx\right|\\&\leqslant\frac{512}{81}|m'_\sigma(\beta_\nu)|\frac{\delta}{4}\sqrt{d_\nu}\\&\leqslant\frac{128}{81}\frac{\delta}{\sqrt{d_\nu}},\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\left|m'_\sigma(\beta_\nu\pm\frac{\delta}{8}\sqrt{d_\nu})\right|&=\left|\int_{\beta_\nu\pm\frac{\delta}{8}\sqrt{d_\nu}}^{\beta_\nu}m'_\sigma(x)dx\right|\\&\geqslant\frac{128}{625}|m'_\sigma(\beta_\nu)|\frac{\delta}{8}\sqrt{d_\nu}\geqslant\frac{16}{625}\frac{\delta}{\sqrt{d_\nu}}.\end{aligned} \tag{3.14}$$

Now, by (3.13) and (3.14), noting that $m_\sigma(x)$ is monotone in $(x_\nu,x_{\nu+1})$, we have (3.5) and (3.6) immediately. □

LEMMA 4. *If* $x\in[\beta_\nu-\frac{\delta}{4}\sqrt{d_\nu},\ \beta_\nu+\frac{\delta}{4}\sqrt{d_\nu}]$, *then*

$$|f(x)|\geqslant\frac{49}{81}|f(\beta_\nu)|. \tag{3.15}$$

Proof. For $x\in[\beta_\nu-\frac{\delta}{4}\sqrt{d_\nu},\ \beta_\nu+\frac{\delta}{4}\sqrt{d_\nu}]$, by (3.6), we deduce that

$$\begin{aligned}\left|f(\beta_\nu)-f(\beta_\nu\pm\frac{\delta}{4}\sqrt{d_\nu})\right|&=|f'(\xi_\nu)|\frac{\delta}{4}\sqrt{d_\nu}=\frac{|m_\sigma(\xi_\nu)|}{1-\sigma\xi_\nu^2}|f(\xi_\nu)|\frac{\delta}{4}\sqrt{d_\nu}\\&\leqslant\frac{\sqrt{d_\nu}}{4\delta}|m_\sigma(\xi_\nu)||f(\xi_\nu)|\leqslant\frac{32}{81}|f(\beta_\nu)|,\end{aligned}$$

where $\xi_v \in [\beta_v - \frac{\delta}{4}\sqrt{d_v},\ \beta_v + \frac{\delta}{4}\sqrt{d_v}]$.

Therefore, by using the monotonicity of $f(x)$ in (x_v, x_{v+1}), we have

$$|f(x)| \geqslant \left| f\left(\beta_v \pm \frac{\delta}{4}\sqrt{d_v}\right)\right| \geqslant \frac{49}{81}|f(\beta_v)|. \quad \square$$

4. Proofs of results

4.1. Proof of Proposition 1

Assume that $W(x)$ is a doubling weight function. For any $2I = [a,\ a+2\eta] \subseteq [-1,1]$, we have

$$\begin{aligned} W([a,\ a+\eta]) &\leqslant LW([a+\eta/4,\ a+3\eta/4]) \leqslant LW([a,\ a+3\eta/4]) \\ &\leqslant L^3 W\left(\left[a,\ a+\left(\frac{3}{4}\right)^3\eta\right]\right) \leqslant L^3 W([a,\ a+\eta/2]). \end{aligned}$$

Similarly, we have

$$W([a+\eta,\ a+2\eta]) \leqslant L^3 W([a+3\eta/2,\ a+2\eta]).$$

Therefore,

$$\begin{aligned} W(2I) &= W([a,\ a+2\eta]) \\ &\leqslant L^3\{W([a,\ a+\eta/2]) + W([a+3\eta/2,\ a+2\eta])\} = L^3 W(2I \setminus I), \end{aligned}$$

which shows that $W(x)$ is an N-doubling weight.

Now, we prove the second assertion. We construct a weight function $W(x)$ as follows:

$$W(x) := \begin{cases} 1, & x \in [-1,-\delta) \cup (\delta,1], \\ 0, & x \in [-\delta,\delta], \end{cases}$$

where $0 < \delta < 1/2$. We shall verify that $W(x)$ is an N-doubling weight function but not a doubling weight function.

Let $I = [-\delta,\delta]$, then $2I = [-2\delta,2\delta] \subset [-1,1]$. Then, $W(I) = 0$, $W(2I) = 2\delta \neq 0$. Therefore, $W(x)$ is not a doubling weight function.

Now we show that $W(x)$ is an N-doubling weight function by considering the following four cases:

Case 1. If $2I \subset [-\delta,\delta]$, then $W(2I) = W(2I \setminus I) = 0$.

Case 2. If $2I = [a,b]$, and $-\delta < a < \delta < b$ (the argument for the case $a < -\delta < b < \delta$ is similar), then $W(2I) = b - a - \delta$, and

$$W(2I \setminus I) > \min\{(b-a)/4, b-\delta\}.$$

Thus, $W(2I) \leqslant 4W(2I \setminus I)$.

Case 3. If $2I = [a,b]$, and $a < -\delta < \delta < b$, then $W(2I) = b - \delta$, and

$$W(2I \setminus I) > \min\{(b-a)/4, b-\delta\} + \min\{(b-a)/4, -\delta-a\}.$$

Thus, we also have $W(2I) \leqslant 4W(2I \setminus I)$ in this case.

Case 4. If $2I = [a,b]$, and $a > \delta$, or $b < -\delta$, then $W(2I) = b - a$, $W(2I \setminus I) = (b-a)/2$. Thus, we still have $W(2I) \leqslant 4W(2I \setminus I)$ in this case.

4.2. Proof of Proposition 2

Assume that $W(x) \in GJW$. Let $2I = [a,b] \subseteq [-1,1]$, then

$$I = \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right] = \left[\frac{3a+b}{4}, \frac{a+b}{2}\right] \cup \left[\frac{a+b}{2}, \frac{a+3b}{4}\right],$$

and

$$2I \setminus I = \left[a, \frac{3a+b}{4}\right] \cup \left[\frac{a+3b}{4}, b\right].$$

We show that $W(x)$ is an N-doubling weight function by considering the following cases.

Case 1. $-1 \leqslant a < b \leqslant 0$. For $x \in [\frac{3a+b}{4}, \frac{a+b}{2}]$, we have

$$\frac{3a+b}{4} \leqslant x \leqslant \frac{a+b}{2} \leqslant \min\left\{0, 1 + \frac{3a+b}{2}\right\},$$

hence,

$$W(x) \approx W\left(\frac{3a+b}{4}\right).$$

While for $x \in [\frac{7a+b}{8}, \frac{3a+b}{4}]$, we see that

$$\frac{7a+b}{8} \leqslant x \leqslant \frac{3a+b}{4} \leqslant \min\left\{0, 1 + \frac{7a+b}{4}\right\},$$

hence,

$$W(x) \approx W\left(\frac{7a+b}{8}\right).$$

Therefore, we have

$$\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} W(x)dx \leqslant L_1 W\left(\frac{3a+b}{4}\right) \leqslant L_2 W\left(\frac{7a+b}{8}\right) \leqslant L_3 \int_{\frac{7a+b}{8}}^{\frac{3a+b}{4}} W(x)dx.$$

Similarly, we have

$$\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} W(x)dx \leqslant L_4 \int_{\frac{a+3b}{4}}^{b} W(x)dx.$$

Therefore, we have $W(2I) \leqslant LW(2I \setminus I)$ in this case.

Case 2. $0 \leqslant a < b \leqslant 1$. In a similar way to case 1, we also have $W(2I) \leqslant LW(2I \setminus I)$.

Case 3. $-1 \leqslant a < 0 < b \leqslant 1$. Without loss of generality, we assume that $\frac{a+b}{2} < 0$.

Subcase 3.1. $\frac{a+b}{2} < 0 < \frac{a+3b}{4}$. In this case, by the result of case 1, we have

$$W\left(\left[\frac{3a+b}{4}, \frac{a+b}{2}\right]\right) \leqslant L_5 W\left(\left[a, \frac{3a+b}{4}\right]\right),$$

and

$$W\left(\left[\frac{a+b}{2}, 0\right]\right) \leqslant L_6 W\left(\left[\frac{3a+b}{4}, \frac{a+b}{2}\right]\right).$$

While by the result of the case 2, we have

$$W\left(\left[0,\frac{3a+b}{4}\right]\right)\leqslant L_7W\left(\left[\frac{3a+b}{4},b\right]\right).$$

Thus, we have $W(2I)\leqslant LW(2I\setminus I)$.

Subcase 3.2. $\frac{a+b}{2}<\frac{a+3b}{4}<0$. By the result of the subcase 3.1, we have

$$\begin{aligned}W\left(\left[\frac{a+3b}{4},0\right]\right) &\leqslant L_8W\left[\left(\frac{a+b}{2},\frac{a+3b}{4}\right]\right)\\ &\leqslant L_9W\left[\left(\frac{3a+b}{4},\frac{a+b}{2}\right]\right)\leqslant W\left(\left[a,\frac{3a+b}{4}\right]\right).\end{aligned}$$

Thus, we also have $W(2I)\leqslant LW(2I\setminus I)$ in this case.

4.3. Proof of Theorem 4

(1) For $v=1,2,\cdots,s-1$, it follows from (3.4) that

$$\begin{aligned}&\left(\int_{x_v}^{\beta_v-\frac{\delta}{4}\sqrt{d_v}}+\int_{\beta_v+\frac{\delta}{4}\sqrt{d_v}}^{x_{v+1}}\right)|f'(x)|^pW(x)dx\\ \geqslant&\left(\int_{x_v}^{\beta_v-\frac{\delta}{4}\sqrt{d_v}}+\int_{\beta_v+\frac{\delta}{4}\sqrt{d_v}}^{x_{v+1}}\right)\left(1-\sigma x^2\right)^p|f'(x)|^p|m_\sigma(x)|^pW(x)dx\\ =&\left(\int_{x_v}^{\beta_v-\frac{\delta}{4}\sqrt{d_v}}+\int_{\beta_v+\frac{\delta}{4}\sqrt{d_v}}^{x_{v+1}}\right)|f(x)|^pW(x)dx\\ \geqslant&\left(\frac{16}{625}\frac{\delta}{\sqrt{d_v}}\right)^p\left(\int_{x_v}^{\beta_v-\frac{\delta}{4}\sqrt{d_v}}+\int_{\beta_v+\frac{\delta}{4}\sqrt{d_v}}^{x_{v+1}}\right)|f(x)|^pW(x)dx\\ \geqslant&(C_\rho)^p(\sqrt{n})^p\left(\int_{x_v}^{\beta_v-\frac{\delta}{4}\sqrt{d_v}}+\int_{\beta_v+\frac{\delta}{4}\sqrt{d_v}}^{x_{v+1}}\right)|f(x)|^pW(x)dx.\end{aligned}\tag{4.1}$$

By the definition of $W(x)$ and the monotonicity of $f(x)$, together with (3.4), (3.5) and (3.15), we deduce that

$$\begin{aligned}&\left(\int_{\beta_v-\frac{\delta}{4}\sqrt{d_v}}^{\beta_v-\frac{\delta}{8}\sqrt{d_v}}+\int_{\beta_v+\frac{\delta}{8}\sqrt{d_v}}^{\beta_v+\frac{\delta}{4}\sqrt{d_v}}\right)\left(1-\sigma x^2\right)^p|f'(x)|^pW(x)dx\\ =&\left(\int_{\beta_v-\frac{\delta}{4}\sqrt{d_v}}^{\beta_v-\frac{\delta}{8}\sqrt{d_v}}+\int_{\beta_v+\frac{\delta}{8}\sqrt{d_v}}^{\beta_v+\frac{\delta}{4}\sqrt{d_v}}\right)|f(x)|^p|m_\sigma(x)|^pW(x)dx\\ \geqslant&\left(\frac{49}{81}\cdot\frac{16}{625}\frac{\delta}{\sqrt{d_v}}\right)^p|f(\beta_v)|^p\left(\int_{\beta_v-\frac{\delta}{4}\sqrt{d_v}}^{\beta_v-\frac{\delta}{8}\sqrt{d_v}}+\int_{\beta_v+\frac{\delta}{8}\sqrt{d_v}}^{\beta_v+\frac{\delta}{4}\sqrt{d_v}}\right)W(x)dx\\ \geqslant&L^{-1}\left(\frac{49}{81}\cdot\frac{16}{625}\frac{\delta}{\sqrt{d_v}}\right)^p|f(\beta_v)|^p\int_{\beta_v-\frac{\delta}{4}\sqrt{d_v}}^{\beta_v+\frac{\delta}{4}\sqrt{d_v}}W(x)dx\end{aligned}$$

$$\geqslant L^{-1}\left(\frac{49}{81}\cdot\frac{16}{625}\frac{\delta}{\sqrt{d_v}}\right)^p\int_{\beta_v-\frac{\delta}{4}\sqrt{d_v}}^{\beta_v+\frac{\delta}{4}\sqrt{d_v}}|f(x)|^pW(x)dx$$
$$\geqslant L^{-1}(C_\rho)^p(\sqrt{n})^p\int_{\beta_v-\frac{\delta}{4}\sqrt{d_v}}^{\beta_v+\frac{\delta}{4}\sqrt{d_v}}|f(x)|^pW(x)dx. \tag{4.2}$$

Combining (4.1) with (4.2) gives

$$\int_{x_v}^{x_{v+1}}|f'(x)|^pW(x)dx \geqslant L^{-1}(C_\rho)^p(\sqrt{n})^p\int_{x_v}^{x_{v+1}}|f(x)|^pW(x)dx. \tag{4.3}$$

(2) If $x_s<1$, by noting $m(x)$ is decreasing in $(x_s,\sigma^{-1/2})$ and $m(\sigma^{-1/2})=0$, and taking $\tau=\frac{1+\sigma^{-1/2}}{2}$, then by (3.3), for any $x\in[x_s,1]$, there is a $\xi\in[x_s,\sigma^{-1/2}]$ such that

$$|m_\sigma(x)| = |m_\sigma(\tau)|+|m'_\sigma(\xi)||x-\tau| \geqslant |\tau-1||m'_\sigma(\xi)|$$
$$\geqslant \frac{\sigma^{-1/2}-1}{2}\frac{\rho}{4(1+\rho)^2}n := C_\rho n. \tag{4.4}$$

Therefore,

$$\int_{x_s}^1|f'(x)|^pW(x)dx \geqslant \int_{x_s}^1(1-\sigma x^2)^p|f'(x)|^pW(x)dx$$
$$= \int_{x_s}^1|f(x)|^p|m_\sigma(x)|^pW(x)dx$$
$$\geqslant (C_\rho n)^p\int_{x_s}^1|f(x)|^pW(x)dx. \tag{4.5}$$

Similarly, if $x_1>-1$, we have

$$\int_{-1}^{x_1}|f'(x)|^pW(x)dx \geqslant (C_\rho n)^p\int_{-1}^{x_1}|f(x)|^pW(x)dx. \tag{4.6}$$

With (4.3), (4.5) and (4.6), by summing over all v, we get (2.1).

4.4. Proof of Theorem 5

(1) Let

$$N_v=\left[16\frac{\beta_v-x_v}{\delta\sqrt{d_v}}\right],\quad v=1,2,\cdots,s-1,$$

where $[x]$ denotes the greatest integer not larger than x, and $\xi_{v,i}=x_v+\frac{i}{16}\delta\sqrt{d_v}$, $i=0,1,\cdots,N_v$. Since

$$x_v+(N_v-2)\frac{1}{16}\delta\sqrt{d_v}\leqslant\beta_v-\frac{1}{8}\delta\sqrt{d_v}$$

for $0 \leqslant i \leqslant N_v - 4$, by applying Lemma 3, we get

$$
\begin{aligned}
&\int_{\xi_{v,i+1}}^{\xi_{v,i+2}} |f'(x)|^p dx \\
\geqslant &\int_{\xi_{v,i+1}}^{\xi_{v,i+2}} (1-\sigma x^2)^p |f'(x)|^p dx \\
= &\int_{\xi_{v,i+1}}^{\xi_{v,i+2}} |f(x)|^p |m_\sigma(x)|^p dx \\
\geqslant &\left(\frac{16\delta}{625\sqrt{d_v}}\right)^p \frac{\delta}{16}\sqrt{d_v}|f(\xi_{v,i+1})|^p \\
\geqslant &\left(\frac{16\delta}{625\sqrt{d_v}}\right)^p \left(\frac{\delta}{16}\sqrt{d_v}\right)^{1-p/q} \left(\int_{\xi_{v,i}}^{\xi_{v,i+1}} |f(x)|^q dx\right)^{p/q}. \qquad (4.7)
\end{aligned}
$$

Applying Lemma 4, we have

$$
\begin{aligned}
&\int_{\beta_v-\frac{\delta}{4}\sqrt{d_v}}^{\beta_v-\frac{\delta}{8}\sqrt{d_v}} |f'(x)|^p dx \\
\geqslant &\int_{\beta_v-\frac{\delta}{4}\sqrt{d_v}}^{\beta_v-\frac{\delta}{8}\sqrt{d_v}} (1-\sigma x^2)^p |f'(x)|^p dx \\
= &\int_{\beta_v-\frac{\delta}{4}\sqrt{d_v}}^{\beta_v-\frac{\delta}{8}\sqrt{d_v}} |f(x)|^p |m_\sigma(x)|^p dx \\
\geqslant &\left(\frac{16\delta}{625\sqrt{d_v}}\right)^p \left|\frac{49}{81}f(\beta_v)\right|^p \frac{\delta}{8}\sqrt{d_v} \\
\geqslant &\frac{1}{2}\left(\frac{49}{81}\cdot\frac{16\delta}{625\sqrt{d_v}}\right)^p \left(\frac{\delta}{4}\sqrt{d_v}\right)^{1-p/q} \left(\int_{\beta_v-\frac{\delta}{4}\sqrt{d_v}}^{\beta_v} |f(x)|^q dx\right)^{p/q}. \qquad (4.8)
\end{aligned}
$$

Noting that

$$
\begin{aligned}
x_v + (N_v - 3)\frac{1}{16}\delta\sqrt{d_v} &\geqslant \beta_v - \frac{\delta}{4}\sqrt{d_v}, \\
x_v + (N_v - 2)\frac{1}{16}\delta\sqrt{d_v} &\leqslant \beta_v - \frac{\delta}{8}\sqrt{d_v},
\end{aligned}
$$

by (4.7) and (4.8), we deduce that

$$
\begin{aligned}
2\int_{x_v}^{\beta_v} |f'(x)|^p dx \geqslant &\left(\sum_{i=0}^{N_v-4}\int_{\xi_{v,i+1}}^{\xi_{v,i+2}} + \int_{\beta_v-\frac{\delta}{4}\sqrt{d_v}}^{\beta_v-\frac{\delta}{8}\sqrt{d_v}}\right) |f'(x)|^p dx \\
\geqslant &\left(\frac{16\delta}{625\sqrt{d_v}}\right)^p \left(\frac{\delta}{16}\sqrt{d_v}\right)^{1-p/q} \sum_{i=0}^{N_v-4} \left(\int_{\xi_{v,i}}^{\xi_{v,i+1}} |f(x)|^q dx\right)^{p/q} \\
&+\frac{1}{2}\left(\frac{49}{81}\cdot\frac{16\delta}{625\sqrt{d_v}}\right)^p \left(\frac{\delta}{4}\sqrt{d_v}\right)^{1-p/q} \left(\int_{\beta_v-\frac{\delta}{4}\sqrt{d_v}}^{\beta_v} |f(x)|^q dx\right)^{p/q}
\end{aligned}
$$

$$\geqslant \frac{1}{2}\left(\frac{49}{81}\cdot\frac{16\delta}{625\sqrt{d_v}}\right)^p\left(\frac{\delta}{16}\sqrt{d_v}\right)^{1-p/q}\left(\int_{x_v}^{\beta_v}|f(x)|^q dx\right)^{p/q}.$$

By (3.4), we have

$$\int_{x_v}^{\beta_v}|f'(x)|^p dx \geqslant \frac{1}{4}(C_\rho)^p(\sqrt{n})^{p-1-p/q}\left(\int_{x_v}^{\beta_v}|f(x)|^q dx\right)^{p/q}. \tag{4.9}$$

In a similar way to (4.9) we have

$$\int_{\beta_v}^{x_{v+1}}|f'(x)|^p dx \geqslant \frac{1}{4}(C_\rho)^p(\sqrt{n})^{p-1-p/q}\left(\int_{\beta_v}^{x_{v+1}}|f(x)|^q dx\right)^{p/q}. \tag{4.10}$$

(2) If $x_s < 1$, taking $\beta_s = 1,\ d_s = |m'_\sigma(\beta_s)|^{-1}$, and copying the proof of (3.12), for $x \in [1-\frac{\delta}{4}\sqrt{d_s}, 1]$, we have

$$\frac{128}{625}|m'_\sigma(1)| \leqslant |m'_\sigma(x)| \leqslant \frac{512}{81}|m'(1)|,$$

thus in view of the monotonicity of $m_\sigma(x)$, we get

$$\left|m_\sigma\left(1-\frac{\delta}{4}\sqrt{d_s}\right)\right| = |m_\sigma(1)| + \left|\int_{1-\frac{\delta}{4}\sqrt{d_s}}^{1} m'_\sigma(x)dx\right| \leqslant |m_\sigma(1)| + \frac{128}{81}\frac{\delta}{\sqrt{d_s}}. \tag{4.11}$$

Let

$$d_s^* = \min\left\{d_s,\ |m(1)|^{-1}\right\}.$$

For any $x \in [1-\frac{\delta^2}{4}\sqrt{d_s^*},\ 1]$, there is a $\xi_s \in [1-\frac{\delta^2}{4}\sqrt{d_s^*},\ 1]$ such that

$$\begin{aligned}\left|f(1)-f\left(1-\frac{\delta^2}{4}\sqrt{d_s^*}\right)\right| &= |f'(\xi_s)|\frac{\delta^2}{4}\sqrt{d_s^*}\\ &= \frac{|m_\sigma(\xi_s)|}{1-\sigma\xi_s^2}|f(\xi_s)|\frac{\delta^2}{4}\sqrt{d_s^*}\\ &\leqslant \frac{|f(1)|}{4}\left(|m_\sigma(1)|+\frac{128}{81}\frac{\delta}{\sqrt{d_s}}\right)\sqrt{d_s^*}\\ &\leqslant \frac{209}{324}|f(1)|.\end{aligned}$$

Thus,

$$|f(x)| \geqslant \left|f\left(1-\frac{\delta^2}{4}\sqrt{d_s^*}\right)\right| \geqslant \frac{115}{324}|f(1)|.$$

From (4.4), by using the same technique as that for (4.8), we obtain that

$$\int_{1-\frac{\delta^2}{4}\sqrt{d_s^*}}^{1-\frac{\delta^2}{8}\sqrt{d_s^*}}|f'(x)|^p dx \geqslant \frac{1}{2}(C_\rho)^p(\sqrt{n})^{p-1-p/q}\left(\int_{1-\frac{\delta^2}{4}\sqrt{d_s^*}}^{1}|f(x)|^q dx\right)^{p/q}.$$

Meanwhile, let $N_s = [\frac{16(1-x_s)}{\delta^2\sqrt{d_s^*}}]$, and $\xi_{s,i} = x_s + \frac{i}{16}\delta^2\sqrt{d_s^*}$, then for $i = 0, 1, \cdots, N_s - 4$, by the same technique as that for (4.7), we get

$$\int_{\xi_{s,i+1}}^{\xi_{s,i+2}} |f'(x)|^p dx \geqslant (C_\rho)^p(\sqrt{n})^{p-1-p/q}\left(\int_{\xi_{s,i}}^{\xi_{s,i+1}} |f'^q dx\right)^{p/q}.$$

Notice that

$$x_s + (N_s - 3)\frac{1}{16}\delta^2\sqrt{d_s^*} \geqslant 1 - \frac{\delta^2}{4}\sqrt{d_s^*},$$

$$x_s + (N_s - 2)\frac{1}{16}\delta^2\sqrt{d_s^*} \leqslant 1 - \frac{\delta^2}{8}\sqrt{d_s^*}.$$

Therefore, by the same technique as that for (4.5), it yields that

$$\int_{x_s}^{1} |f'(x)|^p dx \geqslant \frac{1}{4}(C_\rho)^p(\sqrt{n})^{p-1-p/q}\left(\int_{x_s}^{1} |f(x)|^q dx\right)^{p/q}. \tag{4.12}$$

Similarly, if $x_1 > -1$, we have

$$\int_{-1}^{x_1} |f'(x)|^p W(x)dx \geqslant \frac{1}{4}(C_\rho)^p(\sqrt{n})^{p-1-p/q}\int_{-1}^{x_1} |f(x)|^p W(x)dx. \tag{4.13}$$

By (4.9), (4.10), (4.12), (4.13), and summing over all v, we get (2.2).

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(Received March 4, 2013)

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