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A MULTIDIMENSIONAL DISCRETE HILBERT–TYPE INEQUALITY

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Abstract. In this paper, by using the way of weight coefficients and technique of real analysis, a multidimensional discrete Hilbert-type inequality with parameters and a best possible constant factor is given. The equivalent form, the operator expressions with the norm are also considered.

1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x)$, $g(y) \geqslant 0$, $f \in L^p(\mathbf{R}_+)$, $g \in L^q(\mathbf{R}_+)$, $||f||_p = \{\int_0^\infty f^p(x)dx\}^{\frac{1}{p}} > 0$, $||g||_q > 0$, then we have the following Hardy-Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Assuming that a_m, $b_n \geqslant 0$, $a = \{a_m\}_{m=1}^\infty \in l^p$, $b = \{b_n\}_{n=1}^\infty \in l^q$, $||a||_p = \{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}} > 0$, $||b||_q > 0$, we have the following discrete Hardy-Hilbert's inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} ||a||_p ||b||_q. \tag{2}$$

Inequalities (1) and (2) are important in Analysis and its applications (cf. [1], [2], [3], [4], [5], [6]).

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [7] gave an extension of (1) at $p = q = 2$. In recent years, Yang [3] and [4], gave some extensions of (1) and (2) as follows:

If λ_1, λ_2, $\lambda \in \mathbf{R}$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1) = \int_0^\infty k_\lambda(t,1) t^{\lambda_1 - 1} dt \in \mathbf{R}_+,$$

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$\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(x) = x^{q(1-\lambda_2)-1}$, $f(x)$, $g(y) \geqslant 0$,

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; ||f||_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p dx\}^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\phi}$, $||g||_{q,\psi} > 0$, then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x,y) f(x) g(y) dx dy < k(\lambda_1) ||f||_{p,\phi} ||g||_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing with respect to $x > 0$ $(y > 0)$, then for $a_m, b_n \geqslant 0$,

$$a \in l_{p,\phi} = \left\{ a; ||a||_{p,\phi} := \{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{\frac{1}{p}} < \infty \right\},$$

$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $||a||_{p,\phi}$, $||b||_{q,\psi} > 0$, we have the following inequality:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n) a_m b_n < k(\lambda_1) ||a||_{p,\phi} ||b||_{q,\psi}, \tag{4}$$

where the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for $\lambda = 1$, $k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, (3) reduces to (1), while (4) reduces to (2). Some other results including the multidimensional Hilbert-type integral inequalities are provided by [8]–[21].

About the topic of half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [1]. But they did not prove that the the constant factors are the best possible. However, Yang [22] gave a result with the kernel $\frac{1}{(1+nx)^\lambda}$ by introducing a variable and proved that the constant factor is the best possible. In 2011 Yang [23] gave the following half-discrete Hilbert's inequality with the best possible constant factor $B(\lambda_1, \lambda_2)$:

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} dx < B(\lambda_1, \lambda_2) ||f||_{p,\phi} ||a||_{q,\psi}, \tag{5}$$

where λ_1, $\lambda_2 > 0$, $0 \leqslant \lambda_2 \leqslant 1$, $\lambda_1 + \lambda_2 = \lambda$,

$$B(u,v) = \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{u-1} dt \, (u,v > 0)$$

is the beta function. Zhong et *al* ([24]–[17]) investigated several half-discrete Hilbert-type inequalities with particular kernels.

Applying the way of weight functions and the techniques of discrete and integral Hilbert-type inequalities, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree $-\lambda \in \mathbf{R}$ and a best constant factor $k(\lambda_1)$ is obtained as follows:

$$\int_0^\infty f(x) \sum_{n=1}^\infty k_\lambda(x,n) a_n dx < k(\lambda_1) ||f||_{p,\phi} ||a||_{q,\psi}, \tag{6}$$

which is an extension of (5) (see Yang and Chen [30]). At the same time, a half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best constant factor is given by Yang [31].

In this paper, by using the way of weight coefficients and technique of real analysis, a multidimensional discrete Hilbert's inequality with parameters and a best possible constant factor is given, which is an extension of (4) for $k_\lambda(m,n) = \frac{(\min\{m,n\})^\gamma}{(\max\{m,n\})^{\lambda+\gamma}}$. The equivalent form, the operator expressions with the norm are also considered.

2. Some lemmas

If $i_0,\ j_0 \in \mathbf{N}$($\mathbf{N}$ is the set of positive integers), $\alpha,\ \beta > 0,$ we put

$$||x||_\alpha := \left(\sum_{k=1}^{i_0} |x_k|^\alpha\right)^{\frac{1}{\alpha}} \quad (x = (x_1, \cdots, x_{i_0}) \in \mathbf{R}^{i_0}), \tag{7}$$

$$||y||_\beta := \left(\sum_{k=1}^{j_0} |y_k|^\beta\right)^{\frac{1}{\beta}} \quad (y = (y_1, \cdots, y_{j_0}) \in \mathbf{R}^{j_0}). \tag{8}$$

LEMMA 1. *If $s \in \mathbf{N},\ \gamma,\ M > 0,\ \Psi(u)$ is a non-negative measurable function in $(0,1]$, and*

$$D_M := \left\{x \in \mathbf{R}_+^s; \sum_{i=1}^s x_i^\gamma \leqslant M^\gamma\right\},$$

then we have (cf. [32])

$$\int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right) dx_1 \cdots dx_s = \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du. \tag{9}$$

LEMMA 2. *For $s \in \mathbf{N},\ \gamma > 0,\ \varepsilon > 0,$ we have*

$$\sum_m ||m||_\gamma^{-s-\varepsilon} = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})} + O(1) (\varepsilon \to 0^+). \tag{10}$$

Proof. For $M > s^{1/\gamma},$ we set

$$\Psi(u) = \begin{cases} 0, & 0 < u < \frac{s}{M^\gamma}, \\ (Mu^{1/\gamma})^{-s-\varepsilon}, & \frac{s}{M^\gamma} \leqslant u \leqslant 1. \end{cases}$$

Then by (9), it follows

$$\sum_m ||m||_\gamma^{-s-\varepsilon} \geqslant \int_{\{x\in \mathbf{R}_+^s; x_i\geqslant 1\}} ||x||_\gamma^{-s-\varepsilon} dx$$
$$= \lim_{M\to\infty} \int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma}\right) dx_1\cdots dx_s$$
$$= \lim_{M\to\infty} \frac{M^s\Gamma^s(\frac{1}{\gamma})}{\gamma^s\Gamma(\frac{s}{\gamma})}\int_{s/M^\gamma}^{1}(Mu^{1/\gamma})^{-s-\varepsilon}u^{\frac{s}{\gamma}-1}du = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma}\gamma^{s-1}\Gamma(\frac{s}{\gamma})}.$$

By the above way, we still find

$$0<\sum_{\{m\in\mathbf{N}^s; m_i\geqslant 2\}} ||m||_\gamma^{-s-\varepsilon} \leqslant \int_{\{x\in \mathbf{R}_+^s; x_i\geqslant 1\}} ||x||_\gamma^{-s-\varepsilon} dx = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma}\gamma^{s-1}\Gamma(\frac{s}{\gamma})}.$$

For $s=1$, $0<\sum_{m=1}^{1}||m||_\gamma^{-1-\varepsilon}<\infty$; for $s\geqslant 2$,

$$0<\sum_{\{m\in\mathbf{N}^s;\exists i_0, m_{i_0}=1\}} ||m||_\gamma^{-s-\varepsilon} \leqslant a + \sum_{\{m\in\mathbf{N}^{s-1}; m_i\geqslant 2\}} ||m||_\gamma^{-(s-1)-(1+\varepsilon)}$$
$$\leqslant a+\frac{\Gamma^{s-1}(\frac{1}{\gamma})}{(1+\varepsilon)(s-1)^{(1+\varepsilon)/\gamma}\gamma^{s-2}\Gamma(\frac{s-1}{\gamma})}<\infty (a\in\mathbf{R}_+),$$

$$\sum_m ||m||_\gamma^{-s-\varepsilon} = \sum_{\{m\in\mathbf{N}^s;\exists i_0, m_{i_0}=1\}} ||m||_\gamma^{-s-\varepsilon} + \sum_{\{m\in\mathbf{N}^s; m_i\geqslant 2\}} ||m||_\gamma^{-s-\varepsilon}$$
$$\leqslant O_1(1)+\frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma}\gamma^{s-1}\Gamma(\frac{s}{\gamma})}(\varepsilon\to 0^+). \tag{11}$$

Then we have (10). □

DEFINITION 1. For $-\gamma<\lambda_1\leqslant i_0-\gamma$, $-\gamma<\lambda_2\leqslant j_0-\gamma$, $\lambda_1+\lambda_2=\lambda$, $m=(m_1,\cdots,m_{i_0})\in\mathbf{N}^{i_0}$, $n=(n_1,\cdots,n_{j_0})\in\mathbf{N}^{j_0}$, define two weight coefficients $w_\lambda(\lambda_2,n)$ and $W_\lambda(\lambda_1,m)$ as follows:

$$w_\lambda(\lambda_2,n) := \sum_m \frac{(\min\{||m||_\alpha,||n||_\beta\})^\gamma}{(\max\{||m||_\alpha,||n||_\beta\})^{\lambda+\gamma}}\frac{||n||_\beta^{\lambda_2}}{||m||_\alpha^{i_0-\lambda_1}}, \tag{12}$$
$$W_\lambda(\lambda_1,m) := \sum_n \frac{(\min\{||m||_\alpha,||n||_\beta\})^\gamma}{(\max\{||m||_\alpha,||n||_\beta\})^{\lambda+\gamma}}\frac{||m||_\alpha^{\lambda_1}}{||n||_\beta^{j_0-\lambda_2}}, \tag{13}$$

where, $\sum_m=\sum_{m_{i_0}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}$ and $\sum_n=\sum_{n_{j_0}=1}^{\infty}\cdots\sum_{n_1=1}^{\infty}$.

LEMMA 3. *As the assumptions of Definition 1, then (i) we have*

$$w_\lambda(\lambda_2,n) < K_2(n\in \mathbf{N}^{j_0}), \tag{14}$$

$$W_\lambda(\lambda_1,m) < K_1(m\in \mathbf{N}^{i_0}), \tag{15}$$

where,

$$K_1 = \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\frac{\lambda+2\gamma}{(\lambda_1+\gamma)(\lambda_2+\gamma)},$$

$$K_2 = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\lambda+2\gamma}{(\lambda_1+\gamma)(\lambda_2+\gamma)}; \tag{16}$$

(ii) for $p>1,\ 0<\varepsilon<\frac{p}{2}(\lambda_1+\gamma)$, *setting* $\widetilde{\lambda}_1=\lambda_1-\frac{\varepsilon}{p}, \widetilde{\lambda}_2=\lambda_2+\frac{\varepsilon}{p}$, *we have*

$$0<\widetilde{K}_2(1-\widetilde{\theta}_\lambda(n))<w_\lambda(\widetilde{\lambda}_2,n), \tag{17}$$

where,

$$\widetilde{\theta}_\lambda(n) = \frac{(\widetilde{\lambda}_1+\gamma)(\widetilde{\lambda}_2+\gamma)}{\lambda+2\gamma}\int_0^{i_0^{1/\alpha}/||n||_\beta}\frac{(\min\{v,1\})^\gamma v^{\widetilde{\lambda}_1-1}}{(\max\{v,1\})^{\lambda+\gamma}}dv$$

$$= O\left(\frac{1}{||n||_\beta^{\gamma+\widetilde{\lambda}_1}}\right), \tag{18}$$

$$\widetilde{K}_2 = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\lambda+2\gamma}{(\widetilde{\lambda}_1+\gamma)(\widetilde{\lambda}_2+\gamma)}. \tag{19}$$

Proof. In view of the assumptions,

$$f(x,y) := \frac{(\min\{x,y\})^\gamma x^{\lambda_1-i_0}}{(\max\{x,y\})^{\lambda+\gamma}} = \begin{cases} \frac{x^{\gamma+\lambda_1-i_0}}{y^{\lambda+\gamma}}, 0<x<y, \\ \frac{y^\gamma}{x^{\lambda_2+i_0+\gamma}}, x\geqslant y, \end{cases}$$

is decreasing with respect to $x\in\mathbf{R}_+$, and strict decreasing with respect to $x\geqslant y$.

By the decreasing property and (9), it follows

$$\begin{aligned} w_\lambda(\lambda_2,n) &< \int_{\mathbf{R}_+^{i_0}}\frac{(\min\{||x||_\alpha,||n||_\beta\})^\gamma}{(\max\{||x||_\alpha,||n||_\beta\})^{\lambda+\gamma}}\frac{||n||_\beta^{\lambda_2}}{||x||_\alpha^{i_0-\lambda_1}}dx \\ &= \lim_{M\to\infty}\int_{\mathbf{D}_M}\frac{(\min\{M[\sum_{i=1}^{i_0}(\frac{x_i}{M})^\alpha]^{\frac{1}{\alpha}},||n||_\beta\})^\gamma}{(\max\{M[\sum_{i=1}^{i_0}(\frac{x_i}{M})^\alpha]^{\frac{1}{\alpha}},||n||_\beta\})^{\lambda+\gamma}}\frac{||n||_\beta^{\lambda_2}dx}{M^{i_0-\lambda_1}[\sum_{i=1}^{i_0}(\frac{x_i}{M})^\alpha]^{\frac{i_0-\lambda_1}{\alpha}}} \\ &= \lim_{M\to\infty}\frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})}\int_0^1\frac{(\min\{Mu^{1/\alpha},||n||_\beta\})^\gamma}{(\max\{Mu^{1/\alpha},||n||_\beta\})^{\lambda+\gamma}}\frac{||n||_\beta^{\lambda_2}u^{\frac{i_0}{\alpha}-1}du}{M^{i_0-\lambda_1}u^{(i_0-\lambda_1)/\alpha}} \end{aligned}$$

$$= \lim_{M\to\infty} \frac{M^{\lambda_1}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 \frac{(\min\{Mu^{1/\alpha}, ||n||_\beta\})^\gamma ||n||_\beta^{\lambda_2}}{(\max\{Mu^{1/\alpha}, ||n||_\beta\})^{\lambda+\gamma}} u^{\frac{\lambda_1}{\alpha}-1} du$$

$$\overset{u=||n||_\beta^\alpha M^{-\alpha} v^\alpha}{=} \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \int_0^\infty \frac{(\min\{v,1\})^\gamma v^{\lambda_1-1}}{(\max\{v,1\})^{\lambda+\gamma}} dv$$

$$= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \frac{\lambda+2\gamma}{(\lambda_1+\gamma)(\lambda_2+\gamma)} = K_2.$$

Hence, we have (14). By the same way, we have (15).

By the decreasing property and the same way of obtaining (10), we have

$$w_\lambda(\widetilde{\lambda}_2, n) > \int_{\{x\in\mathbf{R}_+^{i_0}; x_i \geqslant 1\}} \frac{(\min\{||x||_\alpha, ||n||_\beta\})^\gamma}{(\max\{||x||_\alpha, ||n||_\beta\})^{\lambda+\gamma}} \frac{||n||_\beta^{\widetilde{\lambda}_2} dx}{||x||_\alpha^{i_0-\widetilde{\lambda}_1}}$$

$$= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \int_{i_0^{1/\alpha}/||n||_\beta}^\infty \frac{(\min\{v,1\})^\gamma v^{\widetilde{\lambda}_1-1}}{(\max\{v,1\})^{\lambda+\gamma}} dv$$

$$= \widetilde{K}_2(1-\widetilde{\theta}_\lambda(n)) > 0,$$

$$0 < \widetilde{\theta}_\lambda(n) = \frac{(\widetilde{\lambda}_1+\gamma)(\widetilde{\lambda}_2+\gamma)}{\lambda+2\gamma} \int_0^{i_0^{1/\alpha}/||n||_\beta} \frac{(\min\{v,1\})^\gamma v^{\widetilde{\lambda}_1-1}}{(\max\{v,1\})^{\lambda+\gamma}} dv$$

$$\leqslant \frac{(\widetilde{\lambda}_1+\gamma)(\widetilde{\lambda}_2+\gamma)}{\lambda+2\gamma} M \int_0^{i_0^{1/\alpha}/||n||_\beta} v^{\gamma+\widetilde{\lambda}_1-1} dv$$

$$= \frac{\widetilde{\lambda}_2+\gamma}{\lambda+2\gamma} M \frac{i_0^{(\gamma+\widetilde{\lambda}_1)/\alpha}}{||n||_\beta^{\gamma+\widetilde{\lambda}_1}} \quad (M>0).$$

The lemma is proved. □

3. Main results and operator expressions

Setting $\Phi(m) := ||m||_\alpha^{p(i_0-\lambda_1)-i_0} (m \in \mathbf{N}^{i_0})$ and $\Psi(n) := ||n||_\beta^{q(j_0-\lambda_2)-j_0}$ $(n \in \mathbf{N}^{j_0})$, we have

THEOREM 1. *If* $-\gamma < \lambda_1 \leqslant i_0 - \gamma$, $-\gamma < \lambda_2 \leqslant j_0 - \gamma$, $\lambda_1 + \lambda_2 = \lambda$, *then for* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, a_m, $b_n \geqslant 0$, $0 < ||a||_{p,\Phi}$, $||b||_{q,\Psi} < \infty$, *we have the following inequality*

$$I := \sum_n \sum_m \frac{(\min\{||m||_\alpha, ||n||_\beta\})^\gamma}{(\max\{||m||_\alpha, ||n||_\beta\})^{\lambda+\gamma}} a_m b_n$$
$$< K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} ||a||_{p,\Phi} ||b||_{q,\Psi}, \tag{20}$$

where the constant factor

$$K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} \frac{\lambda+2\gamma}{(\lambda_1+\gamma)(\lambda_2+\gamma)} \tag{21}$$

is the best possible.

Proof. By Hölder's inequality (cf. [33]), we have

$$\begin{aligned} I &= \sum_n \sum_m \frac{(\min\{||m||_\alpha, ||n||_\beta\})^\gamma}{(\max\{||m||_\alpha, ||n||_\beta\})^{\lambda+\gamma}} \left[\frac{||m||_\alpha^{(i_0-\lambda_1)/q}}{||n||_\beta^{(j_0-\lambda_2)/p}} a_m\right] \left[\frac{||n||_\beta^{(j_0-\lambda_2)/p}}{||m||_\alpha^{(i_0-\lambda_1)/q}} b_n\right] \\ &\leqslant \left\{\sum_m W_\lambda(\lambda_1, m) ||m||_\alpha^{p(i_0-\lambda_1)-i_0} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_n w_\lambda(\lambda_2, n) ||n||_\beta^{q(j_0-\lambda_2)-j_0} b_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

Then by (14) and (15), we have (20).

For $0 < \varepsilon < \frac{p}{2}(\lambda_1+\gamma)$, $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$, $\widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$, we set

$$\widetilde{a}_m = ||m||_\alpha^{-i_0+\lambda_1-\frac{\varepsilon}{p}}, \widetilde{b}_n = ||n||_\beta^{-j_0+\lambda_2-\frac{\varepsilon}{q}} (m \in \mathbf{N}^{i_0}, n \in \mathbf{N}^{j_0}).$$

Then by (10) and (17), we obtain

$$\begin{aligned} ||\widetilde{a}||_{p,\Phi} ||\widetilde{b}||_{q,\Psi} &= \left\{\sum_m ||m||_\alpha^{p(i_0-\lambda_1)-i_0} \widetilde{a}_m^p\right\}^{\frac{1}{p}} \left\{\sum_n ||n||_\beta^{q(j_0-\lambda_2)-j_0} \widetilde{b}_n^q\right\}^{\frac{1}{q}} \\ &= \left\{\sum_m ||m||_\alpha^{-i_0-\varepsilon}\right\}^{\frac{1}{p}} \left\{\sum_n ||n||_\beta^{-j_0-\varepsilon}\right\}^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{i_0^{\varepsilon/\alpha} \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} + \varepsilon O(1)\right]^{\frac{1}{p}} \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon \widetilde{O}(1)\right]^{\frac{1}{q}}, \end{aligned} \tag{22}$$

$$\begin{aligned} \widetilde{I} &:= \sum_n \left[\sum_m \frac{(\min\{||m||_\alpha, ||n||_\beta\})^\gamma}{(\max\{||m||_\alpha, ||n||_\beta\})^{\lambda+\gamma}} \widetilde{a}_m\right] \widetilde{b}_n \\ &= \sum_n w_\lambda(\widetilde{\lambda}_2, n) ||n||_\beta^{-j_0-\varepsilon} \\ &> \widetilde{K}_2 \sum_n \left(1 - O\left(\frac{1}{||n||_\beta^{\gamma+\widetilde{\lambda}_1}}\right)\right) ||n||_\beta^{-j_0-\varepsilon} \\ &= \widetilde{K}_2 \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \widetilde{O}(1) - O(1)\right]. \end{aligned} \tag{23}$$

If there exists a constant $K \leqslant K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$, such that (20) is valid as we replace $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ by K, then using (22) and (23) we have

$$\begin{aligned}
&(K_2+o(1))\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}+\varepsilon\widetilde{O}(1)-\varepsilon O(1)\right]\\
&<\varepsilon\widetilde{I}(\tau,\sigma)<\varepsilon K||\widetilde{a}||_{p,\varphi}||\widetilde{b}||_{q,\psi}\\
&=K\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}+\varepsilon O(1)\right]^{\frac{1}{p}}\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}+\varepsilon\widetilde{O}(1)\right]^{\frac{1}{q}}.
\end{aligned}$$

For $\varepsilon \to 0^+$, we find

$$\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\lambda+2\gamma}{(\lambda_1+\gamma)(\lambda_2+\gamma)}\leqslant K\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{q}},$$

and then $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \leqslant K$. Hence, $K = K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ is the best possible constant factor of (20). □

THEOREM 2. *As the assumptions of Theorem 1, for $0 < ||a||_{p,\Phi} < \infty$, we have the following inequality with the best constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$:*

$$\begin{aligned}
J &:= \left\{\sum_n ||n||_\beta^{p\lambda_2-j_0}\left(\sum_m \frac{(\min\{||m||_\alpha,||n||_\beta\})^\gamma a_m}{(\max\{||m||_\alpha,||n||_\beta\})^{\lambda+\gamma}}\right)^p\right\}^{\frac{1}{p}}\\
&< K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}||a||_{p,\Phi},
\end{aligned} \tag{24}$$

which is equivalent to (20).

Proof. We set b_n as follows:

$$b_n := ||n||_\beta^{p\lambda_2-j_0}\left(\sum_m \frac{(\min\{||m||_\alpha,||n||_\beta\})^\gamma}{(\max\{||m||_\alpha,||n||_\beta\})^{\lambda+\gamma}}a_m\right)^{p-1}, \quad n \in \mathbf{N}^{j_0}.$$

Then it follows $J^p = ||b||_{q,\Psi}^q$. If $J = 0$, then (24) is trivially valid for $0 < ||a||_{p,\Phi} < \infty$; if $J = \infty$, then it is impossible since the right hand side of (24) is finite. Suppose that $0 < J < \infty$. Then by (20), we find

$$||b||_{q,\Psi}^q = J^p = I < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}||a||_{p,\Phi}||b||_{q,\Psi},$$

namely,

$$||b||_{q,\Psi}^{q-1} = J < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}||a||_{p,\Phi},$$

and then (24) follows.

On the other hand, assuming that (24) is valid, by Hölder's inequality, we have

$$I = \sum_n (\Psi(n))^{\frac{-1}{q}} \left[\sum_m \frac{(\min\{||m||_\alpha, ||n||_\beta\})^\gamma}{(\max\{||m||_\alpha, ||n||_\beta\})^{\lambda+\gamma}} a_m \right] [(\Psi(n))^{\frac{1}{q}} b_n]$$
$$\leqslant J||b||_{q,\Psi}. \tag{25}$$

Then by (24), we have (20). Hence (24) and (20) are equivalent.

By the equivalency, the constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ in (24) is the best possible. Otherwise, we would reach a contradiction by (25) that the constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ in (20) is not the best possible. □

For $p > 1$, we define two real weight normal discrete spaces $\mathbf{l}_{p,\varphi}$ and $\mathbf{l}_{q,\psi}$ as follows:

$$\mathbf{l}_{p,\varphi} := \left\{ a = \{a_m\}; ||a||_{p,\Phi} = \{\sum_m \Phi(m) a_m^p\}^{\frac{1}{p}} < \infty \right\},$$
$$\mathbf{l}_{q,\psi} := \left\{ b = \{b_n\}; ||b||_{q,\Psi} = \{\sum_n \Psi(n) b_n^q\}^{\frac{1}{q}} < \infty \right\}.$$

As the assumptions of Theorem 1, in view of $J < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} ||a||_{p,\Phi}$, we have the following definition:

DEFINITION 2. Define a multidimensional Hilbert-type operator $T : \mathbf{l}_{p,\Phi} \to \mathbf{l}_{p,\Psi^{1-p}}$ as follows: For $a \in \mathbf{l}_{p,\Phi}$, there exists an unique representation $Ta \in \mathbf{l}_{p,\Psi^{1-p}}$, satisfying

$$Ta(n) := \sum_m \frac{(\min\{||m||_\alpha, ||n||_\beta\})^\gamma}{(\max\{||m||_\alpha^\lambda, ||n||_\beta\})^{\lambda+\gamma}} a_m (n \in \mathbf{N}^{j_0}). \tag{26}$$

For $b \in \mathbf{l}_{q,\Psi}$, we define the following formal inner product of Ta and b as follows:

$$(Ta, b) := \sum_n \sum_m \frac{(\min\{||m||_\alpha, ||n||_\beta\})^\gamma}{(\max\{||m||_\alpha^\lambda, ||n||_\beta\})^{\lambda+\gamma}} a_m b_n. \tag{27}$$

Then by Theorem 1 and Theorem 2, for $0 < ||a||_{p,\varphi}$, $||b||_{q,\psi} < \infty$, we have the following equivalent inequalities:

$$(Ta, b) < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} ||a||_{p,\Phi} ||b||_{q,\Psi}, \tag{28}$$
$$||Ta||_{p,\Psi^{1-p}} < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} ||a||_{p,\Phi}. \tag{29}$$

It follows that T is bounded with

$$||T|| := \sup_{a(\neq\theta)\in \mathbf{l}_{p,\Phi}} \frac{||Ta||_{p,\Psi^{1-p}}}{||a||_{p,\Phi}} \leqslant K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}. \tag{30}$$

Since the constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ in (29) is the best possible, we have

COROLLARY 1. *As the assumptions of Theorem 2, T is defined by Definition 2, it follows*

$$||T|| = K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} \frac{\lambda+2\gamma}{(\lambda_1+\gamma)(\lambda_2+\gamma)}. \tag{31}$$

REMARK 1. For $i_0 = j_0 = 1$ in (20), we have inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(\min\{m,n\})^{\gamma} a_m b_n}{(\max\{m,n\})^{\lambda+\gamma}} < \frac{\lambda+2\gamma}{(\lambda_1+\gamma)(\lambda_2+\gamma)} ||a||_{p,\phi} ||b||_{q,\psi}. \tag{32}$$

Hence, (20) is an extension of (4) for $k_\lambda(m,n) = \frac{(\min\{m,n\})^{\gamma}}{(\max\{m,n\})^{\lambda+\gamma}}$.

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