Journal of Mathematical Inequalities
Volume 8, Number 2 (2014), 287–297
doi:10.7153/jmi-08-20

A NOTE ON THE NEUMAN–SÁNDOR MEAN

HUI SUN, TIEHONG ZHAO, YUMING CHU AND BAOYU LIU

(*Communicated by E. Neuman*)

Abstract. In this article, we present several best possible lower bounds for the Neuman-Sándor mean in terms of the geometric combinations of harmonic and quadratic means, geometric and quadratic means, harmonic and contraharmonic means, and geometric and contraharmonic means.

1. Introduction

For $a,b>0$ with $a\neq b$ the Neuman-Sándor mean $M(a,b)$ [1] is defined by

$$M(a,b)=\frac{a-b}{2\sinh^{-1}\left(\frac{a-b}{a+b}\right)}, \tag{1.1}$$

where $\sinh^{-1}(x)=\log(x+\sqrt{x^2+1})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean $M(a,b)$ can be found in the literature [1–14].

Let $H(a,b)=2ab/(a+b)$, $G(a,b)=\sqrt{ab}$, $L(a,b)=(b-a)/(\log b-\log a)$, $P(a,b)=(a-b)/(4\arctan\sqrt{a/b}-\pi)$, $I(a,b)=1/e(b^b/a^a)^{1/(b-a)}$, $A(a,b)=(a+b)/2$, $T(a,b)=(a-b)/[2\arctan((a-b)/(a+b))]$, $Q(a,b)=\sqrt{(a^2+b^2)/2}$ and $C(a,b)=(a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, identric, arithmetic, second Seiffert, quadratic and contraharmonic means of a and b, respectively. Then it is well-known that the inequalities

$$\begin{aligned}&H(a,b)<G(a,b)<L(a,b)<P(a,b)<I(a,b)\\&<A(a,b)<M(a,b)<T(a,b)<Q(a,b)<C(a,b)\end{aligned}$$

hold for $a,b>0$ with $a\neq b$.

Neuman and Sándor [1, 2] proved that the inequalities

$$\frac{\pi}{4\log(1+\sqrt{2})}T(a,b)<M(a,b)<\frac{A(a,b)}{\log(1+\sqrt{2})},$$

Mathematics subject classification (2010): 26E60.

Keywords and phrases: Neuman-Sándor mean, harmonic mean, geometric mean, quadratic mean, contraharmonic mean.

This research was supported by the Natural Science Foundation of China under Grants 11071069 and 11171307, and the Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant T200924.

$$\sqrt{2T^2(a,b)-Q^2(a,b)} < M(a,b) < \frac{T^2(a,b)}{Q(a,b)},$$

$$H(T(a,b),A(a,b)) < M(a,b) < L(A(a,b),Q(a,b)), \quad T(a,b) > H(M(a,b),Q(a,b)),$$

$$M(a,b) < \frac{A^2(a,b)}{P(a,b)}, \quad A^{2/3}(a,b)Q^{1/3}(a,b) < M(a,b) < \frac{2A(a,b)+Q(a,b)}{3},$$

$$\sqrt{A(a,b)T(a,b)} < M(a,b) < \sqrt{A^2(a,b)+T^2(a,b)},$$

$$\frac{G(x,y)}{G(1-x,1-y)} < \frac{L(x,y)}{L(1-x,1-y)} < \frac{P(x,y)}{P(1-x,1-y)}$$

$$< \frac{A(x,y)}{A(1-x,1-y)} < \frac{M(x,y)}{M(1-x,1-y)} < \frac{T(x,y)}{T(1-x,1-y)},$$

$$\frac{1}{A(1-x,1-y)} - \frac{1}{A(x,y)} < \frac{1}{M(1-x,1-y)} - \frac{1}{M(x,y)} < \frac{1}{T(1-x,1-y)} - \frac{1}{T(x,y)},$$

$$A(x,y)A(1-x,1-y) < M(x,y)M(1-x,1-y) < T(x,y)T(1-x,1-y)$$

hold for all $a,b>0$ and $x,y\in(0,1/2]$ with $a\neq b$ and $x\neq y$.

In [15], Neuman proved that the double inequalities

$$Q^{\alpha}(a,b)A^{1-\alpha}(a,b) < M(a,b) < Q^{\beta}(a,b)A^{1-\beta}(a,b)$$

and

$$C^{\lambda}(a,b)A^{1-\lambda}(a,b) < M(a,b) < C^{\mu}(a,b)A^{1-\mu}(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if $\alpha\leqslant 1/3$, $\beta\geqslant 2\left(\log(2+\sqrt{2})-\log 3\right)/\log 2 = 0.373\cdots$, $\lambda\leqslant 1/6$ and $\mu\geqslant\left(\log(2+\sqrt{2})-\log 3\right)/\log 2 = 0.186\cdots$.

In [5, 7, 16], the authors proved that the double inequalities

$$\alpha_1 I(a,b) < M(a,b) < \beta_1 I(a,b),$$

$$\alpha_2 Q(a,b)+(1-\alpha_2)H(a,b) < M(a,b) < \beta_2 Q(a,b)+(1-\beta_2)H(a,b),$$

$$\alpha_3 Q(a,b)+(1-\alpha_3)G(a,b) < M(a,b) < \beta_3 Q(a,b)+(1-\beta_3)G(a,b),$$

$$\alpha_4 C(a,b)+(1-\alpha_4)H(a,b) < M(a,b) < \beta_4 C(a,b)+(1-\beta_4)H(a,b),$$

$$I^{\alpha_5}(a,b)Q^{1-\alpha_5}(a,b) < M(a,b) < I^{\beta_5}(a,b)Q^{1-\beta_5}(a,b),$$

$$I^{\alpha_6}(a,b)C^{1-\alpha_6}(a,b) < M(a,b) < I^{\beta_6}(a,b)C^{1-\beta_6}(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leqslant 1$, $\beta_1\geqslant e/[2\log(1+\sqrt{2})] = 1.5419\cdots$, $\alpha_2\leqslant 7/9=0.777\cdots$, $\beta_2\geqslant 1/[\sqrt{2}\log(1+\sqrt{2})]=0.802\cdots$, $\alpha_3\leqslant 2/3 = 0.666\cdots$, $\beta_3\geqslant 1/[\sqrt{2}\log(1+\sqrt{2})]=0.802\cdots$, $\alpha_4\leqslant 1/[2\log(1+\sqrt{2})]=0.567\cdots$, $\beta_4\geqslant 7/12=0.583\cdots$, $\alpha_5\geqslant 1/2$, $\beta_5\leqslant\log[\sqrt{2}\log(1+\sqrt{2})]/(1-\log\sqrt{2})=0.337\cdots$, $\alpha_6\geqslant 5/7=0.714\cdots$ and $\beta_6\leqslant\log[2\log(1+\sqrt{2})]=0.566\cdots$.

Very recently, inequalities for quotients involving the Neuman-Sándor mean $M(a,b)$ were given in [17].

The main purpose of this paper is to find the least values α_1, α_2, α_3 and α_4 such that the inequalities

$$M(a,b) > H^{\alpha_1}(a,b)Q^{1-\alpha_1}(a,b),$$

$$M(a,b) > G^{\alpha_2}(a,b)Q^{1-\alpha_2}(a,b),$$

$$M(a,b) > H^{\alpha_3}(a,b)C^{1-\alpha_3}(a,b)$$

and

$$M(a,b) > G^{\alpha_4}(a,b)C^{1-\alpha_4}(a,b)$$

hold for all $a,b>0$ with $a \neq b$.

2. Lemmas

In order to establish our main results we need four lemmas, which we present in this section.

LEMMA 2.1. (See [18, Theorem 1.25]). *For $-\infty < a < b < \infty$, let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b), let $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

LEMMA 2.2. (See [19, Lemma 1.1]). *Suppose that the power series $f(x)=\sum_{n=0}^{\infty} a_n x^n$ and $g(x)=\sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r>0$ and $b_n>0$ for all $n\in\{0,1,2,\cdots\}$. Let $h(x)=f(x)/g(x)$, then*

(1) If the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) increasing (decreasing), then $h(x)$ is also (strictly) increasing (decreasing) on $(0,r)$;

(2) If the sequence $\{a_n/b_n\}$ is (strictly) increasing (decreasing) for $0<n\leqslant n_0$ and (strictly) decreasing (increasing) for $n>n_0$, then there exists $x_0\in(0,r)$ such that $h(x)$ is (strictly) increasing (decreasing) on $(0,x_0)$ and (strictly) decreasing (increasing) on (x_0,r).

LEMMA 2.3. *Let*

$$\phi(t)=\frac{[3-\cosh(2t)][\sinh(2t)-2t]}{2t\sinh^2(t)[5+\cosh(2t)]}, \tag{2.1}$$

then $\phi(t)$ is strictly decreasing in $(0,\log(1+\sqrt{2}))$, where $\sinh(t)=(e^t-e^{-t})/2$ and $\cosh(t)=(e^t+e^{-t})/2$ are respectively the hyperbolic sine and cosine functions.

Proof. Let us denote by $\phi_1(t)$ and $\phi_2(t)$ respectively the numerator and denominator of (2.1), then simple computations lead to

$$\phi_1(t) = 3\sinh(2t) - 6t + 2t\cosh(2t) - \frac{1}{2}\sinh(4t), \tag{2.2}$$

$$\phi_2(t) = \frac{t}{2}\left[8\cosh(2t) + \cosh(4t) - 9\right]. \tag{2.3}$$

Using the power series $\sinh(t) = \sum_{n=0}^{\infty} t^{2n+1}/(2n+1)!$ and $\cosh(t) = \sum_{n=0}^{\infty} t^{2n}/(2n)!$, we can express (2.2) and (2.3) as follows

$$\phi_1(t) = \sum_{n=1}^{\infty} \frac{2^{2n+1}(2n+4-2^{2n})}{(2n+1)!} t^{2n+1} = t^3 \sum_{n=0}^{\infty} \frac{2^{2n+4}(n+3-2^{2n+1})}{(2n+3)!} t^{2n}, \tag{2.4}$$

$$\phi_2(t) = \sum_{n=1}^{\infty} \frac{2^{2n}(4+2^{2n-1})}{(2n)!} t^{2n+1} = t^3 \sum_{n=0}^{\infty} \frac{2^{2n+4}(1+2^{2n-1})}{(2n+2)!} t^{2n}. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\phi(t) = \frac{\sum\limits_{n=0}^{\infty} a_n t^{2n}}{\sum\limits_{n=0}^{\infty} b_n t^{2n}} \tag{2.6}$$

with $a_n = 2^{2n+4}(n+3-2^{2n+1})/(2n+3)!$ and $b_n = 2^{2n+4}(1+2^{2n-1})/(2n+2)!$.

Let $c_n = a_n/b_n$, then simple computations lead to

$$c_n = \frac{(n+3)-2^{2n+1}}{(2n+3)(1+2^{2n-1})},$$

$$c_0 = \frac{2}{9} > c_1 = -\frac{4}{15} > c_2 = -\frac{3}{7} < c_3 = -\frac{122}{297}, \tag{2.7}$$

$$\begin{aligned} c_{n+1} - c_n &= \frac{2^{4n+3} - (6n^2+57n+76)2^{2n-1} - 3}{(2n+3)(2n+5)(1+2^{2n-1})(1+2^{2n+1})} \\ &= \frac{[2(4^n-38) + 6(4^n-n^2) + (128\times 4^{n-2} - 57n)]2^{2n-1} - 3}{(2n+3)(2n+5)(1+2^{2n-1})(1+2^{2n+1})} > 0 \end{aligned} \tag{2.8}$$

for all $n > 2$.

Inequalities (2.7) and (2.8) implies that the sequence $\{a_n/b_n\}$ is strictly decreasing in $0 < n \leqslant 2$ and strictly increasing for $n > 2$, then from (2.6) and Lemma 2.2(2) we know that there exists $t_0 > 0$ such that $\phi(t)$ is strictly decreasing on $(0, t_0)$ and strictly increasing in (t_0, ∞).

For convenience, let us denote $t^* = \log(1+\sqrt{2}) = 0.881\cdots$, then we have

$$\sinh(t^*) = 1, \quad \sinh(2t^*) = 2\sqrt{2}, \quad \sinh(3t^*) = 7, \tag{2.9}$$

$$\cosh(t^*) = \sqrt{2}, \quad \cosh(2t^*) = 3, \quad \cosh(3t^*) = 5\sqrt{2}. \tag{2.10}$$

Differentiating (2.1) yields

$$\phi'(t) = \frac{\phi_1{}'(t)\phi_2(t) - \phi_1(t)\phi_2{}'(t)}{\phi_2{}^2(t)}, \tag{2.11}$$

where

$$\phi_1'(t) = 8\sinh(t)[t\cosh(t) - 2\sinh^3(t)], \tag{2.12}$$

$$\phi_2'(t) = \sinh(t)[20t\cosh(t) + 4t\cosh(3t) + 9\sinh(t) + \sinh(3t)]. \tag{2.13}$$

From (2.2) and (2.3) together with (2.9)–(2.13) we get

$$\phi'(t^*) = -\frac{\sqrt{2} - t^*}{\sqrt{2}t^*} < 0. \tag{2.14}$$

It follows from the piecewise monotonicity of $\phi(t)$ and (2.14) that $t_0 > t^*$. This completes the proof of Lemma 2.3. □

LEMMA 2.4. *Let*

$$\varphi(x) = \frac{4}{9}\log(1+x^2) - \log\frac{x}{\sinh^{-1}(x)} + \frac{5}{18}\log(1-x^2). \tag{2.15}$$

Then $\varphi(x) < 0$ *for all* $x \in (0,1)$.

Proof. From (2.15) one has

$$\varphi(0^+) = 0, \tag{2.16}$$

$$\varphi'(x) = \frac{\phi(x)}{x(1-x^4)\sqrt{1+x^2}\sinh^{-1}(x)}, \tag{2.17}$$

where

$$\phi(x) = x - x^5 - \left[1 - \frac{1}{3}x^2 + \frac{4}{9}x^4\right]\sqrt{1+x^2}\sinh^{-1}(x), \tag{2.18}$$

$$\phi(0) = 0, \tag{2.19}$$

$$\phi'(x) = -\frac{xf(x)}{9\sqrt{1+x^2}}, \tag{2.20}$$

where

$$f(x) = x(49x^2 - 3)\sqrt{1+x^2} + (3 + 7x^2 + 20x^4)\sinh^{-1}(x), \tag{2.21}$$

$$f(0) = 0. \tag{2.22}$$

Differentiating (2.21) yields

$$f'(x) = \frac{2x[74x + 108x^3 + (7+40x^2)\sqrt{1+x^2}\sinh^{-1}(x)]}{\sqrt{1+x^2}} > 0 \tag{2.23}$$

for $x \in (0,1)$.

Therefore, $\phi(x) < 0$ for all $x \in (0,1)$ follows easily from (2.19) and (2.20) together with (2.22) and (2.23). □

3. Lower bounds for the Neuman-Sándor mean

In this section we will deal with problems of finding sharp lower bounds for the Neuman-Sándor Mean $M(a,b)$ in terms of the geometric combinations of harmonic mean $H(a,b)$ and quadratic mean $Q(a,b)$, geometric mean $G(a,b)$ and quadratic mean $Q(a,b)$, harmonic mean $H(a,b)$ and contraharmonic mean $C(a,b)$, and geometric mean $G(a,b)$ and contraharmonic mean $C(a,b)$.

Since $H(a,b)$, $G(a,b)$, $M(a,b)$, $Q(a,b)$ and $C(a,b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a > b$. For the later use we denote $x = (a-b)/(a+b) \in (0,1)$ and $t = \sinh^{-1}(x) \in (0,t^*)$ with $t^* = \log(1+\sqrt{2}) = 0.881\cdots$.

THEOREM 3.1. *The inequality*

$$M(a,b) > H^{\alpha}(a,b)Q^{1-\alpha}(a,b) \tag{3.1}$$

holds true for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \geqslant 2/9$.

Proof. First we take the logarithm of each member of (3.1) and next rearrange terms to obtain

$$\frac{\log[Q(a,b)] - \log[M(a,b)]}{\log[Q(a,b)] - \log[H(a,b)]} < \alpha. \tag{3.2}$$

Note that

$$\frac{M(a,b)}{A(a,b)} = \frac{x}{\sinh^{-1}(x)}, \quad \frac{H(a,b)}{A(a,b)} = 1 - x^2, \quad \frac{Q(a,b)}{A(a,b)} = \sqrt{1+x^2}. \tag{3.3}$$

Use of (3.3) followed by a substitution $x = \sinh(t)$ $(0 < t < t^*)$, inequality (3.2) becomes

$$f(t) < \alpha, \tag{3.4}$$

where

$$f(t) = \frac{\log[\cosh(t)] - \log[\sinh(t)/t]}{\log[\cosh(t)] - \log[1-\sinh^2(t)]} := \frac{f_1(t)}{f_2(t)}. \tag{3.5}$$

In order to use Lemma 2.1, we consider the following

$$\frac{f_1'(t)}{f_2'(t)} = \frac{[3-\cosh(2t)][\sinh(2t)-2t]}{2t\sinh^2(t)[5+\cosh(2t)]} := \phi(t), \tag{3.6}$$

where $\phi(t)$ is defined as in Lemma 2.3.

It follows from Lemmas 2.1 and 2.3 together with (3.6) that

$$f(t) = \frac{f_1(t)}{f_2(t)} = \frac{f_1(t) - f_1(0^+)}{f_2(t) - f_2(0)}$$

is strictly decreasing on $(0,t^*)$. Note that

$$\lim_{t\to 0^+} f(t) = \frac{2}{9}. \tag{3.7}$$

Making use of (3.7) and the monotonicity of $\phi(t)$ we conclude that in order for the inequality (3.1) to be valid it is necessary and sufficient that $\alpha \geqslant 2/9$. □

THEOREM 3.2. *The inequality*

$$M(a,b) > G^{\alpha}(a,b)Q^{1-\alpha}(a,b) \tag{3.8}$$

holds true for all $a,b > 0$ *with* $a \neq b$ *if and only if* $\alpha \geqslant 1/3$.

Proof. We will follows lines introduced in the proof of Theorem 3.1. We take the logarithm of each member of (3.8) and next rearrange terms to get

$$\frac{\log[Q(a,b)] - \log[M(a,b)]}{\log[Q(a,b)] - \log[G(a,b)]} < \alpha. \tag{3.9}$$

Use of (3.3) and $G(a,b)/A(a,b) = \sqrt{1-x^2}$ followed by a substitution $x = \sinh(t)$ $(0 < t < t^*)$, inequality (3.9) is equivalent to

$$g(t) < \alpha, \tag{3.10}$$

where

$$g(t) = \frac{\log[\cosh(t)] - \log[\sinh(t)/t]}{\log[\cosh(t)] - \log[1-\sinh^2(t)]/2} := \frac{g_1(t)}{g_2(t)}. \tag{3.11}$$

Equation (3.11) leads to

$$\frac{g_1'(t)}{g_2'(t)} = \frac{[3-\cosh(2t)][\sinh(2t)-2t]}{8t\sinh^2(t)} = \frac{\sum_{n=1}^{\infty}[2^{2n+1}(2n+4-2^{2n})/(2n+1)!]t^{2n+1}}{\sum_{n=1}^{\infty}[2^{2n+2}/(2n)!]t^{2n+1}}$$

$$= \frac{\sum_{n=0}^{\infty}[2^{2n+4}(n+3-2^{2n+1})/(2n+3)!]t^{2n}}{\sum_{n=0}^{\infty}[2^{2n+4}/(2n+2)!]t^{2n}} := \frac{\sum_{n=0}^{\infty} a_n' t^{2n}}{\sum_{n=0}^{\infty} b_n' t^{2n}}, \tag{3.12}$$

$$\frac{a_{n+1}'}{b_{n+1}'} - \frac{a_n'}{b_n'} = -\frac{3+(6n+7)2^{2n+1}}{(2n+3)(2n+5)} < 0 \tag{3.13}$$

for all $n \in \{0,1,2,\cdots\}$.

It follows from Lemmas 2.1(1) and (3.12) together with (3.13) that $g_1'(t)/g_2'(t)$ is strictly decreasing on $(0,t^*)$.

From Lemma 2.1 and (3.11) together with $g_1(0^+) = g_2(0) = 0$ and the monotonicity of $g_1'(t)/g_2'(t)$ we clearly see that $g(t)$ is strictly decreasing on $(0,t^*)$.

Therefore, Theorem 3.2 follows from the monotonicity of $g(t)$ and (3.10) together with the fact that

$$\lim_{t\to 0^+} g(t) = \frac{1}{3}. \quad \square$$

THEOREM 3.3. *The following simultaneous inequality*

$$M(a,b) > H^{\alpha}(a,b)C^{1-\alpha}(a,b) \tag{3.14}$$

holds true for all $a,b>0$ *with* $a \neq b$ *if and only if* $\alpha \geqslant 5/12$.

Proof. We take the logarithm of each member of (3.14) and next rearrange terms to get

$$\frac{\log[C(a,b)]-\log[M(a,b)]}{\log[C(a,b)]-\log[H(a,b)]} < \alpha. \tag{3.15}$$

Use of (3.3) and $C(a,b)/A(a,b) = 1+x^2$ followed by a substitution $x = \sinh(t)$ $(0 < t < t^*)$, inequality (3.15) becomes

$$h(t) < \alpha, \tag{3.16}$$

where

$$h(t) = \frac{\log[\cosh(t)]-\log[\sinh(t)/t]/2}{\log[\cosh(t)]-\log[1-\sinh^2(t)]/2} := \frac{h_1(t)}{h_2(t)}. \tag{3.17}$$

Equation (3.17) gives

$$\begin{aligned}\frac{h_1'(t)}{h_2'(t)} &= \frac{[3-\cosh(2t)][\sinh(2t)+t\cosh(2t)-3t]}{16t\sinh^2(t)}\\ &= \frac{\sum_{n=0}^{\infty}\left[2^{2n+3}\left((3-2^{2n})(2n+3)+3-2^{2n+2}\right)/(2n+3)!\right]t^{2n}}{\sum_{n=0}^{\infty}[2^{2n+5}/(2n+2)!]t^{2n}} := \frac{\sum_{n=0}^{\infty} c_n' t^{2n}}{\sum_{n=0}^{\infty} d_n' t^{2n}},\end{aligned} \tag{3.18}$$

$$\frac{c_{n+1}'}{d_{n+1}'} - \frac{c_n'}{d_n'} = -3\times 2^{2n-2} - \frac{3}{2(2n+3)(2n+5)} - \frac{(6n+7)2^{2n}}{(2n+3)(2n+5)} < 0 \tag{3.19}$$

for all $n \in \{0,1,2,\cdots\}$.

It follows from Lemmas 2.2(1) and (3.18) together with (3.19) that $h_1'(t)/h_2'(t)$ is strictly decreasing on $(0,t^*)$.

From Lemma 2.1 and (3.17) together with $h_1(0^+) = h_2(0) = 0$ and the monotonicity of $h_1'(t)/h_2'(t)$ we clearly see that $h(t)$ is strictly decreasing on $(0,t^*)$.

Therefore, Theorem 3.3 follows from the monotonicity of $h(t)$ and (3.16) together with the fact that

$$\lim_{t\to 0^+} h(t) = \frac{5}{12}. \quad \square$$

THEOREM 3.4. *The following inequality*

$$M(a,b) > G^{\alpha}(a,b)C^{1-\alpha}(a,b) \tag{3.20}$$

is valid for all $a,b>0$ *with* $a \neq b$ *if and only if* $\alpha \geqslant 5/9$.

Proof. Making use of (3.3) and $C(a,b)/A(a,b) = 1+x^2$ together with $G(a,b)/A(a,b) = \sqrt{1-x^2}$ we get

$$\frac{\log[C(a,b)] - \log[M(a,b)]}{\log[C(a,b)] - \log[G(a,b)]} = \frac{\log(1+x^2) - \log[x/\sinh^{-1}(x)]}{\log(1+x^2) - \log\sqrt{1-x^2}}. \tag{3.21}$$

Elaborated computations lead to

$$\lim_{x\to 0^+} \frac{\log(1+x^2) - \log[x/\sinh^{-1}(x)]}{\log(1+x^2) - \log\sqrt{1-x^2}} = \frac{5}{9}. \tag{3.22}$$

Taking the logarithm of (3.20), we consider the difference between the convex combination of $\log G(a,b)$, $\log C(a,b)$ and $\log M(a,b)$ as follows

$$\begin{aligned} &\frac{5}{9}\log G(a,b) + \frac{4}{9}\log C(a,b) - \log M(a,b) \\ =&\frac{5}{9}\log\sqrt{1-x^2} + \frac{4}{9}\log(1+x^2) - \log\frac{x}{\sinh^{-1}(x)} = \varphi(x), \end{aligned} \tag{3.23}$$

where $\varphi(x)$ is defined as in Lemma 2.4.

Therefore, $M(a,b) > G^{5/9}(a,b)C^{4/9}(a,b)$ for all $a,b>0$ with $a \neq b$ follows from (3.23) and Lemma 2.4. This in conjunction with the following statement gives the asserted result.

If $\alpha < 5/9$, then equations (3.21) and (3.22) lead to the conclusion that there exists $0 < \delta_1 < 1$ such that $M(a,b) < G^{\alpha}(a,b)C^{1-\alpha}(a,b)$ for all $a,b>0$ with $(a-b)/(a+b) \in (0,\delta_1)$. □

REMARK 3.1. From the inequalities $M(a,b) < \frac{1}{3}Q(a,b) + \frac{2}{3}A(a,b)$ in [14] and $A(a,b) < Q(a,b) < C(a,b)$, it is not difficult to prove that the inequalities $M(a,b) < H^{\lambda_1}(a,b)Q^{1-\lambda_1}(a,b)$, $M(a,b) < G^{\lambda_2}(a,b)Q^{1-\lambda_2}(a,b)$, $M(a,b) < H^{\lambda_3}(a,b)C^{1-\lambda_3}(a,b)$ and $M(a,b) < G^{\lambda_4}(a,b)C^{1-\lambda_4}(a,b)$ hold for all $a,b>0$ with $a \neq b$ if and only if $\lambda_1 \leqslant 0$, $\lambda_2 \leqslant 0$, $\lambda_3 \leqslant 0$ and $\lambda_4 \leqslant 0$.

REMARK 3.2. All the lower bounds $H^{2/9}(a,b)Q^{7/9}(a,b)$, $G^{1/3}(a,b)Q^{2/3}(a,b)$, $H^{5/12}(a,b)C^{7/12}(a,b)$ and $G^{5/9}(a,b)C^{4/9}(a,b)$ for $M(a,b)$ in Theorems 3.1-3.4 are weaker than the lower bound $Q^{1/3}(a,b)A^{2/3}(a,b)$ given by Neuman in [14]. In fact, elementary computations show that

$$\begin{aligned} &\left[Q^{1/3}(a,b)A^{2/3}(a,b)\right]^9 - \left[H^{2/9}(a,b)Q^{7/9}(a,b)\right]^9 \\ =& Q^3(a,b)\left[A^3(a,b) + H(a,b)Q^2(a,b)\right]\left[A^3(a,b) - H(a,b)Q^2(a,b)\right] \\ =& \frac{\left[A^3(a,b) + H(a,b)Q^2(a,b)\right]Q^3(a,b)}{8(a+b)}(a-b)^4, \end{aligned}$$

$$\begin{aligned} &\left[Q^{1/3}(a,b)A^{2/3}(a,b)\right]^3 - \left[G^{1/3}(a,b)Q^{2/3}(a,b)\right]^3 \\ =& Q(a,b)\left[A^2(a,b) - G(a,b)Q(a,b)\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{Q(a,b)}{A^2(a,b)+G(a,b)Q(a,b)}\left[A^4(a,b)-G^2(a,b)Q^2(a,b)\right] \\
&= \frac{Q(a,b)}{16\left[A^2(a,b)+G(a,b)Q(a,b)\right]}(a-b)^4, \\
&\left[Q^{1/3}(a,b)A^{2/3}(a,b)\right]^{12}-\left[H^{5/12}(a,b)C^{7/12}(a,b)\right]^{12} \\
&= C^2(a,b)\left[A^{10}(a,b)-H^5(a,b)C^5(a,b)\right], \\
&A^2(a,b)-H(a,b)C(a,b)=\frac{(a-b)^4}{4(a+b)^2}, \\
&\left[Q^{1/3}(a,b)A^{2/3}(a,b)\right]^{9}-\left[G^{5/9}(a,b)C^{4/9}(a,b)\right]^{9} \\
&= \frac{Q^3(a,b)}{A^4(a,b)}\left[A^{10}(a,b)-G^5(a,b)Q^5(a,b)\right]
\end{aligned}
$$

and

$$
A^4(a,b)-G^2(a,b)Q^2(a,b)=\frac{1}{16}(a-b)^4.
$$

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(Received May 1, 2013)

Hui Sun
School of Mathematics and Computation Science
Hunan City University
Yiyang 413000, P. R. China
e-mail: sunhui4624117@163.com

Tiehong Zhao
Department of mathematics
Hangzhou Normal University
Hangzhou 310036, P. R. China
e-mail: tiehongzhao@gmail.com

Yuming Chu
School of Mathematics and Computation Science
Hunan City University
Yiyang 413000, P. R. China
e-mail: chuyuming@hutc.zj.cn

Baoyu Liu
School of Science
Hangzhou Dianzi University
Hangzhou 310018, P. R. China
e-mail: 627847649@qq.com