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n–EXPONENTIAL CONVEXITY OF WEIGHTED HERMITE–HADAMARD'S INEQUALITY

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Abstract. In this paper we construct n-exponentially convex functions and exponentially convex functions using the functional defined as the difference of the weighted Hermite-Hadamard's inequality for monotone functions.

1. Introduction

Let f be a convex function on $[a,b]$. One of the most well-known inequalities in mathematics for convex functions is the Hermite-Hadamard integral inequality given in [7] (see also [9]):

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

If the function f is concave, then (1) holds in the reversed direction. It gives an estimate from below and above of the mean value of a convex function. These inequalities for convex functions play an important role in nonlinear analysis. There is a large range of interesting applications of Hermite-Hadamard's inequality given in [7].

In [2] (see also [7]) Fejér established the following weighted generalization of (1).

THEOREM 1.1. *If $f:[a,b]\to\mathbb{R}$ is a convex function, then the following inequality holds*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leqslant \int_a^b f(x)w(x)dx \leqslant \frac{f(a)+f(b)}{2}\int_a^b w(x)dx, \tag{2}$$

where $w:[a,b]\to\mathbb{R}$ is a non-negative function which is integrable and symmetric about $\frac{a+b}{2}$.

G. Zabandan and A. Kilicman gave a different weighted version of the Hermite-Hadamard's inequality in [1]:

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THEOREM 1.2. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable convex function and* $g : [a,b] \to [0,\infty]$ *be a continuous function.*

(i) If g *is decreasing on* $[a,b]$*, then*

$$\frac{1}{\int_a^b g(x)dx} \int_a^b f(x)g(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{3}$$

(ii) If g *is increasing on* $[a,b]$*, then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{\int_a^b g(x)dx} \int_a^b f(x)g(x)dx. \tag{4}$$

Unfortunately, the inequalities (3) and (4) are not valid under the given assumptions. R. Jakšić, Lj. Kvesić and J. Pečarić found some errors in the results given by G. Zabandan and A. Kilicman. In their paper [3] they gave particular examples of functions that satisfy the conditions of Theorem 1.2, but for which the inequalities (3) and (4) are not valid. Moreover, in the same paper they also gave the best possible conditions under which the inequality (3) holds. The following result is given in [3].

THEOREM 1.3. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable function and let* $g : [a,b] \to (0,\infty)$ *be an integrable function.*

(i) If the function $\frac{f'}{g}$ *is increasing, then inequality (3) holds.*

(ii) If the function $\frac{f'}{g}$ *is decreasing, then inequality (3) holds in reversed direction.*

REMARK 1.4. The class of functions for which $f'(x)/x$ is increasing is of special interest because it connects us with the superquadratic functions.

Suppose that a function $\phi : [0,\infty) \to \mathbb{R}$ is continuously differentiable and $\phi(0) \leqslant 0$. If $\phi'(x)/x$ is increasing, then ϕ is superquadratic (see [11]).

COROLLARY 1.5. *Let* $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function.*

(i) If the function $\frac{f'(x)}{x}$ *is increasing, then the following inequality*

$$\frac{2}{b^2-a^2} \int_a^b xf(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{5}$$

holds.

Proof. If we put $g(x) = x$ in Theorem 1.3, we directly get (5). $\square$

In the next section we will give some mean-value theorems and related results. In Section 3 we will construct n-exponentially convex functions and exponentially convex functions by using the functional defined as the difference of the right and the left side of inequality (3) for different classes of functions. In the last section, we will give some interesting examples and construct Stolarsky means.

2. Mean value theorems

To prove mean value theorems of Lagrange and Cauchy type, we need to consider functions ϕ_1 and ϕ_2 defined in the following lemma.

LEMMA 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function and let $g : [a,b] \to \mathbb{R}^+$ be a differentiable integrable function. Denote*

$$G_f(x) = \frac{f''(x)g(x) - f'(x)g'(x)}{g^2(x)}. \tag{6}$$

Let $m, M \in \mathbb{R}$ be such that

$$m \leqslant G_f(x) \leqslant M \quad \textit{for all} \quad x \in [a,b]. \tag{7}$$

Let $\phi_1, \phi_2 \colon [a,b] \to \mathbb{R}$ be the functions defined by

$$\phi_1(x) = M\int_a^x tg(t)dt - f(x) \tag{8}$$

and

$$\phi_2(x) = f(x) - m\int_a^x tg(t)dt. \tag{9}$$

Then $\frac{\phi_1'}{g}$ and $\frac{\phi_2'}{g}$ are increasing functions.

Proof. It is sufficient to show that the first derivatives of $\frac{\phi_1'}{g}$ and $\frac{\phi_2'}{g}$ are positive functions. We have

$$\left(\frac{\phi_1'(x)}{g(x)}\right)' = \left(Mx - \frac{f'(x)}{g(x)}\right)' = M - G_f(x) \geqslant 0,$$

and

$$\left(\frac{\phi_2'(x)}{g(x)}\right)' = \left(\frac{f'(x)}{g(x)} - mx\right)' = G_f(x) - m \geqslant 0.$$

This shows us that $\frac{\phi_1'}{g}, \frac{\phi_2'}{g}$ are increasing functions. $\square$

THEOREM 2.2. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function, $g : [a,b] \to \mathbb{R}^+$ a differentiable integrable function and let $G_f \in C[a,b]$ be as defined in Lemma 2.1. Then there exists $\xi \in [a,b]$ such that*

$$\frac{f(a)+f(b)}{2} - \frac{1}{\int_a^b g(x)dx}\int_a^b f(x)g(x)dx = \alpha G_f(\xi), \tag{10}$$

where

$$\alpha = \left(\frac{\int_a^b tg(t)dt}{2} - \frac{\int_a^b g(x)\int_a^x tg(t)dtdx}{\int_a^b g(x)dx}\right).$$

Proof. Since G_f is continuous on a compact set, it attains its maximum and minimum value on it. Let us consider

$$m = \min\{G_f(x)\}$$

and

$$M = \max\{G_f(x)\}.$$

In Lemma 2.1 we have shown that $\frac{\phi_1'}{g}, \frac{\phi_2'}{g}$, where ϕ_1 and ϕ_2 are defined by (8) and (9), are increasing functions, so we can apply Theorem 1.3 to these functions and obtain the following inequalities:

$$\frac{f(a)+f(b)}{2} - \frac{1}{\int_a^b g(x)dx}\int_a^b f(x)g(x)dx \leqslant \alpha M$$

$$\frac{f(a)+f(b)}{2} - \frac{1}{\int_a^b g(x)dx}\int_a^b f(x)g(x)dx \geqslant \alpha m.$$

Combining both inequalities, we get

$$\alpha m \leqslant \frac{f(a)+f(b)}{2} - \frac{1}{\int_a^b g(x)dx}\int_a^b f(x)g(x)dx \leqslant \alpha M.$$

If $\alpha = 0$ then $\frac{f(a)+f(b)}{2} - \frac{1}{\int_a^b g(x)dx}\int_a^b f(x)g(x)dx = 0$ and (10) holds for all $\xi \in [a,b]$. Otherwise

$$m \leqslant \frac{\frac{f(a)+f(b)}{2} - \frac{1}{\int_a^b g(x)dx}\int_a^b f(x)g(x)dx}{\alpha} \leqslant M.$$

Since G_f is continuous on $[a,b]$, there exists $\xi \in [a,b]$ such that (10) holds and the proof is completed. □

COROLLARY 2.3. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a twice differentiable function and let $G_f \in C[a,b]$ be as defined in Lemma 2.1. Then there exists $\xi \in [a,b]$ such that*

$$\frac{f(a)+f(b)}{2} - \frac{2}{b^2-a^2}\int_a^b xf(x)dx = \alpha G_f(\xi), \tag{11}$$

where $\alpha = \frac{b^4+b^3a-4a^2b^2+ba^3+a^4}{30(b+a)}$ and $G_f(\xi) = \frac{f''(\xi)}{\xi} - \frac{f'(\xi)}{\xi^2}$.

Proof. If we put $g(x) = x$ in Theorem 2.2, we get (11). □

THEOREM 2.4. *Let $f_1, f_2 : [a,b] \to \mathbb{R}$ be twice differentiable functions, $g : [a,b] \to \mathbb{R}^+$ a differentiable integrable functions and $G_{f_1}, G_{f_2} \in C[a,b]$ as defined in Lemma 2.1. Then there exists $\xi \in [a,b]$ such that the following equality is valid:*

$$\frac{\frac{f_1(a)+f_1(b)}{2} - \frac{1}{\int_a^b g(x)dx}\int_a^b f_1(x)g(x)dx}{\frac{f_2(a)+f_2(b)}{2} - \frac{1}{\int_a^b g(x)dx}\int_a^b f_2(x)g(x)dx} = \frac{G_{f_1}(\xi)}{G_{f_2}(\xi)} \tag{12}$$

provided that the denominators are nonzero.

Proof. Let us define $k \in C^2[a,b]$ with

$$k = c_1 f_2 - c_2 f_1,$$

where c_1 and c_2 are defined with

$$c_1 = \frac{f_1(a)+f_1(b)}{2} - \frac{1}{\int_a^b g(x)dx}\int_a^b f_1(x)g(x)dx,$$

$$c_2 = \frac{f_2(a)+f_2(b)}{2} - \frac{1}{\int_a^b g(x)dx}\int_a^b f_2(x)g(x)dx.$$

Now if we apply Theorem 2.2 to the function k we get

$$0 = \left(c_1 G_{f_2}(\xi) - c_2 G_{f_1}(\xi)\right)\alpha. \tag{13}$$

Since $\alpha \neq 0$, otherwise by our assumptions we would have a contradiction with

$$\frac{f_2(a)+f_2(b)}{2} - \frac{1}{\int_a^b g(x)dx}\int_a^b f_2(x)g(x)dx \neq 0,$$

so $\alpha \neq 0$ and from (13) we directly get (12). $\square$

COROLLARY 2.5. *Let $f_1, f_2 : [a,b] \subset (0,\infty) \to \mathbb{R}$ twice differentiable and $G_{f_1}, G_{f_2} \in C[a,b]$ as defined in Corollary 2.3. Then there exists $\xi \in [a,b]$ such that the following equality is valid:*

$$\frac{\frac{f_1(a)+f_1(b)}{2} - \frac{2}{b^2-a^2}\int_a^b x f_1(x)dx}{\frac{f_2(a)+f_2(b)}{2} - \frac{2}{b^2-a^2}\int_a^b x f_2(x))dx} = \frac{\xi f_1''(\xi) - f_1'(\xi)}{\xi f_2''(\xi) - f_2'(\xi)} \tag{14}$$

provided that the denominators are nonzero.

Proof. If we put $g(x) = x$ in Theorem 2.4, we directly get (14). $\square$

3. n-exponential convexity and exponential convexity

We start this section by giving some definitions and notions which are frequently used in the results. Throughout this section I will denote an interval in $\mathbb{R}$. The following results for n-exponentially convex functions have been cited from [6].

DEFINITION 1. A function $f : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

holds for all choices of $\xi_i \in \mathbb{R}$ and every $x_i \in I$, $i = 1, ..., n$.

A function $f : I \longrightarrow \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

REMARK 3.1. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leqslant n$.

By using some linear algebra and definition of positive semi-definite matrices, we get the following proposition.

PROPOSITION 3.2. *If f is an n-exponentially convex function in the Jensen sense then the matrix*

$$\left[f\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$$

is a positive semi-definite matrix for all $k \in \mathbb{N}$, $k \leqslant n$. In particular,

$$\det\left[f\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \geqslant 0$$

for all $k \in \mathbb{N}$, $k \leqslant n$.

DEFINITION 2. A function $f : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $f : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous on I.

LEMMA 3.3. *It is known (and easy to show) that $f : I \to \mathbb{R}^+$ is $\log$-convex in the Jensen sense if and only if*

$$l^2 f(t) + 2lmf\left(\frac{t+r}{2}\right) + m^2 f(r) \geqslant 0$$

holds for each $l, m \in \mathbb{R}$ and $r, t \in I$.

It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense. Also, using basic convexity theory it follows that a positive function is log-convex if and only if it is 2-exponentially convex.

The following lemma is equivalent to the definition of convex function [7, page 2].

LEMMA 3.4. *A function $f : I \to \mathbb{R}$ is convex if and only if the inequality*

$$(x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \geqslant 0$$

holds for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

LEMMA 3.5. *If Φ is a convex function on an interval I and if $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{\Phi(x_2) - \Phi(x_1)}{x_2 - x_1} \leqslant \frac{\Phi(y_2) - \Phi(y_1)}{y_2 - y_1}. \tag{15}$$

If the function Φ is concave then the inequality reverses.

Divided differences are found to be very handy and interesting when we have to operate with diverse functions having various degrees of smoothness. Let $f: I \to \mathbb{R}$ be a function. Then for distinct points $u_i \in I$, $i = 0, 1$, the divided difference of first order is defined as follows:

$$[u_i; f] = f(u_i) \quad (i = 0, 1),$$

$$[u_0, u_1; f] = \frac{f(u_1) - f(u_0)}{u_1 - u_0}.$$

The values of the divided difference are independent of the order of the points u_0, u_1 and may be extended to include the case when the points are equal, that is

$$[u_0, u_0; f] = \lim_{u_1 \to u_0} [u_0, u_1; f] = f'(u_0),$$

provided that f' exists.

REMARK 3.6. One can note that if $[u_0, u_1; f] \geqslant 0$ holds for all $u_0, u_1 \in I$, then f is increasing on I.

Under the assumptions of Theorem 1.3, we consider the following functional

$$\Upsilon(f) = \frac{f(a) + f(b)}{2} - \frac{1}{\int_a^b g(x)dx} \int_a^b f(x)g(x)dx. \tag{16}$$

REMARK 3.7. Under the assumptions of Theorem 1.3, if f'/g is an increasing function on $[a,b]$ then $\Upsilon(f) \geqslant 0$.

In order to obtain our main results regarding the exponential convexity, we need to define different families of functions.

Let $[a,b], J \subseteq \mathbb{R}$ be intervals and let g be a positive integrable function on $[a,b]$. For distinct points $u_0, u_1 \in [a,b]$, we define:

$\mathbf{E}_1 = \{f_t : [a,b] \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1; F_t]$ is n-exponentially convex in the Jensen sense on J, where $F_t(u) = \frac{f_t'(u)}{g(u)}\}$;

$\mathbf{E}_2 = \{f_t : [a,b] \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1; F_t]$ is exponentially convex in the Jensen sense on J, where $F_t(u) = \frac{f_t'(u)}{g(u)}\}$;

$\mathbf{E}_3 = \{f_t : [a,b] \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1; F_t]$ is 2-exponentially convex in the Jensen sense on J, where $F_t(u) = \frac{f_t'(u)}{g(u)}\}$.

THEOREM 3.8. *Let $\Upsilon(f)$ be the linear functional defined in (16) and $f_t \in \mathbf{E}_1$. Then $t \mapsto \Upsilon(f_t)$ is an n-exponentially convex function in the Jensen sense on J. If the function $t \mapsto \Upsilon(f_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Let us consider families of functions $\mathbf{E}_1$. For $\xi_i \in \mathbb{R}$ and $t_i \in J$, $i = 1, \ldots, n$, we define the function

$$h(u) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{t_i + t_j}{2}}(u). \tag{17}$$

We have

$$[u_0,u_1;H] = \sum_{i,j=1}^{n} \xi_i\xi_j \left[u_0,u_1;F_{\frac{t_i+t_j}{2}}\right],$$

where $H(u) = \frac{h'(u)}{g(u)}$ and $F_t(u) = \frac{f_t'(u)}{g(u)}$.

Since $t \mapsto [u_0,u_1;F_t]$ is n-exponentially convex in the Jensen sense on J, the right hand side of the above expression is nonnegative, which by Remark 3.6 implies that $\frac{h'(u)}{g(u)}$ is an increasing function on $[a,b]$.

Thus by Remark 3.7, we have

$$\Upsilon(h) \geqslant 0,$$

thus

$$\sum_{i,j=1}^{n} \xi_i\xi_j \Upsilon\left(f_{\frac{t_i+t_j}{2}}\right) \geqslant 0.$$

Hence, we conclude that the function $t \mapsto \Upsilon(f_t)$ is n-exponentially convex in the Jensen sense on J.

If the function $t \mapsto \Upsilon(f_t)$ is also continuous on J then $t \mapsto \Upsilon(f_t)$ is n-exponentially convex by definition. $\square$

The following corollary is an immediate consequence of the above theorem.

COROLLARY 3.9. *Let $\Upsilon(f)$ be the linear functional defined in (16) and $f_t \in \mathbf{E}_2$. Then $t \mapsto \Upsilon(f_t)$ is an exponentially convex function in the Jensen sense on J. If the function $t \mapsto \Upsilon(f_t)$ is continuous on J then it is exponentially convex on J.*

Proof. For any $n \in \mathbb{N}$ we apply the same steps as in the previous theorem. $\square$

COROLLARY 3.10. *Let $\Upsilon(f)$ be the linear functional defined in (16) and $f_t \in \mathbf{E}_3$. Then the following statements hold:*

(i) If the function $t \mapsto \Upsilon(f_t)$ is continuous on J then it is 2-exponentially convex on J. If the function $t \mapsto \Upsilon(f_t)$ is additionally strictly positive, then it is also log-convex on J, and for $r,s,t \in J$ such that $r < s < t$, we have

$$(\Upsilon(f_s))^{t-r} \leqslant (\Upsilon(f_r))^{t-s}(\Upsilon(f_t))^{s-r}. \tag{18}$$

(ii) If the function $t \mapsto \Upsilon(f_t)$ is strictly positive and differentiable on J then for every $t,r,u,v \in J$ such that $t \leqslant u$, $r \leqslant v$, we have

$$\mathfrak{B}(t,r;\Upsilon) \leqslant \mathfrak{B}(u,v;\Upsilon),$$

where

$$\mathfrak{B}(t,r;\Upsilon) = \begin{cases} \left(\frac{\Upsilon(f_t)}{\Upsilon(f_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{\frac{d}{dt}(\Upsilon(f_t))}{\Upsilon(f_t)}\right), & t = r. \end{cases} \tag{19}$$

Proof. (i) The first part is an immediate consequence of Theorem 3.8 and in the second part log-convexity on J is a consequence of Lemma 3.3. Since $t \mapsto \Upsilon(f_t)$ is strictly positive, so for $r,s,t \in J$ such that $r < s < t$ with $f(t) = \log \Upsilon(f_t)$ in Lemma 3.4 gives

$$(t-s)\log \Upsilon(f_r) + (r-t)\log \Upsilon(f_s) + (s-r)\log \Upsilon(f_t) \geqslant 0.$$

This is equivalent to inequality (18).

(ii) By (i) the function $t \mapsto \Upsilon(f_t)$ is log-convex on J, that is, the function $t \mapsto \log \Upsilon(f_t)$ is convex on J. Thus, by using Lemma 3.5 with $t \leqslant u$, $r \leqslant v$, $t \neq r$, $u \neq v$, we get

$$\frac{\log \Upsilon(f_t) - \log \Upsilon(f_r)}{t-r} \leqslant \frac{\log \Upsilon(f_u)) - \log \Upsilon(f_v)}{u-v}, \tag{20}$$

concluding

$$\mathfrak{B}(t,r;\Upsilon) \leqslant \mathfrak{B}(u,v;\Upsilon).$$

Now, if $t = r \leqslant u$, we apply $\lim_{r \longrightarrow t}$, concluding

$$\mathfrak{B}(t,t;\Upsilon)) \leqslant \mathfrak{B}(u,v;\Upsilon).$$

Other possible cases are treated similarly. □

REMARK 3.11. The results given in Theorem 3.8, Corollary 3.9 and Corollary 3.10 are still valid when the points $u_0, u_1 \in I$ coincide, say $u_1 = u_0$, for a family of differentiable function f_t such that the function $t \mapsto [u_0, u_1; F_t]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense).

4. Stolarsky Means

In this section we will utilize various families of functions in order to construct different examples of log-convex and exponentially convex functions and some related results.

EXAMPLE 4.1. Let $t \in \mathbb{R}$ and $f_t : (0,\infty) \to \mathbb{R}$ be a function defined with

$$f_t(u) = \begin{cases} \frac{1}{t}\int_a^u p^t g(p)dp, & t \neq 0, \\ \int_a^u \log p g(p)dp, & t = 0. \end{cases} \tag{21}$$

Since $\left(\frac{f_t'(u)}{g(u)}\right)' = u^{t-1}$, the mapping $t \mapsto \frac{f_t'(u)}{g(u)}$ is exponentially convex (see [5]).

Analogously as in the proof of Theorem 3.8 we conclude that $t \mapsto [u,u;F_t]$ is exponentially convex, so it is also exponentially convex in the Jensen sense, where $F_t = \frac{f_t'}{g}$.

Also by Corollary 3.9 we have that $t \mapsto \Upsilon(f_t)$ is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex.

For this family of functions, $\mathfrak{B}(t,r;\Upsilon)$ (defined in (19)), is equal to

$$\mathfrak{B}(t,r;\Upsilon)=\begin{cases}\left(\frac{r\left(\int_a^b g(u)du\int_a^b p^t g(p)dp-2\int_a^b g(u)\int_a^u p^t g(p)dpdu\right)}{t\left(\int_a^b g(u)du\int_a^b p^r g(p)dp-2\int_a^b g(u)\int_a^u p^r g(p)dpdu\right)}\right)^{\frac{1}{t-r}}, & t\neq r,\ t,r\neq 0,\\ \left(\frac{\left(\int_a^b g(u)du\int_a^b p^t g(p)dp-2\int_a^b g(u)\int_a^u p^t g(p)dpdu\right)}{t\left(\int_a^b g(u)du\int_a^b \log p g(p)dp-2\int_a^b g(u)\int_a^u \log p g(p)dpdu\right)}\right)^{\frac{1}{t}}, & t\neq r=0,\\ \exp\left(\frac{-1}{t}+\frac{\int_a^b g(u)du\int_a^b \log p p^t g(p)dp-2\int_a^b g(u)\int_a^u \log p p^t g(p)dpdu}{\int_a^b g(u)du\int_a^b p^t g(p)dp-2\int_a^b g(u)\int_a^u p^t g(p)dpdu}\right), & t=r,\ \ t,r\neq 0,\\ \exp\left(\frac{\int_a^b g(u)du\int_a^b (\log p)^2 g(p)dp-2\int_a^b g(u)\int_a^u (\log p)^2 g(p)dpdu}{2\left(\int_a^b g(u)du\int_a^b \log p g(p)dp-2\int_a^b g(u)\int_a^u \log p g(p)dpdu\right)}\right), & t=r=0,\end{cases}$$

Now if we put $g(x)=x$, then for $t\neq r$, $t,r\neq 0,-2,-4$ we have

$$\mathfrak{B}(t,r;\Upsilon)=\left(\frac{r(r+2)(r+4)}{t(t+2)(t+4)}\frac{(t+4)(b^2-a^2)(b^{t+2}+a^{t+2})-4(b^{t+4}-a^{t+4})}{(r+4)(b^2-a^2)(b^{r+2}+a^{r+2})-4(b^{r+4}-a^{r+4})}\right)^{\frac{1}{t-r}} \tag{22}$$

and for $t=r$ and $t\neq 0,-2,-4$,

$$\mathfrak{B}(t,t;\Upsilon)=\exp\left(\frac{-1}{t}+\frac{(b^2-a^2)(t+4)^2\mathbf{A}-4(t+2)(t+4)(b^{t+4}\log b-a^{t+4}\log a)+8(t+3)(b^{t+4}-a^{t+4})+\mathbf{B}}{(t+2)(t+4)\left((t+4)(b^2-a^2)(b^{t+2}+a^{t+2})-4(b^{t+4}-a^{t+4})\right)}\right),$$

where

$$A=\left((t+2)(b^{t+2}\log b-a^{t+2}\log a)-(b^{t+2}-a^{t+2})\right)$$

and

$$B=2(t+4)^2(t+2)(b^2-a^2)a^{t+2}\log a-2(t+4)^2(b^2-a^2)a^{t+2}.$$

However, to get the continuous extension of (22) in order to cover all choices of r and t, we consider the following.

For $t\neq 0,-2,-4$,

$$\mathfrak{B}(t,0;\Upsilon)=\left(\frac{8}{t(t+2)(t+4)}\frac{(t+4)(b^2-a^2)(b^{t+2}+a^{t+2})-4(b^{t+4}-a^{t+4})}{b^4-a^4-4a^2b^2(\ln b-\ln a)}\right)^{\frac{1}{t}},$$

$$\mathfrak{B}(t,-2;\Upsilon)=\left(\frac{-2}{t(t+2)(t+4)}\frac{(t+4)(b^2-a^2)(b^{t+2}+a^{t+2})-4(b^{t+4}-a^{t+4})}{(b^2+a^2)(\log a-\log b)+(b^2-a^2)}\right)^{\frac{1}{t+2}},$$

$$\mathfrak{B}(t,-4;\Upsilon)=\left(\frac{8a^2b^2}{t(t+2)(t+4)}\frac{(t+4)(b^2-a^2)(b^{t+2}+a^{t+2})-4(b^{t+4}-a^{t+4})}{b^4-a^4-4a^2b^2(\log b-\log a)}\right)^{\frac{1}{t+4}}.$$

$$\mathfrak{B}(0,0;\Upsilon)=\exp\left(\frac{b^4(4\ln b-3)-a^4(4\ln a-3)+8a^2b^2((\ln a)^2-\ln a+\ln b-(\ln b)^2)}{4\left(b^4-a^4-4a^2b^2(\log b-\log a)\right)}\right).$$

$$\mathfrak{B}(-2,-2;\Upsilon)=\exp\left(\frac{a^2[\log b(1+\log b)+\log a(1-\log a)]+b^2[\log b(-1+\log b)-\log a(1+\log a)]}{(b^2-a^2)[-2-2(\log b+\log a)]+4b^2\log b-4a^2\log a}\right).$$

$$\mathfrak{B}(-4,-4;\Upsilon)=\exp\left(\frac{b^4[3+4\log a]-a^4[3+4\log b]+8a^2b^2[-\log b(1+\log b)+\log a(1+\log a)]}{4[(b^4-a^4)+4a^2b^2(\log a-\log b)]}\right).$$

Also note that if the function $t \mapsto \Upsilon(f_t)$ is positive and differentiable on $\mathbb{R}$ then for every $t,r,u,v \in \mathbb{R}$ such that $t \leqslant u$, $r \leqslant v$, we have

$$\mathfrak{B}(t,r;\Upsilon) \leqslant \mathfrak{B}(u,v;\Upsilon). \tag{23}$$

If we apply Corollary 2.5 on functions $f_1 = f_t$ and $f_2 = f_r$, where $t \neq r$, we get that there exists some $\xi \in [a,b]$ such that

$$\frac{\frac{f_t(a)+f_t(b)}{2} - \frac{2}{b^2-a^2}\int_a^b x f_t(x)dx}{\frac{f_r(a)+f_r(b)}{2} - \frac{2}{b^2-a^2}\int_a^b x f_r(x)dx} = \xi^{t-r}.$$

Since the function $\xi \mapsto \xi^{t-r}$ is invertible for $t \neq r$, we then have

$$a \leqslant \left(\frac{\frac{f_t(a)+f_t(b)}{2} - \frac{2}{b^2-a^2}\int_a^b x f_t(x)dx}{\frac{f_r(a)+f_r(b)}{2} - \frac{2}{b^2-a^2}\int_a^b x f_r(x)dx}\right)^{\frac{1}{t-r}} \leqslant b,$$

that is

$$a \leqslant B(t,r;\Upsilon) \leqslant b,$$

which together with the fact that $B(t,r;\Upsilon)$ is continuous and monotonous with respect to both of its arguments t and r, shows that $B(t,r;\Upsilon)$ are means of a and b for all $t,r \in \mathbb{R}$.

EXAMPLE 4.2. Let $t \in \mathbb{R}$ and $f_t : (0,\infty) \to \mathbb{R}$ be a function defined as

$$f_t(u) = \frac{-1}{\sqrt{t}}\int_a^u e^{-p\sqrt{t}} g(p)dp. \tag{24}$$

Since $\left(\frac{f_t'(u)}{g(u)}\right)' = e^{-u\sqrt{t}}$, the mapping $t \mapsto \frac{f_t'(u)}{g(u)}$ is exponentially convex (see [5]).

Analogously as in the proof of Theorem 3.8 we conclude that $t \mapsto [u,u;F_t]$ is exponentially convex, so it is exponentially convex in the Jensen sense, where $F_t = \frac{f_t'}{g}$.

Also by Corollary 3.9 we have that $t \mapsto \Upsilon(f_t)$ is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex.

For this family of functions, $\mathfrak{B}(t,r;\Upsilon)$ from (19) is equal to

$$\mathfrak{B}(t,r;\Upsilon) = \begin{cases} \left(\frac{\sqrt{r}\left(\int_a^b g(u)du \int_a^b e^{-p\sqrt{t}} g(p)dp - 2\int_a^b g(u)\int_a^u e^{-p\sqrt{t}} g(p)dpdu\right)}{\sqrt{t}\left(\int_a^b g(u)du \int_a^b e^{-p\sqrt{r}} g(p)dp - 2\int_a^b g(u)\int_a^u e^{-p\sqrt{r}} g(p)dpdu\right)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{-1}{2t} - \frac{\int_a^b g(u)du \int_a^b pe^{-p\sqrt{t}} g(p)dp - 2\int_a^b g(u)\int_a^u pe^{-p\sqrt{t}} g(p)dpdu}{2\sqrt{t}\left(\int_a^b g(u)du \int_a^b e^{-p\sqrt{t}} g(p)dp - 2\int_a^b g(u)\int_a^u e^{-p\sqrt{t}} g(p)dpdu\right)}\right), & t = r, \end{cases}$$

EXAMPLE 4.3. Let $t \in \mathbb{R}$ and $f_t : (0,\infty) \to \mathbb{R}$ be a function defined as

$$f_t(u) = \begin{cases} \frac{1}{t}\int_a^u e^{pt} g(p)dp, & t \neq 0, \\ \int_a^u pg(p)dp, & t = 0. \end{cases} \tag{25}$$

Since $\left(\frac{f_t'(u)}{g(u)}\right)' = e^{ut}$, the mapping $t \mapsto \frac{f_t'(u)}{g(u)}$ is exponentially convex (see [5]).

Analogously as in the proof of Theorem 3.8 we conclude that $t \mapsto [u,u;F_t]$ is exponentially convex, so it is exponentially convex in the Jensen sense, where $F_t = \frac{f_t'}{g}$.

Also by Corollary 3.9 we have that $t \mapsto \Upsilon(f_t)$ is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex.

For this family of functions, $\mathfrak{B}(t,r;\Upsilon)$ from (19) is equal to

$$\mathfrak{B}(t,r;\Upsilon) = \begin{cases} \left(\frac{r\left(\int_a^b g(u)du \int_a^b e^{pt} g(p)dp - 2\int_a^b g(u)\int_a^u e^{pt} g(p)dpdu\right)}{t\left(\int_a^b g(u)du \int_a^b e^{pr} g(p)dp - 2\int_a^b g(u)\int_a^u e^{pr} g(p)dpdu\right)}\right)^{\frac{1}{t-r}}, & t \neq r,\ t,r \neq 0, \\ \left(\frac{\left(\int_a^b g(u)du \int_a^b e^{pt} g(p)dp - 2\int_a^b g(u)\int_a^u e^{pt} g(p)dpdu\right)}{t\left(\int_a^b g(u)du \int_a^b pg(p)dp - 2\int_a^b g(u)\int_a^u pg(p)dpdu\right)}\right)^{\frac{1}{t}}, & t \neq r = 0, \\ \exp\left(\frac{-1}{t} + \frac{\int_a^b g(u)du \int_a^b pe^{pt} g(p)dp - 2\int_a^b g(u)\int_a^u pe^{pt} g(p)dpdu}{\int_a^b g(u)du \int_a^b p^t g(p)dp - 2\int_a^b g(u)\int_a^u p^t g(p)dpdu}\right), & t = r,\ \ t,r \neq 0, \\ \exp\left(\frac{\int_a^b g(u)du \int_a^b p^2 g(p)dp - 2\int_a^b g(u)\int_a^u p^2 g(p)dpdu}{2\left(\int_a^b g(u)du \int_a^b pg(p)dp - 2\int_a^b g(u)\int_a^u pg(p)dpdu\right)}\right), & t = r = 0, \end{cases}$$

EXAMPLE 4.4. Let $t \in (0,\infty)$ and $f_t : (0,\infty) \to \mathbb{R}$ be a function defined as

$$f_t(u) = \begin{cases} \frac{-1}{\log t}\int_a^u t^{-p} g(p)dp, & t \neq 1, \\ \int_a^u pg(p)dp, & t = 1. \end{cases} \tag{26}$$

Since $\left(\frac{f_t'(u)}{g(u)}\right)' = t^{-u}$, the mapping $t \mapsto \frac{f_t'(u)}{g(u)}$ is exponentially convex (see [5]).

Analogously as in the proof of Theorem 3.8 we conclude that $t \mapsto [u,u;F_t]$ is exponentially convex, so it is exponentially convex in the Jensen sense, where $F_t = \frac{f_t'}{g}$.

Also by Corollary 3.9 we have that $t \mapsto \Upsilon(f_t)$ is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex.

For this family of functions, $\mathfrak{B}(t,r;\Upsilon)$ from (19) is equal to

$$\mathfrak{B}(t,r;\Upsilon) = \begin{cases} \left(\frac{\log r\left(\int_a^b g(u)du \int_a^b t^{-p} g(p)dp - 2\int_a^b g(u)\int_a^u t^{-p} g(p)dpdu\right)}{\log t\left(\int_a^b g(u)du \int_a^b r^{-p} g(p)dp - 2\int_a^b g(u)\int_a^u r^{-p} g(p)dpdu\right)}\right)^{\frac{1}{t-r}}, & t \neq r,\ t,r \neq 1, \\ \left(\frac{\left(\int_a^b g(u)du \int_a^b t^{-p} g(p)dp - 2\int_a^b g(u)\int_a^u t^{-p} g(p)dpdu\right)}{-\log t\left(\int_a^b g(u)du \int_a^b pg(p)dp - 2\int_a^b g(u)\int_a^u pg(p)dpdu\right)}\right)^{\frac{1}{t-1}}, & t \neq r = 1, \\ \exp\left(\frac{-1}{t\log t} - \frac{\int_a^b g(u)du \int_a^b pt^{-p} g(p)dp - 2\int_a^b g(u)\int_a^u pt^{-p} g(p)dpdu}{t\left(\int_a^b g(u)du \int_a^b t^{-p} g(p)dp - 2\int_a^b g(u)\int_a^u t^{-p} g(p)dpdu\right)}\right), & t = r,\ \ t,r \neq 1, \\ \exp\left(\frac{\int_a^b g(u)du \int_a^b p^2 g(p)dp - 2\int_a^b g(u)\int_a^u p^2 g(p)dpdu}{-2\left(\int_a^b g(u)du \int_a^b pg(p)dp - 2\int_a^b g(u)\int_a^u pg(p)dpdu\right)}\right), & t = r = 1, \end{cases}$$

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