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SCHUR–HARMONIC CONVEXITY FOR DIFFERENCES OF SOME SPECIAL MEANS IN TWO VARIABLES

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Abstract. In the paper, the authors find Schur-harmonic convexity of linear combinations of differences between some means such as the arithmetic, geometric, harmonic, and root-square means, and establish some inequalities related to these means and differences.

1. Introduction

In 2006, the following chain of inequalities for the binary means is given in [5].

THEOREM 1.1. ([5]) *Let* $a,b \in \mathbb{R}_+ = (0,\infty)$. *Then*

$$H(a,b) \leqslant G(a,b) \leqslant N_1(a,b) \leqslant N_3(a,b) \leqslant N_2(a,b) \leqslant A(a,b) \leqslant S(a,b), \tag{1.1}$$

where

$$A(a,b) = \frac{a+b}{2}, \quad G(a,b) = \sqrt{ab}, \quad H(a,b) = \frac{2ab}{a+b}, \quad N_1(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2,$$
$$S(a,b) = \sqrt{\frac{a^2+b^2}{2}}, \quad N_3(a,b) = \frac{a+\sqrt{ab}+b}{3}, \quad N_2(a,b) = \frac{\sqrt{a}+\sqrt{b}}{2}\sqrt{\frac{a+b}{2}}.$$

The means $A(a,b)$, $G(a,b)$, $H(a,b)$, $S(a,b)$, $N_1(a,b)$, and $N_3(a,b)$ are called the arithmetic, geometric, harmonic, root-square, square-root, and Heron means respectively. The mean $N_2(a,b)$ can be found in [4].

In [2, 3, 5, 8], the differences of means

$$M_{SA}(a,b) = S(a,b) - A(a,b), \qquad M_{SN_2}(a,b) = S(a,b) - N_2(a,b), \tag{1.2}$$
$$M_{SN_3}(a,b) = S(a,b) - N_3(a,b), \qquad M_{SN_1}(a,b) = S(a,b) - N_1(a,b), \tag{1.3}$$
$$M_{SG}(a,b) = S(a,b) - G(a,b), \qquad M_{SH}(a,b) = S(a,b) - H(a,b), \tag{1.4}$$
$$M_{AN_2}(a,b) = A(a,b) - N_2(a,b), \qquad M_{AG}(a,b) = A(a,b) - G(a,b), \tag{1.5}$$
$$M_{AH}(a,b) = A(a,b) - H(a,b), \qquad M_{N_2N_1}(a,b) = N_2(a,b) - N_1(a,b), \tag{1.6}$$
$$M_{N_2G}(a,b) = N_2(a,b) - G(a,b), \tag{1.7}$$

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$$M_{AN_3}(a,b)=A(a,b)-N_3(a,b), \qquad M_{AN_1}(a,b)=A(a,b)-N_1(a,b), \tag{1.8}$$
$$M_{N_2N_3}(a,b)=N_2(a,b)-N_3(a,b), \qquad M_{N_2H}(a,b)=N_2(a,b)-H(a,b), \tag{1.9}$$
$$M_{N_3N_1}(a,b)=N_3(a,b)-N_1(a,b), \qquad M_{N_3G}(a,b)=N_3(a,b)-G(a,b), \tag{1.10}$$
$$M_{N_3H}(a,b)=N_3(a,b)-H(a,b), \qquad M_{N_1G}(a,b)=N_1(a,b)-G(a,b), \tag{1.11}$$
$$M_{N_1H}(a,b)=N_1(a,b)-H(a,b), \qquad M_{GH}(a,b)=G(a,b)-H(a,b) \tag{1.12}$$

and

$$D_{SH-SA}(a,b)=\frac{M_{SH}(a,b)}{3}-M_{SA}(a,b), \tag{1.13}$$
$$D_{AH-SH}(a,b)=\frac{M_{AH}(a,b)}{2}-\frac{M_{SH}(a,b)}{3}, \tag{1.14}$$
$$D_{SG-AH}(a,b)=M_{SG}(a,b)-M_{AH}(a,b), \tag{1.15}$$
$$D_{AG-SG}(a,b)=M_{AG}(a,b)-\frac{M_{SG}(a,b)}{2}, \tag{1.16}$$
$$D_{N_2N_1-AH}(a,b)=M_{N_2N_1}(a,b)-\frac{M_{AH}(a,b)}{8}, \tag{1.17}$$
$$D_{N_2G-N_2N_1}(a,b)=\frac{M_{N_2G}(a,b)}{3}-M_{N_2N_1}(a,b), \tag{1.18}$$
$$D_{AG-N_2G}(a,b)=\frac{M_{AG}(a,b)}{4}-\frac{M_{N_2G}(a,b)}{3}, \tag{1.19}$$
$$D_{AN_2-AG}(a,b)=M_{AN_2}(a,b)-\frac{M_{AG}(a,b)}{4}, \tag{1.20}$$
$$D_{SN_2-SA}(a,b)=\frac{4M_{SN_2}(a,b)}{5}-M_{SA}(a,b), \tag{1.21}$$
$$D_{AN_2-SN_2}(a,b)=4M_{AN_2}(a,b)-\frac{4M_{SN_2}(a,b)}{5}, \tag{1.22}$$
$$D_{SN_1-SH}(a,b)=2M_{SN_1}(a,b)-M_{SH}(a,b), \tag{1.23}$$
$$D_{SG-SN_1}(a,b)=\frac{3M_{SG}(a,b)}{2}-2M_{SN_1}(a,b), \tag{1.24}$$
$$D_{SN_3-SA}(a,b)=\frac{3M_{SN_3}(a,b)}{4}-M_{SA}(a,b), \tag{1.25}$$
$$D_{SN_1-SN_3}(a,b)=\frac{2M_{SN_1}(a,b)}{3}-\frac{3M_{SN_3}(a,b)}{4}. \tag{1.26}$$

were considered and obtained the following theorems.

THEOREM 1.2. ([5]) *The differences of means defined by* (1.2) *to* (1.7) *are non-negative and convex in* $\mathbb{R}_+^2$.

THEOREM 1.3. ([3]) *The differences given in* (1.2) *to* (1.7) *are Schur-geometrically convex in* $\mathbb{R}_+^2$.

THEOREM 1.4. ([8, Theorem 3.1]) *The differences of means listed in equations* (1.2) *to* (1.12) *are Schur-harmonically convex on* $\mathbb{R}^2_+$.

THEOREM 1.5. ([2]) *The differences given by* (1.13) *to* (1.26) *are Schur-geometrically convex in* $\mathbb{R}^2_+$.

In [7], some inequalities for differences of power means in two variables were obtained.

In this paper, we will prove that linear combinations of differences (1.13) to (1.26) are Schur-harmonic in $\mathbb{R}^2_+$ and establish some inequalities of these differences of means.

2. Definitions and a lemma

In order to verify our main results, we need the following definitions and lemma.

It is general knowledge that a set $\Omega \subseteq \mathbb{R}^n$ is said to be convex if

$$\lambda x+(1-\lambda)y=(\lambda x_1+(1-\lambda)y_1,\ldots,\lambda x_n+(1-\lambda)y_n)\in\Omega$$

for every $x,y\in\Omega$ and $\lambda\in[0,1]$.

DEFINITION 2.1. ([1, p. 8 and p. 80] and [6]) Let

$$x=(x_1,\ldots,x_n)\in\mathbb{R}^n \quad \text{and} \quad y=(y_1,\ldots,y_n)\in\mathbb{R}^n.$$

1. The tuple x is said to be majorized by y (in symbols $x\prec y$) if
$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]} \tag{2.1}$$
for $k=1,2,\ldots,n-1$ and
$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i, \tag{2.2}$$
where $x_{[1]}\geqslant\cdots\geqslant x_{[n]}$ and $y_{[1]}\geqslant\cdots\geqslant y_{[n]}$ are rearrangements of x and y in a descending order.

2. A function $\varphi:\Omega\subseteq\mathbb{R}^n\to\mathbb{R}$ is said to be Schur-convex on Ω if $x\prec y$ on Ω implies $\varphi(x)\leqslant\varphi(y)$. A function φ is said to be Schur-concave on Ω if and only if $-\varphi$ is Schur-convex.

DEFINITION 2.2. ([9, Definition 1.3]) Let $\Omega\subset\mathbb{R}^n_+$.

1. A set Ω is said to be harmonically convex if $\frac{xy}{\lambda x+(1-\lambda)y}\in\Omega$ for every $x,y\in\Omega$ and $\lambda\in[0,1]$, where $xy=\sum_{i=1}^n x_iy_i$ and $\frac{1}{x}=\left(\frac{1}{x_1},\cdots,\frac{1}{x_n}\right)$.

2. A function $\varphi:\Omega\to\mathbb{R}_+$ is said to be Schur-harmonically convex on Ω if $\frac{1}{x}\prec\frac{1}{y}$ implies $\varphi(x)\leqslant\varphi(y)$.

LEMMA 2.1. ([9, Lemma 2.4]) *Let* $\Omega \subset \mathbb{R}_+^n$ *be a symmetric and harmonically convex set with inner points and let* $\varphi : \Omega \to \mathbb{R}_+$ *be a continuously symmetric function which is differentiable on* Ω°. *Then* φ *is Schur-harmonically convex on* Ω *if and only if*

$$(x_1 - x_2)\left[x_1^2 \frac{\partial \varphi(x)}{\partial x_1} - x_2^2 \frac{\partial \varphi(x)}{\partial x_2}\right] \geqslant 0, \quad x \in \Omega^\circ. \tag{2.3}$$

3. Main results

Now we start out to state and verify our main results.

THEOREM 3.1. *The differences given by* (1.13) *to* (1.26) *are Schur-harmonically convex functions in* $\mathbb{R}_+^2$.

Proof. It is easy to obtain that

$$\frac{\partial D_{SH-SA}(a,b)}{\partial a} = \frac{1}{2} - \frac{2b^2}{3(a+b)^2} - \frac{2}{3}\frac{a}{\sqrt{2\left(a^2+b^2\right)}}$$

and

$$\frac{\partial D_{SH-SA}(a,b)}{\partial b} = \frac{1}{2} - \frac{2a^2}{3(a+b)^2} - \frac{2}{3}\frac{b}{\sqrt{2\left(a^2+b^2\right)}}.$$

Hence,

$$\begin{aligned}
&(a-b)\left[a^2\frac{\partial D_{SH-SA}(a,b)}{\partial a} - b^2\frac{\partial D_{SH-SA}(a,b)}{\partial b}\right] \\
&= \frac{2(a-b)^2}{3}\left[\frac{3(a+b)}{4} - \frac{a^2+ab+b^2}{\sqrt{2\left(a^2+b^2\right)}}\right] \\
&= \frac{\sqrt{2}\,(a-b)^2\left[a^4+2a^3b+2ab^3+b^4-6a^2b^2\right]}{6\sqrt{a^2+b^2}\left[3\sqrt{2}\,(a+b)\sqrt{a^2+b^2}+4\left(a^2+ab+b^2\right)\right]} \\
&\geqslant 0,
\end{aligned}$$

Thus, by Lemma 2.1, it follows that D_{SH-SA} is Schur-harmonically convex in $\mathbb{R}_+^2$.

Since

$$\frac{\partial D_{SG-AH}(a,b)}{\partial a} = \frac{a}{\sqrt{2\left(a^2+b^2\right)}} - \frac{b}{2\sqrt{ab}} - \frac{1}{2} + \frac{2b^2}{(a+b)^2}$$

and

$$\frac{\partial D_{SG-AH}(a,b)}{\partial b} = \frac{b}{\sqrt{2\left(a^2+b^2\right)}} - \frac{a}{2\sqrt{ab}} - \frac{1}{2} + \frac{2a^2}{(a+b)^2},$$

we have

$$
\begin{aligned}
&(a-b)\left[a^2\frac{\partial D_{SG-AH}(a,b)}{\partial a}-b^2\frac{\partial D_{SG-AH}(a,b)}{\partial b}\right]\\
&=\frac{(a-b)^2}{2}\left[\frac{\sqrt{2}\left(a^2+ab+b^2\right)}{\sqrt{a^2+b^2}}-\left(a+\sqrt{ab}+b\right)\right]\\
&=\frac{(a-b)^2[\left(a^2+b^2\right)^2+ab(a+b)^2-2\sqrt{ab}\,(a+b)\left(a^2+b^2\right)]}{2\sqrt{a^2+b^2}\left[\sqrt{2}\,(a^2+ab+b^2)+\left(a+b+\sqrt{ab}\right)\sqrt{a^2+b^2}\right]}\\
&\geqslant 0.
\end{aligned}
$$

Therefore, by Lemma 2.1, it follows that D_{SG-AH} is Schur-harmonically convex in $\mathbb{R}_+^2$.

Because

$$
\frac{\partial D_{AG-SG}(a,b)}{\partial a}=\frac{1}{2}\left[1-\frac{b}{2\sqrt{ab}}-\frac{a}{\sqrt{2\left(a^2+b^2\right)}}\right]
$$

and

$$
\frac{\partial D_{AG-SG}(a,b)}{\partial b}=\frac{1}{2}\left[1-\frac{a}{2\sqrt{ab}}-\frac{b}{\sqrt{2\left(a^2+b^2\right)}}\right],
$$

we have

$$
\begin{aligned}
&(a-b)\left[a^2\frac{\partial D_{AG-SG}(a,b)}{\partial a}-b^2\frac{\partial D_{AG-SG}(a,b)}{\partial b}\right]\\
&=\frac{(a-b)^2}{2}\left[a+b-\frac{\sqrt{ab}}{2}-\frac{a^2+ab+b^2}{\sqrt{2\left(a^2+b^2\right)}}\right]\\
&\geqslant\frac{(a-b)^2}{2}\left[\frac{3(a+b)}{4}-\frac{a^2+ab+b^2}{\sqrt{2\left(a^2+b^2\right)}}\right]\\
&\geqslant 0.
\end{aligned}
$$

By Lemma 2.1, it follows that D_{AG-SG} is Schur-harmonically convex in $\mathbb{R}_+^2$.

Since

$$
\begin{aligned}
\frac{\partial D_{N_2N_1-AH}(a,b)}{\partial a}=&\frac{1}{4\sqrt{a}}\sqrt{\frac{a+b}{2}}+\frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}\\
&-\frac{1}{4}-\frac{b}{4\sqrt{ab}}-\frac{1}{8}\left[\frac{1}{2}-\frac{2b^2}{(a+b)^2}\right]
\end{aligned}
$$

and

$$\frac{\partial D_{N_2N_1-AH}(a,b)}{\partial b}=\frac{1}{4\sqrt{b}}\sqrt{\frac{a+b}{2}}+\frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}-\frac{1}{4}-\frac{a}{4\sqrt{ab}}-\frac{1}{8}\left[\frac{1}{2}-\frac{2a^2}{(a+b)^2}\right],$$

we have

$$\begin{aligned}&(a-b)\left[a^2\frac{\partial D_{N_2N_1-AH}(a,b)}{\partial a}-b^2\frac{\partial D_{N_2N_1-AH}(a,b)}{\partial b}\right]\\&=\frac{(a-b)^2}{16\left(\sqrt{a}+\sqrt{b}\right)}\left[2\sqrt{2}\sqrt{a+b}\left(2a+3\sqrt{ab}+2b\right)-\left(\sqrt{a}+\sqrt{b}\right)\left(5a+4\sqrt{ab}+5b\right)\right].\end{aligned}$$

Because

$$2\sqrt{2}\sqrt{a+b}\left(2a+3\sqrt{ab}+2b\right)-\left(\sqrt{a}+\sqrt{b}\right)\left(5a+4\sqrt{ab}+5b\right)\geqslant 0,$$

that is,

$$7\left(a^3+b^3\right)+6\left(a^2+b^2\right)\sqrt{ab}\geqslant 3\left(a^2b+ab^2\right)+20ab\sqrt{ab},$$

we have

$$(a-b)\left[a^2\frac{\partial D_{N_2N_1-AH}(a,b)}{\partial a}-b^2\frac{\partial D_{N_2N_1-AH}(a,b)}{\partial b}\right]\geqslant 0.$$

From Lemma 2.1, it follows that $D_{N_2N_1-AH}(a,b)$ is Schur-harmonically convex in $\mathbb{R}^2_+$.

It is not difficult to obtain that

$$\frac{\partial D_{N_2G-N_2N_1}(a,b)}{\partial a}=\frac{1}{4}+\frac{b}{12\sqrt{ab}}-\frac{1}{6\sqrt{a}}\sqrt{\frac{a+b}{2}}-\frac{1}{6}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}$$

and

$$\frac{\partial D_{N_2G-N_2N_1}(a,b)}{\partial b}=\frac{1}{4}+\frac{a}{12\sqrt{ab}}-\frac{1}{6\sqrt{b}}\sqrt{\frac{a+b}{2}}-\frac{1}{6}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}.$$

Consequently,

$$\begin{aligned}&(a-b)\left[a^2\frac{\partial D_{N_2G-N_2N_1}(a,b)}{\partial a}-b^2\frac{\partial D_{N_2G-N_2N_1}}{\partial b}\right]\\&=\frac{(a-b)^2}{12\left(\sqrt{a}+\sqrt{b}\right)}\left[\left(\sqrt{a}+\sqrt{b}\right)\left(3a+\sqrt{ab}+3b\right)-\sqrt{2(a+b)}\left(2a+3\sqrt{ab}+2b\right)\right].\end{aligned}$$

It is clear that replacing $\sqrt{a}$ and $\sqrt{b}$ by a and b in

$$\left(\sqrt{a}+\sqrt{b}\right)\left(3a+\sqrt{ab}+3b\right)-\sqrt{2(a+b)}\left(2a+3\sqrt{ab}+2b\right)\geqslant 0$$

reduces to

$$a^3+b^3+2ab\sqrt{ab} \geqslant 2a^2b+2ab^2.$$

Let $f(u)=u^6-2u^4+2u^3-2u^2+1$ for $u \geqslant 1$. Then

$$f'(u)=6u^5-8u^3+6u^2-4u, \quad f''(u)=30u^4-24u^2+12u-4>0,$$

and $f'(u) \geqslant f'(1)=0$ for $u \geqslant 1$. So $f(u) \geqslant f(1)=0$ for $u \geqslant 1$. Without loss of generality, assume that $b \geqslant a$ and let $u=\sqrt{\frac{b}{a}}$, then $f(u) \geqslant 0$ becomes

$$a^3+b^3+2ab\sqrt{ab}-2a^2b-2ab^2 \geqslant 0.$$

As a result,

$$(a-b)\left[a^2\frac{\partial D_{N_2G-N_2N_1}(a,b)}{\partial a}-b^2\frac{\partial D_{N_2G-N_2N_1}}{\partial b}\right] \geqslant 0.$$

From Lemma 2.1, it follows that $D_{N_2G-N_2N_1}(a,b)$ is Schur-harmonically convex in $\mathbb{R}_+^2$.

Since

$$\frac{\partial D_{SN_2-SA}(a,b)}{\partial a}=\frac{1}{2}-\frac{a}{5\sqrt{2(a^2+b^2)}}-\frac{1}{5}\sqrt{\frac{a+b}{2a}}-\frac{\sqrt{a}+\sqrt{b}}{5\sqrt{2(a+b)}}$$

and

$$\frac{\partial D_{SN_2-SA}(a,b)}{\partial b}=\frac{1}{2}-\frac{b}{5\sqrt{2(a^2+b^2)}}-\frac{1}{5}\sqrt{\frac{a+b}{2b}}-\frac{\sqrt{a}+\sqrt{b}}{5\sqrt{2(a+b)}},$$

we have

$$\begin{aligned}
&(a-b)\left[a^2\frac{\partial D_{SN_2-SA}(a,b)}{\partial a}-b^2\frac{\partial D_{SN_2-SA}(a,b)}{\partial b}\right]\\
&=\frac{\sqrt{2}(a-b)^2}{20\left(\sqrt{a}+\sqrt{b}\right)\sqrt{a^2+b^2}}\Big[5\sqrt{2}(a+b)\left(\sqrt{a}+\sqrt{b}\right)\sqrt{a^2+b^2}\\
&\quad-2\left(\sqrt{a}+\sqrt{b}\right)\left(a^2+ab+b^2\right)-2\sqrt{a+b}\sqrt{a^2+b^2}\left(2a+3\sqrt{ab}+2b\right)\Big]\\
&=\frac{\sqrt{2}(a-b)^2}{40\left(\sqrt{a}+\sqrt{b}\right)\sqrt{a^2+b^2}}\Big\{\left(\sqrt{a}+\sqrt{b}\right)\left[3\sqrt{2}(a+b)\sqrt{a^2+b^2}-4\left(a^2+ab+b^2\right)\right]\\
&\quad+\sqrt{a^2+b^2}\left[7\sqrt{2}(a+b)\left(\sqrt{a}+\sqrt{b}\right)-4\sqrt{a+b}\left(2a+3\sqrt{ab}+2b\right)\right]\Big\}\\
&\geqslant 0.
\end{aligned}$$

By Lemma 2.1, it follows that $D_{SN_2-SA}(a,b)$ is Schur-harmonically convex in $\mathbb{R}_+^2$.

Notice that

$$D_{AH-SH}(a,b)=\frac{D_{SH-SA}(a,b)}{2}, \qquad D_{AN_2-AG}(a,b)=3D_{AG-N_2G}(a,b),$$

$$D_{SG-SN_1}(a,b)=D_{AG-SG}(a,b), \qquad D_{AG-N_2G}(a,b)=\frac{D_{N_2G-N_2N_1}(a,b)}{2},$$

$$D_{AN_2-SN_2}(a,b) = 4D_{SN_2-SA}(a,b), \qquad D_{SN_1-SH}(a,b) = D_{SG-AH}(a,b),$$
$$D_{SN_3-SA}(a,b) = \frac{D_{AG-SG}(a,b)}{2}, \qquad D_{SN_1-SN_3}(a,b) = \frac{D_{AG-SG}(a,b)}{6}.$$

So, the differences

$$D_{AH-SH}(a,b), \quad D_{AN_2-AG}(a,b), \quad D_{SG-SN_1}(a,b), \quad D_{AG-N_2G}(a,b),$$
$$D_{AN_2-SN_2}(a,b), \quad D_{SN_1-SH}(a,b), \quad D_{SN_3-SA}(a,b), \quad D_{SN_1-SN_3}(a,b)$$

are Schur-harmonically convex in $\mathbb{R}_+^2$. The proof of Theorem 3.1 is complete. □

COROLLARY 3.1. *For $a,b \in \mathbb{R}_+$ and $0 \leqslant t \leqslant 1$, we have*

$$A(a,b) - \frac{2}{3}S(a,b) \geqslant \frac{A(p_{a,b}(t),q_{a,b}(t)) - \frac{2}{3}S(p_{a,b}(t),q_{a,b}(t))}{p_{a,b}(t)q_{a,b}(t)} \geqslant 0, \tag{3.1}$$

$$S(a,b) - 2N_1(a,b) \geqslant \frac{S(p_{a,b}(t),q_{a,b}(t)) - 2N_1(p_{a,b}(t),q_{a,b}(t))}{p_{a,b}(t)q_{a,b}(t)} \geqslant 0, \tag{3.2}$$

$$\begin{aligned} &A(a,b) - \frac{1}{2}G(a,b) - \frac{1}{2}S(a,b) \\ &\geqslant \frac{A(p_{a,b}(t),q_{a,b}(t)) - \frac{1}{2}G(p_{a,b}(t),q_{a,b}(t)) - \frac{1}{2}S(p_{a,b}(t),q_{a,b}(t))}{p_{a,b}(t)q_{a,b}(t)} \\ &\geqslant 0, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &N_2(a,b) - \frac{5}{8}A(a,b) - \frac{1}{2}G(a,b) \\ &\geqslant \frac{N_2(p_{a,b}(t),q_{a,b}(t)) - \frac{5}{8}A(p_{a,b}(t),q_{a,b}(t)) - \frac{1}{2}G(p_{a,b}(t),q_{a,b}(t))}{p_{a,b}(t)q_{a,b}(t)} \\ &\geqslant 0, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\frac{1}{2}A(a,b) + \frac{1}{6}G(a,b) - \frac{2}{3}N_2(a,b) \\ &\geqslant \frac{\frac{1}{2}A(p_{a,b}(t),q_{a,b}(t)) + \frac{1}{6}G(p_{a,b}(t),q_{a,b}(t)) - \frac{2}{3}N_2(p_{a,b}(t),q_{a,b}(t))}{p_{a,b}(t)q_{a,b}(t)} \\ &\geqslant 0, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &A(a,b) - \frac{1}{5}S(a,b) - \frac{4}{5}N_2(a,b) \\ &\geqslant \frac{A(p_{a,b}(t),q_{a,b}(t)) - \frac{1}{5}S(p_{a,b}(t),q_{a,b}(t)) - \frac{4}{5}N_2(p_{a,b}(t),q_{a,b}(t))}{p_{a,b}(t)q_{a,b}(t)} \\ &\geqslant 0, \end{aligned} \tag{3.6}$$

where $p_{a,b}(t) = \frac{t}{a} + \frac{1-t}{b}$ *and* $q_{a,b}(t) = \frac{1-t}{a} + \frac{t}{b}$.

Proof. We only prove the inequality (3.1), since the rest can be proved similarly. It is easy to see that

$$\left(\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right),\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\right) \prec \left(\frac{t}{a}+\frac{1-t}{b},\frac{1-t}{a}+\frac{t}{b}\right) \prec \left(\frac{1}{a},\frac{1}{b}\right), \quad 0 \leqslant t \leqslant 1.$$

By Theorem 3.1, the difference D_{SH-SA} is Schur-harmonically convex in $\mathbb{R}^2_+$. Hence,

$$\begin{aligned} D_{SH-SA}(a,b) &= \frac{a+b}{2} - \frac{2ab}{3(a+b)} - \frac{2}{3}\sqrt{\frac{a^2+b^2}{2}} \\ &\geqslant \frac{1}{2}\left[\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}+\left(\frac{1-t}{a}+\frac{t}{b}\right)^{-1}\right] - \frac{2ab}{3(a+b)} \\ &\quad - \frac{2}{3}\sqrt{\frac{1}{2}\left[\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2}+\left(\frac{1-t}{a}+\frac{t}{b}\right)^{-2}\right]} \\ &\geqslant 0 \end{aligned}$$

which is equivalent to the inequality (3.1). The proof of Corollary 3.1 is complete. □

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