n–EXPONENTIAL CONVEXITY OF SOME DYNAMIC HARDY–TYPE FUNCTIONALS

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Abstract. Recently, some dynamic Hardy-type inequalities with certain kernels are studied in [5] with the help of arbitrary time scales. We use the positive linear functionals obtained from the results of [5] to give non trivial examples of \( n \)-exponentially convex functions.

1. Introduction and preliminary results

The theory of time scales can be studied in [1, 2, 3] and the well-known Hardy inequality as presented in [7] is investigated in [6, 11, 12, 13] under more general settings. Some of Hardy-type inequalities are extended on time scales (see [15, 16, 20]).

We start with some notions of time scales. Any nonempty closed subset of \( \mathbb{R} \) is called a time scale \( \mathbb{T} \). A time scale \( \mathbb{T} \) may or may not be connected, keeping in mind the disconnection of time scales the forward and backward jump operators \( \sigma, \rho : \mathbb{T} \rightarrow \mathbb{T} \) are defined by

\[
\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.
\]

In general, \( \sigma(t) \geq t \) and \( \rho(t) \leq t \). The mappings \( \mu, \nu : \mathbb{T} \rightarrow [0, +\infty) \) defined by

\[
\mu(t) = \sigma(t) - t
\]

and

\[
\nu(t) = t - \rho(t)
\]

are called, respectively, the forward and backward graininess functions. For further properties including the concept of delta differentiation, we refer the reader to [3, 4].

Let \( n \in \mathbb{N} \) be fixed. For each \( i \in \{1, 2, \ldots, n\} \), let \( \mathbb{T}_i \) denote a time scale and let \( \sigma_i, \rho_i \) and \( \Delta_i \) denote the forward jump operator, the backward jump operator, and the delta differentiation operator, respectively. Let us set

\[
\Omega^n = \{a = (a_1, a_2, \ldots, a_n) : a_i \in \mathbb{T}_i, 1 \leq i \leq n\}.
\]


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We call $\Omega^n$ an $n$-dimensional time scale.

If $a \in \mathbb{T}$, where $\mathbb{T}$ is an arbitrary time scale, then the set $[a, \infty) = \{ t \in \mathbb{T} : a \leq t \}$ is $\Delta$-measurable.

An extended real-valued function $f : \Omega^n \to \mathbb{R} = [-\infty, \infty]$ is $\Delta$-measurable if for every $\alpha \in \mathbb{R}$, the set $f^{-1}((-\infty, \alpha)) = \{ t \in \Omega^n : f(t) < \alpha \}$ is $\Delta$-measurable. Note that $f$ is $\Delta$-measurable iff for each open set $G \subset \mathbb{R}$, the set $f^{-1}(G) = \{ t \in \Omega^n : f(t) \in G \}$ is $\Delta$-measurable. Moreover, if $f : \Omega^n \to \mathbb{R}$ is $\Delta$-measurable and $g : I \to \mathbb{R}$ with $I \subset \mathbb{R}$ is a continuous function, then $g \circ f : \Omega^n \to \mathbb{R}$ is $\Delta$-measurable.

Let $V = (a, b)$ be an $n$-dimensional time scale interval in $\Omega^n$ and let $f$ be a bounded real-valued function on $V$. If $f$ is Riemann $\Delta$-integrable over $V$, then $f$ is Lebesgue $\Delta$-integrable over $V$ and

$$R \int_V f(t) \Delta t = L \int_V f(t) \Delta t,$$

where $R$ and $L$ indicate the Riemann and Lebesgue $\Delta$-integrals, respectively. In particular, if $(a, b) \subset \mathbb{T}$ contains only isolated points, then

$$\int_a^b f(t) \Delta t = \sum_{t \in [a, b)} (\sigma(t) - t) f(t),$$

where $\mathbb{T}$ is an arbitrary time scale.

Let $(\Omega, \mathcal{M}, \mu_\Delta)$ and $(\Lambda, \mathcal{L}, \lambda_\Delta)$ be two finite dimensional time scale measure spaces. We consider the measure space $(\Omega \times \Lambda, \mathcal{M} \times \mathcal{L}, \mu_\Delta \times \lambda_\Delta)$, where $\mathcal{M} \times \mathcal{L}$ is $\sigma$-algebra product generated by the family $\{E \times F : E \in \mathcal{M}, F \in \mathcal{L} \}$ and

$$(\mu_\Delta \times \lambda_\Delta)(E \times F) = \mu_\Delta(E) \lambda_\Delta(F).$$

Recently in [5] the following extension of Hardy-type inequality for arbitrary time scales is constructed.

**Theorem 1.1.** Assume

$$(\Omega, \mathcal{M}, \mu_\Delta) \text{ and } (\Lambda, \mathcal{L}, \lambda_\Delta) \text{ are two time scale measure spaces,}$$

$$k : \Omega \times \Lambda \to \mathbb{R}_+ \text{ is such that } K(x) := \int_\Lambda k(x, y) \Delta y < \infty, \ x \in \Omega$$

and

$$\xi : \Omega \to \mathbb{R}_+ \text{ is such that } w(y) := \int_\Omega \frac{k(x, y) \xi(x)}{K(x)} \Delta x < \infty, \ y \in \Lambda.$$  

If $\Phi \in C(I, \mathbb{R})$ is convex, where $I \subset \mathbb{R}$ is an interval, then

$$\int_\Omega \xi(x) \Phi((A_k f)(x)) \Delta x \leq \int_\Lambda w(y) \Phi(f(y)) \Delta y.$$
holds for all $\lambda_\Delta$-integrable $f : \Lambda \to \mathbb{R}$ such that $f(\Lambda) \subset I$, where

$$(A_k f)(x) = \frac{1}{K(x)} \int_{\Lambda} k(x, y) f(y) \Delta y.$$  

In the following, the entries of a vector $x \in \mathbb{R}^n$ are called $x_i$, where $1 \leq i \leq n$.

**THEOREM 1.2.** Let $\mathbb{T}$ be a time scale and assume

$$a_i, b_i \in \mathbb{T}, \ 0 \leq a_i < b_i \leq \infty, \ 1 \leq i \leq n, \ \Omega = \Lambda := \times_{i=1}^n [a_i, b_i) \mathbb{T}, \quad (2)$$

and

$$u : \Omega \to \mathbb{R}_+ \text{ is such that } v(y) := \int_{\Omega} \frac{y_1 \cdots y_n k(x, y) u(x)}{\sigma(x_1) \cdots \sigma(x_n) K(x)} \Delta x < \infty, \ y \in \Lambda. \quad (5)$$

If $\Phi \in C(I, \mathbb{R})$ is convex, where $I \subset \mathbb{R}$ is an interval, then

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(x) \Phi((\hat{A}_k f)(x)) \frac{\Delta x_1 \cdots \Delta x_n}{\sigma(x_1) \cdots \sigma(x_n)} \leq \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} v(y) \Phi(f(y)) \frac{\Delta y_1 \cdots \Delta y_n}{y_1 \cdots y_n}$$

holds for all $\lambda_\Delta$-integrable $f : \Lambda \to \mathbb{R}$ such that $f(\Lambda) \subset I$, where

$$(\hat{A}_k f)(x) := \frac{1}{K(x)} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} k(x, y) f(y) \Delta y_1 \cdots \Delta y_n.$$  

**COROLLARY 1.3.** Assume (4), (2) and (5) with the kernel $k$ such that

$$k(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0 \quad \text{if} \quad a_i \leq \sigma(x_i) \leq y_i \leq b_i, \ 1 \leq i \leq n.$$  

If $\Phi \in C(I, \mathbb{R})$ is convex, where $I \subset \mathbb{R}$ is an interval, then (6) holds for all $\lambda_\Delta$-integrable $f : \Lambda \to \mathbb{R}$ such that $f(\Lambda) \subset I$, where

$$K(x) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} k(x_1, \ldots, x_n, y_1, \ldots, y_n) \Delta y_1 \cdots \Delta y_n,$$

$$v(y) = y_1 \cdots y_n \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \frac{k(x_1, \ldots, x_n, y_1, \ldots, y_n) u(x_1, \ldots, x_n)}{\sigma(x_1) \cdots \sigma(x_n) K(x_1, \ldots, x_n)} \Delta x_1 \cdots \Delta x_n$$

and

$$(\hat{A}_k f)(x) = \frac{1}{K(x)} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} k(x, y_1, \ldots, y_n) f(y_1, \ldots, y_n) \Delta y_1 \cdots \Delta y_n.$$  

**THEOREM 1.4.** ([5]) Assume (4) and

$\xi : \Omega \to \mathbb{R}_+$ is such that $\bar{w}(y) := \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \frac{\xi(x_1, \ldots, x_n)}{\prod_{i=1}^n (\sigma(x_i) - a_i)} \Delta x_1 \cdots \Delta x_n < \infty, \ y \in \Lambda.$
If \( \Phi \in C(I, \mathbb{R}) \) is convex, where \( I \subset \mathbb{R} \) is an interval, then
\[
\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \xi(x_1, \ldots, x_n) \Phi \left( \left( \tilde{A}f \right)(x_1, \ldots, x_n) \right) \Delta x_1 \cdots \Delta x_n \\
\leq \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \bar{w}(y_1, \ldots, y_n) \Phi (f(y_1, \ldots, y_n)) \Delta y_1 \cdots \Delta y_n
\]
holds for all \( \lambda_\Delta \)-integrable \( f : \Lambda \to \mathbb{R} \) such that \( f(\Lambda) \subset I \), where
\[
(\tilde{A}f)(x) := \frac{1}{n} \prod_{i=1}^{n} \left( \sigma(x_i) - a_i \right) \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_n}^{\sigma(x_n)} f(y_1, \ldots, y_n) \Delta y_1 \cdots \Delta y_n.
\]

**Remark 1.5.** Under the considerations of Theorem 1.1, we have
\[
\Upsilon_1(\Phi) := \int_{\Lambda} w(y) \Phi (f(y)) \Delta y - \int_{\Omega} \xi(x) \Phi (A_k f)(x) \Delta x \geq 0.
\]

From Theorem 1.2, we have
\[
\Upsilon_2(\Phi) := \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} v(y) \Phi (f(y)) \frac{\Delta y_1 \cdots \Delta y_n}{y_1 \cdots y_n} - \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(x) \Phi (\left( \tilde{A}f \right)(x)) \frac{\Delta x_1 \cdots \Delta x_n}{\sigma(x_1) \cdots \sigma(x_n)} \geq 0.
\]

From Theorem 1.4, we have
\[
\Upsilon_3(\Phi) := \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \bar{w}(y_1, \ldots, y_n) \Phi (f(y_1, \ldots, y_n)) \Delta y_1 \cdots \Delta y_n - \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \xi(x_1, \ldots, x_n) \Phi \left( \left( \tilde{A}f \right)(x_1, \ldots, x_n) \right) \Delta x_1 \cdots \Delta x_n \geq 0.
\]

For simplicity, we use \( \Upsilon(\Phi) \) instead of \( \Upsilon_i(\Phi) \) \( \forall \ i \in \{1, 2, 3\} \).

Hence, for any convex function \( \Phi \in C(I, \mathbb{R}) \),
\[
\Upsilon(\Phi) \geq 0.
\]

**2. n-exponential convexity**

The notion of \( n \)-exponentially convex function and the following properties of \( n \) -exponentially convex function defined on an interval \( I \subset \mathbb{R} \), are given in [17].

**Definition 1.** A function \( g : I \to \mathbb{R} \) is called \( n \)-exponentially convex in the Jensen sense if
\[
\sum_{i,j=1}^{n} a_i a_j g \left( \frac{x_i + x_j}{2} \right) \geq 0
\]
holds for every \( a_i \in \mathbb{R} \) and every \( x_i \in I, i \in \{1, 2, \ldots, n\} \).

A function \( g : I \to \mathbb{R} \) is \( n \)-exponentially convex if it is \( n \)-exponentially convex in the Jensen sense and continuous on \( I \).

**Remark 2.1.** From the definition it is clear that \( 1 \)-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, \( n \)-exponentially convex functions in the Jensen sense are \( m \)-exponentially convex in the Jensen sense for every \( m \in \mathbb{N}, m \leq n \).

**Definition 2.** A function \( g : I \to \mathbb{R} \) is exponentially convex in the Jensen sense, if it is \( n \)-exponentially convex in the Jensen sense for all \( n \in \mathbb{N} \). A function \( g : I \to \mathbb{R} \) is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

**Remark 2.2.** It is easy to see that a positive function \( g : I \to \mathbb{R} \) is log-convex in the Jensen sense if and only if it is \( 2 \)-exponentially convex in the Jensen sense, that is

\[
 a_1^2 g(x) + 2a_1a_2 g \left( \frac{x + y}{2} \right) + a_2^2 g(y) \geq 0
\]

holds for every \( a_1, a_2 \in \mathbb{R} \) and \( x, y \in I \).

Similarly, if \( g \) is \( 2 \)-exponentially convex, then \( g \) is log-convex. Conversely, if \( g \) is log-convex and continuous, then \( g \) is \( 2 \)-exponentially convex.

Divided differences are fertile to study functions having different degree of smoothness.

**Definition 3.** The second order divided difference of a function \( g : I \to \mathbb{R} \) at mutually different points \( y_0, y_1, y_2 \in I \) is defined recursively by

\[
 [y; g] = g(y), \quad i \in \{0, 1, 2\}
\]

\[
 [y, y_{i+1}; g] = \frac{g(y_{i+1}) - g(y_i)}{y_{i+1} - y_i}, \quad i \in \{0, 1\}
\]

\[
 [y_0, y_1, y_2; g] = \frac{[y_1, y_2; g] - [y_0, y_1; g]}{y_2 - y_0}. \quad (8)
\]

**Remark 2.3.** The value \([y_0, y_1, y_2; g] \) is independent of the order of the points \( y_0, y_1, \) and \( y_2 \). By taking limits this definition may be extended to include the cases in which any two or all three points coincide as follows: \( \forall \ y_0, y_1, y_2 \in I \) such that \( y_2 \neq y_0 \)

\[
 \lim_{y_1 \to y_0} [y_0, y_1, y_2; g] = [y_0, y_0, y_2; g] = \frac{g(y_2) - g(y_0) - g'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}
\]
provided that $g'$ exists, and furthermore, taking the limits $y_i \to y_0, i \in \{1, 2\}$ in (8), we get

$$[y_0, y_0, y_0; g] = \lim_{y_i \to y_0} [y_0, y_1, y_2; g] = \frac{g''(y_0)}{2}$$

for $i \in \{1, 2\}$

provided that $g''$ exists on $I$.

In [9], the authors describe the $n$-exponential convexity for the functionals obtained from the inequalities of Hardy and Boas types.

In this paper we utilize the functional $\Upsilon(\Phi)$ given in Remark 1.5 to establish the $n$-exponential convexity via theory of time scales. Therefore our work is a continuation of results in [9].

**Theorem 2.4.** Assume $J \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of continuous functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0, y_1, y_2; \phi_t] \ (t \in J)$ is $n$-exponentially convex in the Jensen sense on $J$ for every three mutually different points $y_0, y_1, y_2 \in I$. Consider $\Upsilon(\Phi)$ as given in Remark 1.5. Then $t \to \Upsilon(\phi_t) \ (t \in J)$ is an $n$-exponentially convex function in the Jensen sense on $J$.

Also the function $t \to \Upsilon(\phi_t)$ (for any $t \in J$) is continuous, therefore it is $n$-exponentially convex on $J$.

**Proof.** Let $t_k, t_l \in J$, $t_{kl} := \frac{t_k + t_l}{2}$ and $b_k, b_l \in \mathbb{R}$ for $k, l \in \{1, 2, \ldots, n\}$, and define the function $\omega$ on $I$ by

$$\omega := \sum_{k, l=1}^n b_k b_l \phi_{t_{kl}}.$$ 

Then $\omega$ is continuous on $I$ being the linear combination of continuous functions. Also by hypothesis the function $t \to [y_0, y_1, y_2; \phi_t] \ (t \in J)$ is $n$-exponentially convex in the Jensen sense, therefore we have

$$[y_0, y_1, y_2; \omega] = \sum_{k, l=1}^n b_k b_l [y_0, y_1, y_2; \phi_{t_{kl}}] \geq 0,$$

which implies that $\omega$ is a convex function on $I$. Therefore we have $\Upsilon(\omega) \geq 0$, which yields by the linearity of $\Upsilon$, that

$$\sum_{k, l=1}^n b_k b_l \Upsilon(\phi_{t_{kl}}) \geq 0.$$

We conclude that the function $t \to \Upsilon(\phi_t) \ (t \in J)$ is an $n$-exponentially convex function in the Jensen sense on $J$. \hspace{1cm} \square

As a consequence of the above theorem we can give the following corollaries.
COROLLARY 2.5. Assume $J \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of continuous functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \mapsto [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is exponentially convex in the Jensen sense on $J$ for every three mutually different points $y_0, y_1, y_2 \in I$. Consider $\Upsilon(\Phi)$ as given in Remark 1.5. Then $t \mapsto \Upsilon(\phi_t)$ $(t \in J)$ is an exponentially convex function in the Jensen sense on $J$.

As the function $t \mapsto \Upsilon(\phi_t)$ $(t \in J)$ is continuous, therefore exponentially convex on $J$.

COROLLARY 2.6. Assume $J \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t : t \in J\}$ is a family of continuous functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \mapsto [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is 2-exponentially convex in the Jensen sense on $J$ for every three mutually different points $y_0, y_1, y_2 \in I$. Consider $\Upsilon(\Phi)$ as given in Remark 1.5. Then the following two statements hold:

(i) As the function $t \mapsto \Upsilon(\phi_t)$ $(t \in J)$ is continuous, therefore it is 2-exponentially convex on $J$, and thus log-convex, i.e.,

$$\Upsilon^{(r-p)}(\phi_q) \leq \Upsilon^{(r-q)}(\phi_p)\Upsilon^{(q-p)}(\phi_r)$$

for $p, q, r \in J$ such that $p < q < r$.

(ii) If the function $t \mapsto \Upsilon(\phi_t)$ $(t \in J)$ is positive, then for every $s, t, u, v \in J$, such that $s \leq u$ and $t \leq v$, we have

$$u_{s,t}(\Upsilon, \Lambda) \leq u_{u,v}(\Upsilon, \Lambda),$$

where

$$u_{s,t}(\Upsilon, \Lambda) := \begin{cases} \left( \frac{\Upsilon(\phi_s)}{\Upsilon(\phi_t)} \right)^{\frac{1}{s-t}}; & s \neq t, \\ \exp \left( \frac{d}{ds} \Upsilon(\phi_s) \right); & s = t \end{cases}$$

for $\phi_s, \phi_t \in \Lambda$. Also we consider that the function $t \mapsto \Upsilon(\phi_t)$ is differentiable when $t = s$.

Proof.

(i) See Remark 2.2 and Theorem 2.4.

(ii) From the definition of a convex function $\psi$ on $J$, we have the following inequality (see [18, page 2])

$$\frac{\psi(s) - \psi(t)}{s - t} \leq \frac{\psi(u) - \psi(v)}{u - v},$$

where $s, t, u, v \in J$ and $s < t < u < v$. Taking $\psi = \Upsilon$ and $\phi_t$, we get

$$\Upsilon(s, \lambda) - \Upsilon(t, \lambda) \leq \Upsilon(u, \lambda) - \Upsilon(v, \lambda),$$

for $\lambda \in (0, 1)$. Thus

$$\Upsilon_{s,t}(\Upsilon, \Lambda) \leq \Upsilon_{u,v}(\Upsilon, \Lambda),$$

for $s \leq u$ and $t \leq v$.

Proof.

(i) See Remark 2.2 and Theorem 2.4.

(ii) From the definition of a convex function $\psi$ on $J$, we have the following inequality (see [18, page 2])

$$\frac{\psi(s) - \psi(t)}{s - t} \leq \frac{\psi(u) - \psi(v)}{u - v},$$

where $s, t, u, v \in J$ and $s < t < u < v$. Taking $\psi = \Upsilon$ and $\phi_t$, we get

$$\Upsilon(s, \lambda) - \Upsilon(t, \lambda) \leq \Upsilon(u, \lambda) - \Upsilon(v, \lambda),$$

for $\lambda \in (0, 1)$. Thus

$$\Upsilon_{s,t}(\Upsilon, \Lambda) \leq \Upsilon_{u,v}(\Upsilon, \Lambda),$$

for $s \leq u$ and $t \leq v$.
∀s,t,u,v ∈ J such that s ≤ u, t ≤ v, s ≠ t, u ≠ v.

By (i), s → \( \Upsilon(\phi_s) \), s ∈ J is log-convex, and hence using \( \psi(s) = \log \Upsilon(\phi_s) \), s ∈ J in (12), we have

\[
\frac{\log \Upsilon(\phi_s) - \log \Upsilon(\phi_t)}{s-t} \leq \frac{\log \Upsilon(\phi_u) - \log \Upsilon(\phi_v)}{u-v}
\]

(13)

for s ≤ u, t ≤ v, s ≠ t, u ≠ v, which is equivalent to (10). For s = t or u = v, (10) follows from (13) by taking limit. □

REMARK 2.7. Note that the results from Theorem 2.4, Corollary 2.5, Corollary 2.6 are valid when two of the points \( y_0, y_1, y_2 \in I \) coincide, say \( y_1 = y_0 \), for a family of differentiable functions \( \phi_t \) such that the function \( t \mapsto [y_0, y_1, y_2; \phi_t] \) is \( n \)-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and moreover, they are also valid when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 2.3 and suitable characterization of convexity.

The following result is given in [8].

THEOREM 2.8. Assume \( J \subset \mathbb{R} \) is an interval, and assume \( \Lambda = \{ \phi_t \mid t \in J \} \) is a family of twice differentiable functions defined on an interval \( I \subset \mathbb{R} \) such that the function \( t \mapsto \phi_t''(x) \) (t ∈ J) is exponentially convex for every fixed \( x \in I \). Then the function \( t \mapsto [y_0, y_1, y_2; \phi_t] \) (t ∈ J) is exponentially convex in the Jensen sense for any three points \( y_0, y_1, y_2 \in I \).

3. Applications to Cauchy means

In this section, first we give the mean value theorems corresponding to the Hardy-type functional \( \Upsilon(\Phi) \) given in Remark 1.5.

THEOREM 3.1. Let \( [a,b] \subset \mathbb{R} \) and consider the linear functional \( \Upsilon(\Phi) \) as defined in Remark 1.5 for \( \Phi = g \in (C^2[a,b], \mathbb{R}) \), then there exists \( \xi \in [a,b] \) such that

\[
\Upsilon(g) = \frac{1}{2} g''(\xi) \Upsilon(x^2).
\]

Proof. The idea of proof is same as given in [8]. □

THEOREM 3.2. Let \( [a,b] \subset \mathbb{R} \) and consider the linear functional \( \Upsilon(\Phi) \) as defined in Remark 1.5 for \( g, h \in C^2[a,b] \). Then there exists \( \xi \in [a,b] \) such that

\[
\frac{\Upsilon(g)}{\Upsilon(h)} = \frac{g''(\xi)}{h''(\xi)},
\]

(14)

provided that \( \Upsilon(h) \neq 0 \).
Proof. The idea of proof is same as given in [8]. □

Suppose that \( \frac{g''}{h''} \) has inverse function. Then (14) gives

\[
\xi = \left( \frac{g''}{h''} \right)^{-1} \left( \frac{\Upsilon(g)}{\Upsilon(h)} \right).
\] (15)

Now, we generate new Cauchy means with the help of some classes of functions from [17].

**Example 3.3.** Assume \( I = \mathbb{R} \) and consider the class of continuous convex functions

\( \Lambda_1 := \{ \phi : \mathbb{R} \to [0, \infty) \mid t \in \mathbb{R} \} \),

where

\[
\phi_t(x) := \begin{cases} 
\frac{1}{t^2} e^{tx}; & t \neq 0, \\
\frac{1}{2} x^2; & t = 0.
\end{cases}
\]

Then \( t \mapsto \phi''_t(x) \ (t \in \mathbb{R}) \) is exponentially convex for every fixed \( x \in \mathbb{R} \) (see [10]), thus by Theorem 2.8, the function \( t \mapsto [y_0, y_1, y_2; \phi_t], \ t \in \mathbb{R} \) is exponentially convex in the Jensen sense for every three mutually different points \( y_0, y_1, y_2 \in \mathbb{R} \).

By applying Corollary 2.5 with \( \Lambda = \Lambda_1 \), we get the exponential convexity of \( t \mapsto \Upsilon(\phi_t) \ (t \in \mathbb{R}) \) in the Jensen sense. This mapping is also differentiable, therefore exponentially convex, and the expression in (11) has the form

\[
u_{s,t}(\Upsilon, \Lambda_1) = \begin{cases} 
\left( \frac{\Upsilon(\phi_s)}{\Upsilon(\phi_t)} \right)^{\frac{1}{s-t}}; & s \neq t, \\
\exp \left( \frac{\Upsilon(id \phi_s) - 2}{s} \right); & s = t \neq 0, \\
\exp \left( \frac{\Upsilon(id \phi_0)}{3\Upsilon(\phi_0)} \right); & s = t = 0,
\end{cases}
\]

where “\( id \)” means the identity function on \( \mathbb{R} \).

From (10) we have the monotonicity of the functions \( u_{s,t}(\Upsilon, \Lambda_1) \) in both parameters.

Suppose \( \Upsilon(\phi_t) > 0 \ (t \in \mathbb{R}) \), and let

\( \mathcal{M}_{s,t}(\Upsilon, \Lambda_1) := \log u_{s,t}(\Upsilon, \Lambda_1); \ s, t \in \mathbb{R} \).

Then from (15) we have

\[ a \leq \mathcal{M}_{s,t}(\Upsilon, \Lambda_1) \leq b, \]

and thus \( \mathcal{M}_{s,t}(\Upsilon, \Lambda_1) \ (s, t \in \mathbb{R}) \) are means. The monotonicity of these means is followed by (10).
Example 3.4. Assume \( I = (0, \infty) \) and consider the class of continuous convex functions
\[
\Lambda_2 = \{ \psi_t : (0, \infty) \to \mathbb{R} \mid t \in \mathbb{R} \},
\]
where
\[
\psi_t(x) := \begin{cases} 
\frac{x'}{t(t-1)}; & t \neq 0, 1, \\
-\log x; & t = 0, \\
x \log x; & t = 1.
\end{cases}
\]

Then \( t \mapsto \psi''_t(x) = x^{r-2} = e^{(t-2) \log x} \) \((t \in \mathbb{R})\) is exponentially convex for every fixed \( x \in (0, \infty) \).

By similar arguments as given in Example 3.3 we get the exponential convexity of \( t \mapsto \Psi_t(x) \) \((t \in \mathbb{R})\) in the Jensen sense. This mapping is differentiable too, therefore exponentially convex. From (11) we have the following Cauchy means

\[
M_{s,t} = \left( \frac{t(t-1)}{s(s-1)} \int_{\Lambda} w(y)(f(y))^{s} \Delta y - \int_{\Omega} \xi(x) (R(x))^{s} \Delta x \right)^{\frac{1}{s}}; \quad s \neq 0, 1,
\]

Then \( s \neq 0, 1, \)

\[
M_{s,0} = \left( -\frac{1}{s(s-1)} \int_{\Lambda} w(y)(f(y))^{s} \Delta y - \int_{\Omega} \xi(x) (R(x))^{s} \Delta x \right)^{\frac{1}{s}}; \quad s \neq 0, 1,
\]

\[
M_{s,1} = \left( \frac{1}{s(s-1)} \int_{\Lambda} w(y)(f(y))^{s} \Delta y - \int_{\Omega} \xi(x) (R(x))^{s} \Delta x \right)^{\frac{1}{s-1}}; \quad s \neq 0, 1,
\]

\[
M_{0,1} = -\frac{\int_{\Lambda} w(y)(f(y)) \Delta y - \int_{\Omega} \xi(x) (R(x)) \Delta x}{\int_{\Lambda} w(y)(f(y)) \Delta y - \int_{\Omega} \xi(x) (R(x)) \Delta x},
\]

\[
M_{1,1} = \exp \left( -1 + \frac{1}{2} \frac{\int_{\Lambda} w(y)(f(y))^2 \Delta y - \int_{\Omega} \xi(x) (R(x))^2 \Delta x}{\int_{\Lambda} w(y)(f(y))^2 \Delta y - \int_{\Omega} \xi(x) (R(x))^2 \Delta x} \right),
\]

\[
M_{0,0} = \exp \left( 1 + \frac{1}{2} \frac{\int_{\Lambda} w(y)(f(y))^2 \Delta y - \int_{\Omega} \xi(x) (R(x))^2 \Delta x}{\int_{\Lambda} w(y)(f(y))^2 \Delta y - \int_{\Omega} \xi(x) (R(x))^2 \Delta x} \right),
\]

where
\[
R(x) \doteq (A_kf)(x) = \frac{1}{K(x)} \int_{\Lambda} k(x,y) f(y) \Delta y
\]

and \( w(y) := \int_{\Omega} \frac{k(x,y) \xi(x)}{K(x)} \Delta x \).

The means \( M_{s,t} \) \((s, t \in \mathbb{R})\) are continuous, symmetric and monotone in both parameters (using (10)).
For the class $\Lambda_2$, we have

$$
Y_1(\phi_p) = \begin{cases} 
\frac{1}{p(p-1)} \left( \int_{\Lambda} w(y) f^p(y) \Delta y - \int_{\Omega} \xi(x)(R(x))^p \Delta x \right); & p \neq 0, 1, \\
- \int_{\Lambda} w(y) \log f(y) \Delta y + \int_{\Omega} \xi(x) \log (R(x)) \Delta x; & p = 0, \\
\int_{\Lambda} w(y) f(y) \log f(y) \Delta y - \int_{\Omega} \xi(x) R(x) \log (R(x)) \Delta x; & p = 1.
\end{cases}
$$

(16)

For (16), (9) gives the following improvement result; for $p = 0 < q < 1 = r$, we have

$$
\frac{1}{q(q-1)} \left( \int_{\Lambda} w(y) f^q(y) \Delta y - \int_{\Omega} \xi(x)(R(x))^q \Delta x \right)
\leq \left( - \int_{\Lambda} w(y) \log f(y) \Delta y + \int_{\Omega} \xi(x) \log (R(x)) \Delta x \right)^{1-q}
\times \left( \int_{\Lambda} w(y) f(y) \log f(y) \Delta y - \int_{\Omega} \xi(x) R(x) \log (R(x)) \Delta x \right)^q.
$$

(17)

If $q < 0 < 1$ or $0 < 1 < q$, then we have reverse inequality in (17).

Observe that (17) is a refinement of inequality given in [5, Corollary 3.3].

Similar results can be written for $i \in \{2, 3\}$.

Particularly, for $i = 3, n = 1$, we can write

$$
Y_3(\phi_p) = \begin{cases} 
\frac{1}{p(p-1)} \left( \int_{a}^{b} \tilde{w}(y) f^p(y) \Delta y - \int_{a}^{b} \tilde{\xi}(x) \left( \tilde{R}(x) \right)^p \Delta x \right); & p \neq 0, 1, \\
- \int_{a}^{b} \tilde{w}(y) \log f(y) \Delta y + \int_{a}^{b} \tilde{\xi}(x) \log \left( \tilde{R}(x) \right) \Delta x; & p = 0, \\
\int_{a}^{b} \tilde{w}(y) f(y) \log f(y) \Delta y - \int_{a}^{b} \tilde{\xi}(x) \tilde{R}(x) \log \left( \tilde{R}(x) \right) \Delta x; & p = 1.
\end{cases}
$$

(18)

where

$$
\tilde{R}(x) = (\tilde{A}_k f)(x) = \frac{1}{\sigma(x) - a} \int_{a}^{\sigma(x)} f(y) \Delta y
$$

and

$$
\tilde{w}(y) = \int_{y}^{\infty} \frac{\tilde{\xi}(x) \Delta x}{\sigma(x) - a}.
$$
For $0 < q < 1$, using (18) in (9) we have the following inequality

$$\frac{1}{q(q-1)} \left( \int_a^b \tilde{w}(y)f^q(y)\Delta y - \int_a^b \xi(x)(\tilde{R}(x))^q \Delta x \right)$$

$$\leq \left( - \int_a^b \tilde{w}(y) \log f(y)\Delta y + \int_a^b \xi(x) \log (\tilde{R}(x)) \Delta x \right)^{1-q} \times \left( \int_a^b \tilde{w}(y)f(y) \log f(y)\Delta y - \int_a^b \xi(x) \tilde{R}(x) \log (\tilde{R}(x)) \Delta x \right)^q.$$  \hspace{1cm} (19)

If $q < 0 < 1$ or $0 < 1 < q$, then we have reverse inequality in (19).

### 4. Applications to time scales consists of isolated points

Now, we consider some particular cases corresponding to examples from [5].

Let us take $\Omega = \Lambda = [a, b] \subseteq (0, \infty) \cap \mathbb{T}$, $a \geq 0$. Further assume that time scale $\mathbb{T}$ consists of isolated points.

In this case (18) takes the form

$$\Gamma_3(\phi_p) = \begin{cases} \frac{1}{p(p-1)} \left( \sum_{[a,b)} \tilde{w}(y)(f(y))^p \mu(y) - \sum_{[a,b)} \xi(x)(\tilde{R}(x))^p \mu(x) \right); & p \neq 0, 1, \\ - \log \left( \prod_{[a,b)} (f(y)) \tilde{w}(y) \mu(y) \right) + \log \left( \prod_{[a,b)} (\tilde{R}(x)) \xi(x) \mu(x) \right); & p = 0, \\ \log \left( \prod_{[a,b)} (f(y)) \tilde{w}(y) f(y) \mu(y) \right) - \log \left( \prod_{[a,b)} (\tilde{R}(x)) \xi(x) \tilde{R}(x) \mu(x) \right); & p = 1, \end{cases}$$  \hspace{1cm} (20)

where

$$\tilde{R}(x) = \left( \frac{1}{\sigma(x) - a} \sum_{y \in [a,x]} f(y) \mu(y) \right)$$

and $\tilde{w}(y) = \left( \sum_{x \in [y, \infty)} \xi(x) \frac{\mu(x)}{\sigma(x) - a} \right)$.

Also, for $0 < q < 1$, (19) takes the form

$$\frac{1}{q(q-1)} \left( \sum_{[a,b)} \tilde{w}(y)(f(y))^q \mu(y) - \sum_{[a,b)} \xi(x)(\tilde{R}(x))^q \mu(x) \right)$$

$$\leq \left( \log \left( \prod_{[a,b)} (f(y)) \tilde{w}(y) \mu(y) \right) \right)^{1-q} \times \left( \log \left( \prod_{[a,b)} (\tilde{R}(x)) \xi(x) \tilde{R}(x) \mu(x) \right) \right)^q.$$  \hspace{1cm} (21)
For $\mathbb{T} = h\mathbb{N} = \{hn : n \in \mathbb{N}\}$ with $h > 0$, $a = h$, and $\xi(x) = \frac{1}{\sigma(x)}$, (20) takes the form

$$\Psi_3(\phi_p) = \begin{cases} \frac{1}{q(q-1)} \left( \sum_{n=1}^{\infty} \frac{(f(nh))^p}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1} \left( \tilde{R}(nh) \right)^p \right); & p \neq 0, 1, \\ -\log \left( \prod_{n=1}^{\infty} \frac{(f(nh))^{1/n}}{n} \right) + \log \left( \prod_{n=1}^{\infty} \left( \tilde{R}(nh) \right)^{1/(n+1)} \right); & p = 0, \\ \log \left( \prod_{n=1}^{\infty} \frac{(f(nh))^{f(nh)}}{n} \right) - \log \left( \prod_{n=1}^{\infty} \left( \tilde{R}(nh) \right)^{f(nh)/(n+1)} \right); & p = 1. \end{cases}$$

For $0 < q < 1$, (21) takes the form

$$\frac{1}{q(q-1)} \left( \sum_{n=1}^{\infty} \frac{(f(nh))^q}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1} \left( \tilde{R}(nh) \right)^q \right) \leq \left( \log \left( \prod_{n=1}^{\infty} \left( \tilde{R}(nh) \right)^{1/(n+1)} \right) \right)^{1-q} \left( \log \left( \prod_{n=1}^{\infty} \frac{(f(nh))^{f(nh)}}{n} \right) \right)^q,$$

where

$$\tilde{R}(nh) = \frac{1}{n} \sum_{k=1}^{n} f(kh).$$

For $\mathbb{T} = \mathbb{N}^2 = \{n^2 : n \in \mathbb{N}\}$ with $a = 1$ and

$$\xi(x) = \frac{2(\sigma(x) - 1)}{(\sigma(x) - x)^2(2\sqrt{x} + 3)},$$

(20) takes the form

$$\Psi_3(\phi_p) = \begin{cases} \frac{1}{p(p-1)} \left( \sum_{n=1}^{\infty} \frac{(f(n^2))^p}{n} - \sum_{n=1}^{\infty} \frac{2n(n+2)}{(2n+1)(2n+3)} \left( \tilde{R}(n^2) \right)^p \right); & p \neq 0, 1, \\ -\log \left( \prod_{n=1}^{\infty} f(n^2) \right) + \log \left( \prod_{n=1}^{\infty} \left( \tilde{R}(n^2) \right)^{2n(n+2)/(2n+1)(2n+3)} \right); & p = 0, \\ \log \left( \prod_{n=1}^{\infty} f(n^2)^{f(n^2)} \right) - \log \left( \prod_{n=1}^{\infty} \left( \tilde{R}(n^2) \right)^{2n(n+2)/(2n+1)(2n+3)} \right); & p = 1. \end{cases}$$
For $0 < q < 1$, (21) takes the form

$$\frac{1}{q(q-1)} \left( \sum_{n=1}^{\infty} (f(n^2))^q - \sum_{n=1}^{\infty} \frac{2n(n+2)}{(2n+1)(2n+3)} (\tilde{R}(n^2))^q \right)$$

$$\leq \left( \log \left( \prod_{n=1}^{\infty} (\tilde{R}(n^2))^{\frac{2n(n+2)}{(2n+1)(2n+3)}} f(n^2) \right) \right)^{1-q} \left( \log \left( \prod_{n=1}^{\infty} \frac{f(n^2) f(n^2)}{(\tilde{R}(n^2))^{\frac{2n(n+2)}{(2n+1)(2n+3)}}} \right) \right)^q,$$

(23)

where

$$\tilde{R}(n^2) = \frac{\sum_{k=1}^{n} (2k+1) f(k^2)}{n(n+2)}.$$

For $T = \lambda^N = \{\lambda^n : n \in \mathbb{N}\}$ with $\lambda > 1$, $a = \lambda$ and

$$\xi(x) = \frac{\sigma(x) - a}{\sigma(x)(\sigma(x) - x)},$$

(20) takes the form

$$\Upsilon_3(\phi_p) = \begin{cases} 
\frac{1}{p(p-1)} \left( \sum_{n=1}^{\infty} (f(\lambda^n))^p - \sum_{n=1}^{\infty} \lambda^{-n}(\lambda^n - 1)(\tilde{R}(\lambda^n))^p \right); & p \neq 0, 1, \\
- \log \left( \prod_{n=1}^{\infty} f(\lambda^n) \right) + \log \left( \prod_{n=1}^{\infty} (\tilde{R}(\lambda^n))^\lambda^{-n}(\lambda^n - 1) \right); & p = 0, \\
\log \left( \prod_{n=1}^{\infty} (f(\lambda^n))^{f(\lambda^n)} \right) - \log \left( \prod_{n=1}^{\infty} (\tilde{R}(\lambda^n))^\lambda^{-n}(\lambda^n - 1)\tilde{R}(\lambda^n) \right); & p = 1.
\end{cases}$$

For $0 < q < 1$, (21) takes the form

$$\frac{1}{q(q-1)} \left( \sum_{n=1}^{\infty} (f(\lambda^n))^q - \sum_{n=1}^{\infty} \lambda^{-n}(\lambda^n - 1)(\tilde{R}(\lambda^n))^q \right)$$

$$\leq \left( \log \left( \prod_{n=1}^{\infty} (\tilde{R}(\lambda^n))^{\lambda^{-n}(\lambda^n - 1)} f(\lambda^n) \right) \right)^{1-q} \left( \log \left( \prod_{n=1}^{\infty} \frac{f(\lambda^n) f(\lambda^n)}{(\tilde{R}(\lambda^n))^{\lambda^{-n}(\lambda^n - 1)\tilde{R}(\lambda^n)}} \right) \right)^q,$$

(24)

where

$$\tilde{R}(\lambda^n) = \frac{(\lambda - 1) \sum_{k=1}^{n} \lambda^{k-1} f(\lambda^k)}{\lambda^n - 1}.$$
**Remark 4.1.** (a) If \( q < 0 < 1 \) or \( 0 < 1 < q \), then we have reverse inequalities in (22), (23) and (24).

(b) The inequalities (22), (23) and (24) are refinements of (5.9), first inequality given in Example 5.11, and (5.11) of [5] respectively.

**Example 4.2.** Assume \( I = (0, \infty) \) and consider the class of continuous convex functions
\[
\Lambda_3 = \{ \eta_t : (0, \infty) \to (0, \infty) \mid t \in (0, \infty) \},
\]
where
\[
\eta_t(x) := \begin{cases} 
  t^{-x} \log^2 t; & t \neq 1, \\
  \frac{x^2}{2}; & t = 1.
\end{cases}
\]

\( t \mapsto \eta''_t(x) \) \( (t \in (0, \infty)) \) is exponentially convex for every fixed \( x \in (0, \infty) \), being the restriction of the Laplace transform of a nonnegative function (see [10] or [19] page 210).

We can get the exponential convexity of \( t \mapsto \Upsilon(\eta_t) \) \( (t \in \mathbb{R}^+) \) as in Example 3.3. For the class \( \Lambda_3 \), (11) has the form
\[
u_{s,t}(\Upsilon, \Lambda_3) = \begin{cases} 
  \left( \frac{\Upsilon(\eta_s)}{\Upsilon(\eta_t)} \right)^{\frac{t}{s}}; & s \neq t, \\
  \exp \left( -\frac{2}{s \log s} - \frac{\Upsilon(id \eta_s)}{s \Upsilon(\eta_s)} \right); & s = t \neq 1, \\
  \exp \left( -\frac{\Upsilon(id \eta_1)}{3 \Upsilon(\eta_1)} \right); & s = t = 1.
\end{cases}
\]

The monotonicity of \( \nu_{s,t}(\Upsilon, \Lambda_3) \) \( (s,t \in (0, \infty)) \) comes from (10).

Suppose \( \Upsilon(\eta_t) > 0 \) \( (t \in (0, \infty)) \), and define
\[
\mathcal{M}_{s,t}(\Upsilon, \Lambda_3) := -L(s,t) \log \nu_{s,t}(\Upsilon, \Lambda_3), \quad s,t \in (0, \infty),
\]
where \( L(s,t) \) is the well known logarithmic mean
\[
L(s,t) := \begin{cases} 
  \frac{s-t}{\log s - \log t}; & s \neq t, \\
  t; & s = t.
\end{cases}
\]

From (15) we have
\[
a \leq \mathcal{M}_{s,t}(\Upsilon, \Lambda_3) \leq b, \quad s,t \in (0, \infty),
\]
and therefore we get means.

**Example 4.3.** Assume \( I = (0, \infty) \) and consider the class of continuous convex functions
\[
\Lambda_4 = \{ \gamma_t : (0, \infty) \to (0, \infty) \mid t \in (0, \infty) \},
\]
where
\[ \gamma(x) := e^{-x\sqrt{t}}. \]

\[ t \mapsto \gamma''(x) = e^{-x\sqrt{t}}, \quad t \in (0, \infty) \]
is exponentially convex for every fixed \( x \in (0, \infty) \), being the restriction of the Laplace transform of a non-negative function (see [10] or [19] page 214).

As before \( t \mapsto \Upsilon(\psi_t) \) \( (t \in \mathbb{R}^+) \) is exponentially convex and differentiable. For the class \( \Lambda_4 \), (11) becomes

\[
\begin{cases}
\left( \frac{\Upsilon(\gamma_s)}{\Upsilon(\gamma_t)} \right)^{\frac{1}{s-t}}; & s \neq t, \\
\exp \left( -\frac{1}{t} - \frac{\Upsilon(id \gamma)}{2\sqrt{t}\Upsilon(\gamma)} \right); & s = t,
\end{cases}
\]

where ‘\( id \)’ means the identity function on \( (0, \infty) \). The monotonicity of \( u_{s,t}(\Upsilon, \Lambda_4) \) \( (s, t \in (0, \infty)) \) is followed by (10).

Suppose \( \Upsilon(\eta_t) > 0 \) \( (t \in (0, \infty)) \) and define

\[
M_{s,t}(\Upsilon, \Lambda_4) := -\left( \sqrt{s} + \sqrt{t} \right) \log u_{s,t}(\Upsilon, \Lambda_4), \quad s, t \in (0, \infty).
\]

Then (15) yields that
\[ a \leq M_{s,t}(\Upsilon, \Lambda_4) \leq b, \]
thus we have new means.

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