Journal of
Mathematical
Inequalities
Volume 8, Number 2 (2014), 359–368 doi:10.7153/jmi-08-26

ON SOME MEANS DERIVED FROM THE SCHWAB–BORCHARDT MEAN II

EDWARD NEUMAN

(*Communicated by J. Pečarić*)

Abstract. Sharp bounds for four bivariate means derived from the Schwab-Borchardt mean are obtained. The bounding quantities are either geometric or arithmetic convex combinations of two generating means. The four means discussed in this paper have been introduced and studied in [13].

1. Introduction

History of bivariate means and their inequalities is long and laden with a detail. Among numerous well known means the Schwab-Borchardt mean could be called intriguing and elegant. There are reasons why we called this mean intriguing. Several well known bivariate means can be represented as the Schwab-Borchardt mean of two means. Means obtained this way include the logarithmic mean, two Seiffert means, the Neuman-Sándor mean and four other means introduced in [13]. A problem which was recently investigated extensively deals with finding sharp bounds for these means, where the bounding expressions depend usually on two simple means. For more details, see, e.g., [5, 6, 7, 9, 10, 11, 12, 17, 18, 19, 20, 21, 22] and the references therein. It is worth mentioning that the inequalities involving the Schwab-Borchardt mean can be found in [3, 4, 8, 14, 15].

Results obtained in this paper complement in a natural way those established in author's work [13]. The present paper is organized as follows. Bivariate means used in the subsequent parts of this work are introduced in Section 2. In particular, definitions of four means derived from the Schwab-Borchardt mean are included in this section. Those means have been defined and studied in [13]. Lemmas needed in the proofs of the main results are either cited or proven in Section 3. The main results are presented in Section 4. They deal with sharp lower and upper bounds for those means. The bounding quantities used are either the geometric convex combinations or the arithmetic convex combinations of what we called the generating means. The latter term is explained in the next section in the paragraph which follows (5).

Mathematics subject classification (2010): 26E60, 26D05, 26D07.
Keywords and phrases: Bivariate means, Schwab-Borchardt mean, convex combinations, inequalities.

2. Bivariate means used in this paper

For the reader's convenience we provide below definitions of several bivariate means used in the subsequent sections of this paper.

Let a and b be positive numbers. In order to avoid trivialities we will always assume that $a \neq b$. The unweighted arithmetic mean A of a and b is defined as

$$A = \frac{a+b}{2}.$$

Another unweighted bivariate means used in this paper are the harmonic mean H and the contra-harmonic mean C which are defined in usual way

$$H = \frac{2ab}{a+b}, \quad C = \frac{a^2+b^2}{a+b}. \tag{1}$$

One can easily verify that the means listed in (1) can be expressed in terms of A. We have

$$H = A(1-v^2), \quad C = A(1+v^2), \tag{2}$$

where

$$v = \frac{a-b}{a+b}. \tag{3}$$

Clearly $0 < |v| < 1$.

In order to facilitate presentation let us recall definition of the Schwab-Borchardt mean SB

$$SB(a,b) \equiv SB = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)} & \text{if } a < b, \\[2ex] \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)} & \text{if } b < a \end{cases} \tag{4}$$

(see, e.g., [3], [4]). This mean has been studied extensively in [14], [15], and in [8]. It is well known that the mean $SB(a,b)$ is strictly increasing in both a and b, nonsymmetric and homogeneous of degree 1 in its variables.

Other bivariate means used in this paper are derived from the Schwab-Borchardt mean. They are defined as follows [13]:

$$\begin{aligned} S_{AH} &= SB(A,H), \quad & S_{HA} &= SB(H,A), \\ S_{CA} &= SB(C,A), \quad & S_{AC} &= SB(A,C). \end{aligned} \tag{5}$$

We will call the pairs of means $\{A,H\}$ and $\{C,A\}$ the generating means of the four Schwab-Borchardt means defined in (5).

In the last section of this paper we will deal with optimal bounds for two pairs of means $\{S_{HA}, S_{AC}\}$ and $\{S_{AH}, S_{CA}\}$. The bounding quantities are either the geometric or arithmetic convex combinations of two numbers. Recall that the following quantity $x^{\lambda} y^{\mu}$ is called the geometric convex combination of positive numbers x and y while $\lambda x + \mu y$ is called the arithmetic convex combination of x and y, where the numbers λ and μ are such that $0 \leqslant \lambda, \mu \leqslant 1$ and $\lambda + \mu = 1$.

3. Lemmas

In this section we give some lemmas which are needed in the sequel.

The following one, often called L'Hospital's - type rule for monotonicity, can be found, e.g. in [2].

LEMMA A. *Let the functions f and g be continuous on $[c,d]$, differentiable on (c,d) and such that $g'(t) \neq 0$ on (c,d). If $\frac{f'(t)}{g'(t)}$ is (strictly) increasing (decreasing) on (c,d), then the functions $\frac{f(t)-f(d)}{g(t)-g(d)}$ and $\frac{f(t)-f(c)}{g(t)-g(c)}$ are also (strictly) increasing (decreasing) on (c,d).*

Also, we will need the following monotonicity result [16].

LEMMA B. *Suppose that the power series $f(t)=\sum_{n=0}^{\infty} a_n t^n$ and $g(t)=\sum_{n=0}^{\infty} b_n t^n$ $(b_n > 0$ for all $n \geqslant 0)$ both converge for $|t| < \infty$. Then the function $f(t)/g(t)$ is (strictly) increasing (decreasing) for $t > 0$ if the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) increasing (decreasing).*

For the latter use we also record the following result (see [23]).

LEMMA C. *For $t \in (0,\infty)$ a function*

$$\varphi_1(t) = \frac{\ln\left(\dfrac{\sinh t}{t}\right)}{\ln(\cosh t)} \tag{6}$$

is strictly increasing.

The next lemma reads as follows.

LEMMA 1. *The following function*

$$\varphi_2(t) = \frac{\ln\left(\dfrac{\sin t}{t}\right)}{\ln(\cos t)} \tag{7}$$

is strictly decreasing on the interval $(0,\pi/2)$.

Proof. Let $0 < t < \pi/2$. Then

$$\ln\left(\frac{\sin t}{t}\right) = \sum_{n=1}^{\infty} a_n t^{2n},$$

where

$$a_n = \frac{|B_{2n}|2^{2n-1}}{n(2n)!},$$

$n = 1,2,\ldots$ and B_{2n} is the Bernoulli number (see [1, 4.3.71]). Also, if $0 < t < \pi/2$, then

$$\ln(\cos t) = \sum_{n=1}^{\infty} b_n t^{2n},$$

where

$$b_n = \frac{|B_{2n}|(2^{2n}-1)}{n(2n)!},$$

$n = 1,2,\ldots$ (see [1, 4.3.72]). In order to apply Lemma B we have to investigate monotonicity of the sequence $\{a_n/b_n\}$. Using formulas given above we have

$$\frac{a_n}{b_n} = \frac{1}{2(1-4^{-n})},$$

where n is a positive integer. Clearly the sequence in question is strictly decreasing and so is function $\varphi_2(t)$. $\square$

We shall also utilize the following.

LEMMA 2. *Let*

$$\varphi_3(t) = \frac{t-\sin t}{\tan t - \sin t}. \tag{8}$$

Then the function $\varphi_3(t)$ *is strictly decreasing on the interval* $(0,\pi/2)$.

Proof. Let

$$\frac{f(t)}{g(t)} := \varphi_3(t) = \frac{t-\sin t}{\tan t - \sin t}.$$

Differentiation yields

$$\frac{f'(t)}{g'(t)} = \frac{1-\cos t}{\sec^2 t - \cos t} = \frac{1}{\sec^2 t + \sec t + 1} =: h(t).$$

Taking into account that the function $\sec t$ is strictly increasing on the interval $(0,\pi/2)$ we conclude that the function $h(t)$ is strictly decreasing on the same interval. We appeal now to Lemma A, to obtain the desired result. $\square$

The next lemma will also be used in the following section.

LEMMA 3. *We define*

$$\varphi_4(t) = \frac{t-\tanh t}{\sinh t - \tanh t}. \tag{9}$$

Then the function $\varphi_4(t)$ *is strictly decreasing for all* $t > 0$.

Proof. We follow the lines of proof of Lemma 2. Let

$$\frac{f(t)}{g(t)} := \varphi_4(t) = \frac{t-\tanh t}{\sinh t - \tanh t}.$$

Hence

$$\frac{f'(t)}{g'(t)} = \frac{1-\operatorname{sech}^2 t}{\cosh t - \operatorname{sech}^2 t} = \frac{\cosh t + 1}{\cosh^2 t + \cosh t + 1} = 1 - \frac{\cosh^2 t}{\cosh^2 t + \cosh t + 1}.$$

Dividing numerator and denominator of the last fraction by $\cosh^2 t$ we obtain

$$\frac{f'(t)}{g'(t)} = 1 - \frac{1}{1+\operatorname{sech} t + \operatorname{sech}^2 t} =: k(t).$$

Taking into account that the function $\operatorname{sech} t$ is strictly decreasing for all $t > 0$ we conclude that the function $k(t)$ is strictly decreasing on the stated domain. Utilizing Lemma A again we obtain the desired result. □

4. Main results

To this end we will assume that the letters a and b stand for two positive and unequal numbers.

Before we will state and prove the main results we recall formulas for means defined in (5). From [13, (12)]

$$S_{HA} = A\frac{\sin p}{p}, \tag{10}$$

where $\cos p = 1 - v^2$ $(0 < p < \pi/2)$ and v is defined in (3). Also,

$$S_{AC} = A\frac{\tan q}{q} \tag{11}$$

where $\sec q = 1 + v^2$ $(0 < q < \pi/3)$ (see [13, (25)]). We are in a position to state and prove the following.

THEOREM 1. *The two-sided inequalities*

$$H^{\alpha_1}A^{1-\alpha_1} < S_{HA} < H^{\beta_1}A^{1-\beta_1} \tag{12}$$

and

$$A^{\alpha_2}C^{1-\alpha_2} < S_{AC} < A^{\beta_2}C^{1-\beta_2} \tag{13}$$

hold true if

$$\frac{1}{3} \leqslant \alpha_1 \leqslant 1 \qquad \text{and} \qquad \beta_1 = 0 \tag{14}$$

and

$$\frac{1}{3} \leqslant \alpha_2 \leqslant 1 \qquad \text{and} \qquad 0 \leqslant \beta_2 < \left(\ln\frac{4\pi^2}{27}\right)/\ln 4 = 0.274\ldots, \tag{15}$$

respectively.

Proof. It is easy to see that the inequality (12) is equivalent to

$$(H/A)^{\alpha_1} < S_{HA}/A < (H/A)^{\beta_1}.$$

Taking logarithms and using the fact that $H/A < 1$ we can write the last two-sided inequality as follows

$$\beta_1 < \varphi_2(p) < \alpha_1, \tag{16}$$

where $0 < p < \pi/2$ and φ_2 is defined in (7). It is elementary task to show that

$$\varphi_2(0^+) = 1/3 \qquad \text{and} \qquad \varphi_2(\pi/2^-) = 0.$$

Utilizing (16) and the fact that the function φ_2 is strictly decreasing on its domain (see Lemma 1) we obtain conditions (14). In order to prove that the two-sided inequalities (13) are valid if conditions (15) are satisfied we follow the lines introduced above in this proof. First we rewrite (13) as

$$(A/C)^{\alpha_2} < S_{AC}/C < (A/C)^{\beta_2}.$$

Since $A/C < 1$, the last inequality is equivalent to

$$\beta_2 < \varphi_2(q) < \alpha_2, \tag{17}$$

where $q \in (0, \pi/3)$. Taking into account that

$$\varphi_2(0^+) = 1/3 \qquad \text{and} \qquad \varphi_2(\pi/3^-) = \left(\ln\frac{4\pi^2}{27}\right)/\ln 4,$$

and also that the function $\varphi_2(q)$ is strictly decreasing on the stated domain, one obtains the desired result (15). The proof is complete. $\square$

We will now determine geometric convex combinations as the lower and upper bounds for another pair of means derived from the Schwab-Borchardt mean. The means of interest are now S_{AH} and S_{CA}. It has been demonstrated in [13, (11)] that

$$S_{AH} = A\frac{\tanh r}{r}, \tag{18}$$

where $\operatorname{sech} r = 1 - v^2$ $(r > 0)$ and also that

$$S_{CA} = A\frac{\sinh s}{s}, \tag{19}$$

where $\cosh s = 1 + v^2$ $(0 < s < \gamma := \cosh^{-1}(2))$ (see [13, (24)]). The proof is complete.

Our next result reads as follows.

THEOREM 2. *The following inequalities*

$$A^{\alpha_3}H^{1-\alpha_3} < S_{AH} < A^{\beta_3}H^{1-\beta_3} \tag{20}$$

hold true if

$$0 \leqslant \alpha_3 \leqslant \frac{1}{3} \qquad \text{and} \qquad \beta_3 = 1. \tag{21}$$

Also,

$$C^{\alpha_4}A^{1-\alpha_4} < S_{CA} < C^{\beta_4}A^{1-\beta_4} \tag{22}$$

if

$$0 \leqslant \alpha_4 \leqslant \frac{1}{3} \qquad \text{and} \qquad \left(\ln\frac{\sqrt{3}}{\gamma}\right)/\ln 2 \leqslant \beta_4 \leqslant 1. \tag{23}$$

Let us note that $\left(\ln \frac{\sqrt{3}}{\gamma}\right)/\ln 2 = 0.395\ldots$.

Proof. For the proof of (20) with conditions of its validity as stated above let us rewrite (20) in the form

$$\alpha_3 < \frac{\ln(S_{AH}/H)}{\ln(A/H)} < \beta_3.$$

Using (18), (2) and the fact that $1 - v^2 = \operatorname{sech} r$ we can write the last two-sided inequality in the form

$$\alpha_3 < \varphi_1(r) < \beta_3,$$

where $r > 0$. Elementary computations yield $\varphi_1(0^+) = 1/3$ and $\lim_{r\to\infty} \varphi_1(r) = 1$. Taking into account that the function φ_1 is strictly increasing on the stated domain (see Lemma C) we obtain

$$\alpha_3 \leqslant \frac{1}{3} \leqslant \varphi_1(r) \leqslant 1 = \beta_3.$$

This yields the desired result. The second part of the thesis can be established in a similar fashion. First we express (22) as

$$\alpha_4 < \frac{\ln(S_{CA}/A)}{\ln(C/A)} < \beta_4.$$

Using (19) and (2) together with the formula $1 + v^2 = \cosh s$ we can write the last two-sided inequality as follows

$$\alpha_4 < \varphi_1(s) < \beta_4,$$

where $0 < s < \gamma$. Taking into account that the function φ_1 is strictly increasing on the stated domain and also that $\lim_{s\to 0} \varphi_1(s) = 1/3$ and $\varphi_1(\gamma) = \left(\ln \frac{\sqrt{3}}{\gamma}\right)/\ln 2$ we obtain

$$\alpha_4 \leqslant \frac{1}{3} \leqslant \varphi_1(s) \leqslant \left(\ln \frac{\sqrt{3}}{\gamma}\right)/\ln 2 \leqslant \beta_4.$$

This completes the proof. $\square$

The remaining two theorems deal with sharp bounds for the reciprocals of the four means defined in (5). The bounding quantities are the arithmetic convex combinations of the reciprocals of the generating means.

THEOREM 3. *The two-sided inequalities*

$$\frac{\alpha_5}{H} + \frac{1-\alpha_5}{A} < \frac{1}{S_{HA}} < \frac{\beta_5}{H} + \frac{1-\beta_5}{A} \tag{24}$$

hold true if

$$\alpha_5 = 0 \qquad \text{and} \qquad \frac{1}{3} \leqslant \beta_5 \leqslant 1. \tag{25}$$

Also, the inequalities

$$\frac{\alpha_6}{A} + \frac{1-\alpha_6}{C} < \frac{1}{S_{AC}} < \frac{\beta_6}{A} + \frac{1-\beta_6}{C} \tag{26}$$

are valid if

$$0 \leqslant \alpha_6 \leqslant \frac{2\pi}{3\sqrt{3}} - 1 = 0.209... \qquad \text{and} \qquad \frac{1}{3} \leqslant \beta_6 \leqslant 1. \tag{27}$$

Proof. In order to prove that inequalities (24) are valid if the conditions (25) are satisfied we write the former as

$$\alpha_5\left(\frac{1}{H} - \frac{1}{A}\right) < \frac{1}{S_{HA}} - \frac{1}{A} < \beta_5\left(\frac{1}{H} - \frac{1}{A}\right).$$

Substituting (10) and (2) into last inequalities and taking into account that $1 - v^2 = \cos p$ we obtain, after a little algebra,

$$\alpha_5 < \varphi_3(p) < \beta_5,$$

where $p \in (0, \pi/2)$. Easy computations and the fact that the function φ_3 is strictly decreasing on the stated domain yield

$$0 = \varphi_3((\pi/2)^-) \leqslant \varphi_3(p) \leqslant \varphi_3(0^+) = \frac{1}{3}.$$

Conditions (25) of validity of inequalities (24) now follow. In a similar fashion one can demonstrate that the inequalities (26) hold true if the conditions (27) are satisfied. First using (11), (2) and $1 + v^2 = \sec q$ $(0 < q < \pi/3)$ we can write (26) in the equivalent form as

$$\alpha_6 < \varphi_3(q) < \beta_6.$$

Simple computations together with the monotonicity property of φ_3 yield

$$\frac{2\pi}{3\sqrt{3}} - 1 = \varphi_3((\pi/3)^-) \leqslant \varphi_3(q) \leqslant \varphi_3(0^+) = \frac{1}{3}.$$

The conditions (27) now follow. □

We close this section with the following.

THEOREM 4. *In order for the inequalities*

$$\frac{\alpha_7}{H} + \frac{1-\alpha_7}{A} < \frac{1}{S_{AH}} < \frac{\beta_7}{H} + \frac{1-\beta_7}{A} \tag{28}$$

to be valid it suffices that

$$\alpha_7 = 0 \qquad \text{and} \qquad \frac{2}{3} \leqslant \beta_7 \leqslant 1. \tag{29}$$

Also, the inequalities

$$\frac{\alpha_8}{A}+\frac{1-\alpha_8}{C}<\frac{1}{S_{CA}}<\frac{\beta_8}{A}+\frac{1-\beta_8}{C} \tag{30}$$

are hold true if

$$0\leqslant\alpha_8\leqslant\frac{2}{3}\gamma-1=0.520\ldots \qquad \textit{and} \qquad \frac{2}{3}\leqslant\beta_8\leqslant 1. \tag{31}$$

Proof. To obtain the first part of the thesis we write (28) in the equivalent form

$$\alpha_7\left(\frac{1}{H}-\frac{1}{A}\right)<\frac{1}{S_{AH}}-\frac{1}{A}<\beta_7\left(\frac{1}{H}-\frac{1}{A}\right).$$

Making use of (18) and (2) we can rewrite the last two-sided inequality in the form

$$\alpha_7<\varphi_4(r)<\beta_7,$$

where $\operatorname{sech} r=1-v^2 \ (r>0)$ and v is defined in (3). Computing limits of $\varphi_4(r)$ as $r\to 0^+$ and as $r\to+\infty$ and taking into account that the function φ_4 is strictly decreasing on the stated domain we obtain

$$0=\varphi_4(+\infty)\leqslant\varphi_4(r)\leqslant\varphi_4(0^+)=2/3.$$

The conditions (29) now follow. Finally, in order to prove that the inequalities (30) are satisfied if conditions (31) are valid we follow the steps used above to obtain

$$\alpha_8\left(\frac{1}{A}-\frac{1}{C}\right)<\frac{1}{S_{CA}}-\frac{1}{C}<\beta_8\left(\frac{1}{A}-\frac{1}{C}\right).$$

With the aid of (19) and (2) we write the above inequalities as

$$\alpha_8<\varphi_4(s)<\beta_8,$$

where $\cosh s=1+v^2 \ (0<s<\gamma)$. Computing limits of $\varphi_4(s)$ at $s=0$ and at $s=\gamma$ and next utilizing monotonicity of φ_4 we obtain

$$\frac{2}{3}\gamma-1=\varphi_4(\gamma^-)\leqslant\varphi_4(s)\leqslant\varphi_4(0^+)=2/3.$$

This yields the desired conditions (31). The proof is complete. □

Acknowledgements.

The author is indebted to an anonymous referee for constructive remarks on the first draft of this paper and also for calling his attention to papers [5, 6, 7, 17] and [20].

REFERENCES

[1] M. ABRAMOWIZ, A. S. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, Inc., New York, 1972.
[2] H. ALZER, S.-L. QIU, *Monotonicity theorems and inequalities for complete elliptic integrals*, J. Comput. Appl. Math. **172** (2004), 289–312.
[3] J. M. BORWEIN, P. B. BORWEIN, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, John Wiley and Sons, New York, 1987.
[4] B. C. CARLSON, *Algorithms involving arithmetic and geometric means*, Amer. Math. Monthly **78** (1971), 496–505.
[5] Y.-M. CHU AND B.-Y. LONG, *Bounds of the Neuman-Sándor mean using power and identric means*, Abstr. Appl. Anal., 2013, Article ID 832591, 6 pages.
[6] Y.-M. CHU, B.-Y. LONG, W.-M. GONG AND Y.-Q. SONG, *Sharp bounds for Seiffert and Neuman-Sándor means in terms of generalized logarithmic means*, J. Inequal. Appl. **2013**, 10 (2013), 13 pages.
[7] Z.-Y. HE, W.-M. QIAN, Y.-L. JIANG, Y.-Q. SONG AND Y.-M. CHU, *Bounds for the combination of Neuman-Sándor, arithmetic, and second Seiffert means in terms of contraharmonic mean*, Abstr. Appl. Anal., 2013, Article ID 903982, 5 pages.
[8] E. NEUMAN, *Inequalities for the Schwab-Borchardt mean and their applications*, J. Math. Inequal. **5**, 4 (2011), 601–609.
[9] E. NEUMAN, *A note on a certain bivariate mean*, J. Math. Inequal. **6**, 4 (2012), 637–643.
[10] E. NEUMAN, *A one-parameter family of bivariate means*, J. Math.Inequal. **7**, 3 (2013), 399–412.
[11] E. NEUMAN, *Sharp inequalities involving Neuman-Sándor and logarithmic means*, J. Math. Inequal. **7**, 3 (2013), 413–419.
[12] E. NEUMAN, *Inequalities involving certain bivariate means II*, J. Inequal. Spec. Funct., in press.
[13] E. NEUMAN, *On some means derived from the Schwab-Borchardt mean*, J. Math. Inequal., to appear.
[14] E. NEUMAN, J. SÁNDOR, *On the Schwab-Borchardt mean*, Math. Pannon. **14**, 2 (2003), 253–266.
[15] E. NEUMAN, J. SÁNDOR, *On the Schwab-Borchardt mean II*, Math. Pannon. **17**, 1 (2006), 49–59.
[16] S. PONNUSAMY, M. VUORINEN, *Asymptotic expansions and inequalities for hypergeometric functions*, Mathematika **44** (1997), 43–64.
[17] W.-M. QIAN AND Y.-M. CHU, *On certain inequalities for Neuman-Sándor mean*, Abstr. Appl. Anal., 2013, Article ID 790783, 6 pages.
[18] Y.-Q. SONG, W.-M. QIAN, Y.-L. JIANG AND Y.-M. CHU, *Optimal lower generalized logarithmic mean bound for the Seiffert mean*, J. Appl. Math., 2013, Article ID 273653, 5 pages.
[19] M.-K. WANG, Y.-M. CHU, B.-Y. LIU, *Sharp inequalities for the Neuman-Sándor mean in terms of arithmetic and contra-harmonic means*, preperint, arXiv: 1209.5825 vol 1 [math CA] 26 Sept. 2012.
[20] M.-K. WANG, Y.-F. QIU AND Y.-M. CHU, *Sharp bounds for Seiffert means in terms of Lehmer means*, J. Math. Inequal. **4** (2010), 581–586.
[21] A. WITKOWSKI, *Interpolations of Schwab-Borchardt means*, Math. Inequal. Appl. **16**, 1 (2013), 193–206.
[22] T.-H. ZHAO, Y.-M. CHU, Y.-L. JIANG AND Y.-M. LI, *Best possible bounds for Neuman-Sándor mean by identric, quadratic and contraharmonic means*, Abstr. Appl. Anal., 2013, Article ID 348326, 12 pages.
[23] T.-H. ZHAO, Y.-M. CHU, B.-Y. LIU, *Optimal bounds for the Neuman-Sándor mean in terms of arithmetic and contra-harmonic means*, Abstract Appl. Anal. Volume **2012**, Article ID 302635, 9 pages.

(Received May 31, 2013)

Edward Neuman
Mathematical Research Institute
144 Hawthorn Hollow
Carbondale, IL 62903
USA
e-mail: edneuman76@gmail.com