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ORIGIN–SYMMETRIC BODIES OF REVOLUTION WITH MINIMAL MAHLER VOLUME IN $\mathbb{R}^3$–A NEW PROOF

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Abstract. In [22], Meyer and Reisner proved the Mahler conjecture for rovelution bodies. In this paper, using a new method, we prove that among *origin-symmetric bodies of revolution* in $\mathbb{R}^3$, cylinders have the minimal Mahler volume. Further, we prove that among *parallel sections homothety bodies* in $\mathbb{R}^3$, 3-cubes have the minimal Mahler volume.

1. Introduction

The well-known Mahler's conjecture (see, e.g., [11], [18], [29] for references) states that, for any origin-symmetric convex body K in $\mathbb{R}^n$,

$$\mathscr{P}(K) \geqslant \mathscr{P}(C^n) = \frac{4^n}{n!}, \tag{1.1}$$

where C^n is an n-cube and $\mathscr{P}(K) = Vol(K)Vol(K^*)$, which is known as the *Mahler volume* of K.

For $n = 2$, Mahler [19] himself proved the conjecture, and in 1986 Reisner [26] showed that equality holds only for parallelograms. For $n = 2$, a new proof of inequality (1.1) was obtained by Campi and Gronchi [4]. Recently, Lin and Leng [17] gave a new and intuitive proof of the inequality (1.1) in $\mathbb{R}^2$.

For some special classes of origin-symmetric convex bodies in $\mathbb{R}^n$, a sharper estimate for the lower bound of $\mathscr{P}(K)$ has been obtained. If K is a convex body which is symmetric around all coordinate hyperplanes, Saint Raymond [28] proved that $\mathscr{P}(K) \geqslant 4^n/n!$; the equality case was discussed in [20, 27]. When K is a zonoid (limits of finite Minkowski sums of line segments), Meyer and Reisner (see, e.g., [12, 25, 26]) proved that the same inequality holds, with equality if and only if K is an n-cube. For the case of polytopes with at most $2n+2$ vertices (or facets) (see, e.g., [2] for references), Lopez and Reisner [15] proved the inequality (1.1) for $n \leqslant 8$ and the minimal bodies are characterized. Recently, Nazarov, Petrov, Ryabogin and Zvavitch [24]

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proved that the cube is a strict local minimizer for the Mahler volume in the class of origin-symmetric convex bodies endowed with the Banach-Mazur distance.

Bourgain and Milman [3] proved that there exists a universal constant $c > 0$ such that $\mathscr{P}(K) \geqslant c^n \mathscr{P}(B)$, which is now known as the reverse Santaló inequality. Very recently, Kuperberg [14] found a beautiful new approach to the reverse Santaló inequality. What's especially remarkable about Kuperberg's inequality is that it provides an explicit value for c.

Another variant of the Mahler conjecture without the assumption of origin-symmetry states that, for any convex body K in $\mathbb{R}^n$,

$$\mathscr{P}(K) \geqslant \frac{(n+1)^{(n+1)}}{(n!)^2}, \tag{1.2}$$

with equality conjectured to hold only for simplices. For $n = 2$, Mahler himself proved this inequality in 1939 (see, e.g., [5, 6, 16] for references) and Meyer [21] obtained the equality conditions in 1991. Recently, Meyer and Reisner [23] have proved inequality (1.2) for polytopes with at most $n+3$ vertices. Very recently, Kim and Reisner [13] proved that the simplex is a strict local minimum for the Mahler volume in the Banach-Mazur space of n-dimensional convex bodies.

Strong functional versions of the Blaschke-Santaló inequality and its reverse form have been studied recently (see, e.g., [1, 7, 8, 9, 10, 22]).

The Mahler conjecture is still open even in the three-dimensional case. Terence Tao in [30] made an excellent remark about the open question.

To state our results, we first give some definitions. In the coordinate plane XOY of $\mathbb{R}^3$, let

$$D = \{(x,y) : -a \leqslant x \leqslant a, |y| \leqslant f(x)\}, \tag{1.3}$$

where $f(x)$ ($[-a,a]$, $a > 0$) is a concave, even and nonnegative function. An *origin-symmetric body of revolution* R is defined as the convex body generated by rotating D around the X-axis in $\mathbb{R}^3$. $f(x)$ is called its *generating function* and D is its *generating domain*. If the generating domain of R is a rectangle (the generating function of R is a constant function), R is called a *cylinder*. If the generating domain of R is a diamond (the generating function $f(x)$ of R is a linear function on $[-a,0]$ and $f(-a) = 0$), R is called a *bicone*.

In this paper, we prove that cylinders have the minimal Mahler volume for origin-symmetric bodies of revolution in $\mathbb{R}^3$.

THEOREM 1.1. *For any origin-symmetric body of revolution K in $\mathbb{R}^3$, we have*

$$\mathscr{P}(K) \geqslant \frac{4\pi^2}{3}, \tag{1.4}$$

and the equality holds if and only if K is a cylinder or bicone.

REMARK 1. In [22], for the Schwarz rounding $\tilde{K}$ of a convex body K in $\mathbb{R}^n$, Meyer and Reisner gave a lower bound for $\mathscr{P}(\tilde{K})$. Especially, for a general body of revolution K in $\mathbb{R}^3$, they proved

$$\mathscr{P}(K) \geqslant \frac{4^4\pi^2}{3^5}, \tag{1.5}$$

with equality if and only if K is a cone and $|AO|/|AD| = 3/4$ (where, A is the vertex of the cone and AD is the height and O is the Santaló point of K).

The following Theorem 1.2 is the functional version of the Theorem 1.1.

THEOREM 1.2. *Let $f(x)$ be a concave, even and nonnegative function defined on $[-a,a]$, $a > 0$, and for $x' \in [-\frac{1}{a}, \frac{1}{a}]$ define*

$$f^*(x') = \inf_{x\in[-a,a]} \frac{1-x'x}{f(x)}.$$

Then, we have

$$\left(\int_{-a}^{a}(f(x))^2dx\right)\left(\int_{-\frac{1}{a}}^{\frac{1}{a}}(f^*(x'))^2dx'\right) \geqslant \frac{4}{3}, \tag{1.6}$$

with equality if and if $f(x) = f(0)$ or $f^*(x') = 1/f(0)$.

Let C be an origin-symmetric convex body in the coordinate plane YOZ of $\mathbb{R}^3$ and $f(x)$ $(x \in [-a,a],\ a > 0)$ is a concave, even and nonnegative function. A *parallel sections homothety body* is defined as the convex body

$$K = \bigcup_{x\in[-a,a]} \{f(x)C + xv\},$$

where $v = (1,0,0)$ is a unit vector in the positive direction of the X-axis, $f(x)$ is called its *generating function* and C is its *homothetic section*.

Applying Theorem 1.2, we prove that among parallel sections homothety bodies in $\mathbb{R}^3$, 3-cubes have the minimal Mahler volume.

THEOREM 1.3. *For any parallel sections homothety body K in $\mathbb{R}^3$, we have*

$$\mathscr{P}(K) \geqslant \frac{4^3}{3!}, \tag{1.7}$$

and the equality holds if and only if K is a 3-cube or octahedron.

2. Definitions, notation, and preliminaries

As usual, S^{n-1} denotes the unit sphere, and B^n the unit ball centered at the origin, O the origin and $\|\cdot\|$ the norm in Euclidean n-space $\mathbb{R}^n$. The symbol for the set of all natural numbers is $\mathbb{N}$. Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. Let $\mathscr{K}_o^n$ denote the subset of $\mathscr{K}^n$ that contains the origin in its interior. For $u \in S^{n-1}$, we denote by $u^\perp$ the $(n-1)$-dimensional subspace orthogonal to u. For x, $y \in \mathbb{R}^n$, $x \cdot y$ denotes the inner product of x and y.

Let $\operatorname{int} K$ denote the interior of K. Let $\operatorname{conv} K$ denote the convex hull of K. we denote by $V(K)$ the n-dimensional volume of K. The notation for the usual orthogonal projection of K on a subspace S is $K|S$.

If $K \in K_o^n$, we define the *polar body* K^* of K by

$$K^* = \{x \in \mathbb{R}^n : \ x \cdot y \leqslant 1 \ , \forall y \in K\}.$$

REMARK 2. If P is a polytope, i.e., $P = \operatorname{conv}\{p_1, \cdots, p_m\}$, where p_i $(i = 1, \cdots, m)$ are vertices of polytope P. By the definition of the polar body, we have

$$\begin{aligned} P^* &= \{x \in \mathbb{R}^n : x \cdot p_1 \leqslant 1, \cdots, x \cdot p_m \leqslant 1\} \\ &= \bigcap_{i=1}^{m} \{x \in \mathbb{R}^n : x \cdot p_i \leqslant 1\}, \end{aligned} \tag{2.1}$$

which implies that P^* is an intersection of m closed halfspaces with exterior normal vectors p_i $(i = 1, \cdots, m)$ and the distance of hyperplane

$$\{x \in \mathbb{R}^n : x \cdot p_i = 1\}$$

from the origin is $1/\|p_i\|$.

Associated with each convex body K in $\mathbb{R}^n$ is its *support function* $h_K : \mathbb{R}^n \to [0, \infty)$, defined for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{y \cdot x : y \in K\}, \tag{2.2}$$

and its *radial function* $\rho_K : \mathbb{R}^n \backslash \{0\} \to (0, \infty)$, defined for $x \neq 0$, by

$$\rho_K(x) = \max\{\lambda \geqslant 0 \ : \ \lambda x \in K\}. \tag{2.3}$$

For K, $L \in \mathscr{K}^n$, the *Hausdorff distance* is defined by

$$\delta(K,L) = \min\{\lambda \geqslant 0 : \ K \subset L + \lambda B^n, \ L \subset K + \lambda B^n\}. \tag{2.4}$$

A *linear transformation* (or *affine transformation*) of $\mathbb{R}^n$ is a map ϕ from $\mathbb{R}^n$ to itself such that $\phi x = Ax$ (or $\phi x = Ax + t$, respectively), where A is an $n \times n$ matrix and $t \in \mathbb{R}^n$. It is known that Mahler volume of K is invariant under affine transformation.

For $K \in \mathscr{K}_o^n$, if $(x_1, x_2, \cdots, x_n) \in K$, we have $(\varepsilon_1 x_1, \cdots, \varepsilon_n x_n) \in K$ for any signs $\varepsilon_i = \pm 1$ $(i = 1, \cdots, n)$, then K is a *1-unconditional convex body*. In fact, K is symmetric with respect to all coordinate planes.

The following Lemma 2.1 will be used to calculate the volume of an origin-symmetric body of revolution. Since the lemma is an elementary conclusion in calculus, we omit its proof.

LEMMA 2.1. *In the coordinate plane XOY, let*

$$D = \{(x,y) : a \leqslant x \leqslant b, |y| \leqslant f(x)\},$$

where $f(x)$ is a linear, nonnegative function defined on $[a,b]$. Let R be a body of revolution generated by D. Then

$$V(R) = \frac{\pi}{3}(b-a)\left[f(a)^2 + f(a)f(b) + f(b)^2\right]. \tag{2.5}$$

3. Main result and its proof

In the paper, we consider convex bodies in a three-dimensional Cartesian coordinate system with origin O and its three coordinate axes are denoted by X-axis, Y-axis, and Z-axis.

LEMMA 3.1. *If $K \in \mathscr{K}_0^3$, then for any $u \in S^2$, we have*

$$K^* \cap u^{\perp} = (K|u^{\perp})^*. \tag{3.1}$$

On the other hand, if $K' \in \mathscr{K}_0^3$ satisfies

$$K' \cap u^{\perp} = (K|u^{\perp})^* \tag{3.2}$$

for any $u \in S^2 \cap v_0^{\perp}$ (v_0 is a fixed vector), then,

$$K' = K^*. \tag{3.3}$$

Proof. Firstly, we prove (3.1).

Let $x \in u^{\perp}$, $y \in K$ and $y' = y|u^{\perp}$, since the hyperplane $u^{\perp}$ is orthogonal to the vector $y - y'$, then

$$y \cdot x = (y' + y - y') \cdot x = y' \cdot x + (y - y') \cdot x = y' \cdot x.$$

If $x \in K^* \cap u^{\perp}$, for any $y' \in K|u^{\perp}$, there exists $y \in K$ such that $y' = y|u^{\perp}$, then $x \cdot y' = x \cdot y \leqslant 1$, thus $x \in (K|u^{\perp})^*$. Thus, we have $K^* \cap u^{\perp} \subset (K|u^{\perp})^*$.

If $x \in (K|u^{\perp})^*$, then for any $y \in K$ and $y' = y|u^{\perp}$, $x \cdot y = x \cdot y' \leqslant 1$, thus $x \in K^*$, and since $x \in u^{\perp}$, thus $x \in K^* \cap u^{\perp}$. Thus, we have $(K|u^{\perp})^* \subset K^* \cap u^{\perp}$.

Next we prove (3.3).

Let $S^1 = S^2 \cap v_0^{\perp}$. For any vector $v \in S^2$, there exists a $u \in S^1$ satisfying $v \in u^{\perp}$. Since $K' \cap u^{\perp} = (K|u^{\perp})^*$ and $K^* \cap u^{\perp} = (K|u^{\perp})^*$, thus $K' \cap u^{\perp} = K^* \cap u^{\perp}$. Hence, we have $\rho_{K'}(v) = \rho_{K^*}(v)$. Since $v \in S^2$ is arbitrary, we get $K' = K^*$. $\square$

LEMMA 3.2. *In the coordinate plane XOY, let P be a 1-unconditional convex body. Let R and R' be two origin-symmetric bodies of revolution generated by P and P^*, respectively. Then $R' = R^*$.*

Proof. Let $v_0 = \{1,0,0\}$ and $S^1 = S^2 \cap v_0^{\perp}$, for any $u \in S^1$, we have $R|u^{\perp} = R \cap u^{\perp}$. Since $R' \cap u^{\perp} = P^* = (R \cap u^{\perp})^*$ for any $u \in S^1$, thus $R' \cap u^{\perp} = (R|u^{\perp})^*$ for any $u \in S^1$. By Lemma 3.1, we have $R' = R^*$. □

LEMMA 3.3. *For any origin-symmetric body of revolution R, there exists a linear transformation ϕ satisfying*

(i) ϕR is an origin-symmetric body of revolution;

(ii) $\phi R \subset C^3 = [-1,1]^3$, where C^3 is the unit cube in $\mathbb{R}^3$.

Proof. Let $f(x)$ $(x \in [-a,a])$ be the generating function of R.

For vector $v = (1,0,0)$ and any $t \in [-a,a]$, the set $R \cap (v^{\perp} + tv)$ is a disk in the plane $v^{\perp} + tv$ with the point $(t,0,0)$ as the center and $f(t)$ as the radius.

Next, for a 3×3 diagonal matrix $A = \begin{bmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$, where $b,c \in \mathbb{R}^+$, let $\phi R = \{Ax : x \in R\}$, we prove that ϕR is still an origin-symmetric body of revolution.

For $t' \in [-ab,ab]$, if $(t',y',z') \in \phi R \cap (v^{\perp} + t'v)$, there is $(t,y,z) \in R \cap (v^{\perp} + tv)$ satisfying $t' = bt,\ y' = cy,\ z' = cz$. Hence, we have

$$\|(t',y',z') - (t',0,0)\| = c\|(t,y,z) - (t,0,0)\| \leqslant cf(t),$$

which implies that $\phi R \cap (v^{\perp} + t'v) \subset B'$, where B' is a disk in the plane $v^{\perp} + t'v$ with $(t',0,0)$ as the center and $cf(t'/b)$ as the radius.

On the other hand, if $(t',y',z') \in B'$, then $\|(t',y',z') - (t',0,0)\| \leqslant cf(t'/b)$. Let $t = t'/b$, $y = y'/c$ and $z = z'/c$. Noting $t' \in [-ab,ab]$, we have $t \in [-a,a]$ and

$$\|(t,y,z) - (t,0,0)\| = \frac{1}{c}\|(t',y',z') - (t',0,0)\| \leqslant f(t).$$

Hence, we have $(t,y,z) \in R \cap (v^{\perp} + tv)$, which implies that $(t',y',z') = (bt,cy,cz) \in \phi R \cap (v^{\perp} + t'v)$. Thus, $B' \subset \phi R \cap (v^{\perp} + t'v)$. Therefore, we have $\phi R \cap (v^{\perp} + t'v) = B'$. It follows that ϕR is an origin-symmetric body of revolution and its generating function is $F(x) = cf(x/b)$, $x \in [-ab,ab]$.

Set $b = 1/a$ and $c = 1/f(0)$, we obtain $\phi R \subset C^3 = [-1,1]^3$. □

REMARK 3. By Lemma 3.3 and the affine invariance of Mahler volume, to prove our theorems, we need only consider the origin-symmetric body of revolution R whose generating domain P satisfies $T \subset P \subset Q$, where

$$T = \{(x,y) : |x| + |y| \leqslant 1\} \text{ and } Q = \{(x,y) : \max\{|x|,|y|\} \leqslant 1\}.$$

In the following lemmas, let $\triangle ABD$ denote $\mathrm{conv}\{A,B,D\}$, where $A = (-1,1)$, $B = (0,1)$ and $D = (-1,0)$.

LEMMA 3.4. *Let P be a 1-unconditional polygon in the coordinate plane XOY satisfying*

$$P \cap \{(x,y) : x \leqslant 0,\ y \geqslant 0\} = \mathrm{conv}\{O,D,A_2,A_1,B\},$$

where A_1 lies on the line segment AB and $A_2 \in \mathrm{int}\triangle ABD$, R the origin-symmetric body of revolution generated by P. Then

$$\mathscr{P}(R) \geqslant \min\{\mathscr{P}(R_1), \mathscr{P}(R_2)\} \tag{3.4}$$

and

$$\mathscr{P}(R) \geqslant \frac{4\pi^2}{3}, \tag{3.5}$$

where R_1 and R_2 are origin-symmetric bodies of revolution generated by 1-unconditional polygons P_1 and P_2 satisfying

$$P_1 \cap \{(x,y) : x \leqslant 0,\ y \geqslant 0\} = \mathrm{conv}\{O, D, A_2, B\}$$

and

$$P_2 \cap \{(x,y) : x \leqslant 0,\ y \geqslant 0\} = \mathrm{conv}\{O, D, C, B\},$$

respectively, where C is the point of intersection between two lines A_2D and AB.

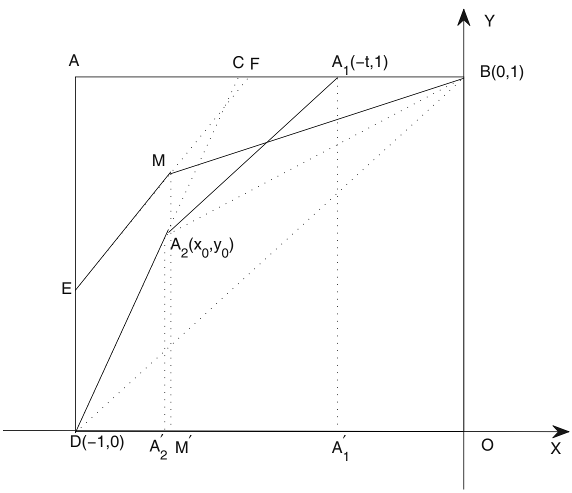

Figure 3.1: *P and P^* in the second quadrant.*

Proof. In Figure 3.1, let $A_2 = (x_0, y_0)$ and $A_1 = (-t, 1)$, then

$$C = \left(\frac{x_0 - y_0 + 1}{y_0}, 1\right) \text{ and } 0 \leqslant t \leqslant \frac{-x_0 + y_0 - 1}{y_0}.$$

From Remark 2, we can get P^*, which satisfies

$$P^* \cap \{(x,y) : x \leqslant 0,\ y \geqslant 0\} = \mathrm{conv}\{M, E, D, O, B\},$$

where E lies on the line segment AD and $M \in \operatorname{int}\triangle ABD$. Let F be the point of intersection between two lines EM and AB. Let

$$F_1(t) = \frac{1}{2}V(R),\ \ F_2(t) = \frac{1}{2}V(R^*)\ \text{ and }\ F(t) = F_1(t)F_2(t).$$

Firstly, we prove (3.4). The proof consists of three steps for good understanding.

First step. *We calculate the first and second derivatives of the functions $F(t)$.*
Since $EF \perp OA_2$ and the distance of the line EF from O is $1/\|OA_2\|$, we have the equation of the line EF

$$y = -\frac{x_0}{y_0}x + \frac{1}{y_0}. \tag{3.6}$$

Similarly, since $BM \perp OA_1$ and the distance of the line BM from O is $1/\|OA_1\|$, we get the equation of the line BM

$$y = tx + 1. \tag{3.7}$$

Using equations (3.6) and (3.7), we obtain

$$M = (x_M, y_M) = \left(\frac{1-y_0}{ty_0+x_0}, \frac{x_0+t}{ty_0+x_0}\right) \tag{3.8}$$

and

$$E = (x_E, y_E) = \left(-1, \frac{x_0+1}{y_0}\right). \tag{3.9}$$

Noting that

$$\begin{aligned} &P \cap \{(x,y) : x \leqslant 0,\ y \geqslant 0\} \\ &= \operatorname{conv}\{D, A_2, A_2'\} \cup \operatorname{conv}\{A_1, A_2, A_2', A_1'\} \cup \operatorname{conv}\{O, B, A_1, A_1'\}, \end{aligned}$$

where A_1' and A_2' are the orthogonal projections of points A_1 and A_2, respectively, on the X-axis, and applying Lemma 2.1, we have

$$\begin{aligned} F_1(t) &= \frac{\pi}{3}y_0^2(x_0+1) + \frac{\pi}{3}(-t-x_0)(y_0^2+y_0+1) + \pi t \\ &= \frac{\pi}{3}(-y_0^2-y_0+2)t + \frac{\pi}{3}(y_0^2 - x_0y_0 - x_0). \end{aligned} \tag{3.10}$$

Thus, we have

$$F_1'(t) = \frac{\pi}{3}(-y_0^2-y_0+2). \tag{3.11}$$

Noting that

$$P^* \cap \{(x,y) : x \leqslant 0,\ y \geqslant 0\}$$

$$= \mathrm{conv}\{D,E,M,M'\} \cup \mathrm{conv}\{M,M',O,B\},$$

where M' is the orthogonal projection of point M on the X-axis, and applying Lemma 2.1, we obtain

$$\begin{aligned} F_2(t) &= \frac{\pi}{3}(x_M - x_E)(y_E^2 + y_E y_M + y_M^2) + \frac{\pi}{3}(-x_M)(y_M^2 + y_M + 1) \\ &= \frac{\pi}{3}\left(\frac{1-y_0}{ty_0+x_0}+1\right)\left[\left(\frac{x_0+1}{y_0}\right)^2 + \left(\frac{x_0+1}{y_0}\right)\left(\frac{x_0+t}{ty_0+x_0}\right) + \left(\frac{x_0+t}{ty_0+x_0}\right)^2\right] \\ &\quad + \frac{\pi}{3}\left(\frac{y_0-1}{ty_0+x_0}\right)\left[\left(\frac{x_0+t}{ty_0+x_0}\right)^2 + \left(\frac{x_0+t}{ty_0+x_0}\right)+1\right] \\ &= \frac{\pi}{3}\frac{\Delta_1 t^3 + \Delta_2 t^2 + \Delta_3 t + \Delta_4}{y_0^2(ty_0+x_0)^3}, \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} \Delta_1 &= y_0^3(x_0^2 + 3x_0 + 3), \\ \Delta_2 &= y_0^2(3x_0^3 + 9x_0^2 + 9x_0 + y_0^3 - 3y_0 + 2), \\ \Delta_3 &= 3y_0[x_0^4 + 3x_0^3 + 3x_0^2 + x_0(y_0^3 - y_0^2 - y_0 + 1)], \\ \Delta_4 &= x_0^2(x_0^3 + 3x_0^2 + 3x_0 + 2y_0^3 - 3y_0^2 + 1). \end{aligned}$$

Thus, we have

$$\begin{aligned} F_2'(t) &= \frac{\pi}{3}\frac{(3\Delta_1 x_0 - \Delta_2 y_0)t^2 + (2\Delta_2 x_0 - 2\Delta_3 y_0)t + (\Delta_3 x_0 - 3\Delta_4 y_0)}{y_0^2(ty_0+x_0)^4} \\ &= \frac{\pi}{3}(y_0-1)^2\frac{-y_0(y_0+2)t^2 - 2x_0(2y_0+1)t - 3x_0^2}{(ty_0+x_0)^4}. \end{aligned} \tag{3.13}$$

Then, we have

$$\begin{aligned} F'(t) &= F_1'(t)F_2(t) + F_1(t)F_2'(t) \\ &= \frac{\pi^2}{9}\frac{\Lambda_1 t^4 + \Lambda_2 t^3 + \Lambda_3 t^2 + \Lambda_4 t + \Lambda_5}{y_0^2(ty_0+x_0)^4}, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} \Lambda_1 &= y_0^4[x_0^2(-y_0^2 - y_0 + 2) + 3x_0(-y_0^2 - y_0 + 2) + 3(-y_0^2 - y_0 + 2)], \\ \Lambda_2 &= y_0^3[4x_0^3(-y_0^2 - y_0 + 2) + 12x_0^2(-y_0^2 - y_0 + 2) + 12x_0(-y_0^2 - y_0 + 2)], \\ \Lambda_3 &= y_0^2[6x_0^4(-y_0^2 - y_0 + 2) + 18x_0^3(-y_0^2 - y_0 + 2) + 18x_0^2(-y_0^2 - y_0 + 2) \\ &\quad + x_0(y_0^5 - 2y_0^4 + 8y_0^2 - 13y_0 + 6) + (-y_0^6 + 3y_0^4 - 2y_0^3)], \\ \Lambda_4 &= y_0[4x_0^5(-y_0^2 - y_0 + 2) + 12x_0^4(-y_0^2 - y_0 + 2) + 12x_0^3(-y_0^2 - y_0 + 2) \\ &\quad + x_0^2(2y_0^5 - 4y_0^4 + 4y_0^3 + 4y_0^2 - 14y_0 + 8) + x_0(-4y_0^6 + 6y_0^5 - 2y_0^3)], \\ \Lambda_5 &= x_0^6(-y_0^2 - y_0 + 2) + 3x_0^5(-y_0^2 - y_0 + 2) + 3x_0^4(-y_0^2 - y_0 + 2) \end{aligned}$$

$$+x_0^3(y_0^5-2y_0^4+4y_0^3-4y_0^2-y_0+2)+x_0^2(-3y_0^6+6y_0^5-3y_0^4).$$

Simplifying the above equation, we get

$$\begin{aligned}F'(t) = \frac{\pi^2}{9}\frac{-y_0^2-y_0+2}{y_0^2(ty_0+x_0)^3}\Big\{&(x_0^2+3x_0+3)y_0^3t^3+3x_0(x_0^2+3x_0+3)y_0^2t^2\\&+[3x_0^4+9x_0^3+9x_0^2+x_0(-y_0^3+3y_0^2-5y_0+3)+y_0^3(y_0-1)]y_0t\\&+\Big[x_0^5+3x_0^4+3x_0^3+x_0^2\frac{-y_0^4+y_0^3-3y_0^2+y_0+2}{y_0+2}+x_0\frac{3y_0^5-3y_0^4}{y_0+2}\Big]\Big\}. \qquad (3.15)\end{aligned}$$

From (3.14), we can get

$$\begin{aligned}F''(t) &= \frac{\pi^2}{9}\frac{(4\Lambda_1x_0-\Lambda_2y_0)t^3+(3\Lambda_2x_0-2\Lambda_3y_0)t^2+(2\Lambda_3x_0-3\Lambda_4y_0)t+(\Lambda_4x_0-4\Lambda_5y_0)}{y_0^2(ty_0+x_0)^5}\\&= \frac{\pi^2}{9}\frac{\Gamma_1t^2+\Gamma_2t+\Gamma_3}{y_0^2(ty_0+x_0)^5}, \qquad (3.16)\end{aligned}$$

where

$$\begin{aligned}\Gamma_1 &= -2x_0y_0^3(y_0^5-2y_0^4+8y_0^2-13y_0+6)-2y_0^6(-y_0^3+3y_0-2),\\ \Gamma_2 &= x_0^2y_0^2(-4y_0^5+8y_0^4-12y_0^3+4y_0^2+16y_0-12)+x_0y_0^5(10y_0^3-18y_0^2+6y_0+2),\\ \Gamma_3 &= x_0^3y_0^2(-2y_0^4+4y_0^3-12y_0^2+20y_0-10)+x_0^2y_0^4(8y_0^3-18y_0^2+12y_0-2).\end{aligned}$$

Simplifying the above equation, we get

$$\begin{aligned}F''(t) &= \frac{\pi^2}{9}\frac{\frac{\Gamma_1}{y_0}t+\left(\frac{\Gamma_2}{y_0}-\frac{x_0\Gamma_1}{y_0^2}\right)}{y_0^2(ty_0+x_0)^4}\\&= \frac{\pi^2}{9}\frac{(y_0-1)^2}{(ty_0+x_0)^4}\{[-2x_0(y_0+2)(y_0^2-2y_0+3)+2y_0^3(y_0+2)]t\\&\quad+[x_0^2(-2y_0^2-10)+x_0y_0^2(8y_0-2)]\}. \qquad (3.17)\end{aligned}$$

Second step. *We prove that*

$$(i)\ F'\Big(\frac{-x_0+y_0-1}{y_0}\Big)\leqslant 0 \text{ for } (x_0,y_0)\in\mathscr{D}_1$$

and

$$(ii)\ F''\Big(\frac{-x_0+y_0-1}{y_0}\Big)\leqslant 0 \text{ for } (x_0,y_0)\in\mathscr{D}_2,$$

where

$$\begin{aligned}\mathscr{D}_1 &= \Big\{(x,y): -1\leqslant x\leqslant y-1,\ \frac{-1+\sqrt{5}}{2}\leqslant y\leqslant 1\}\\&\quad\cup\{(x,y): -1\leqslant x\leqslant \frac{y^3+2y^2+3y-6}{(2-y)(y+3)},\ 0\leqslant y\leqslant \frac{-1+\sqrt{5}}{2}\Big\} \qquad (3.18)\end{aligned}$$

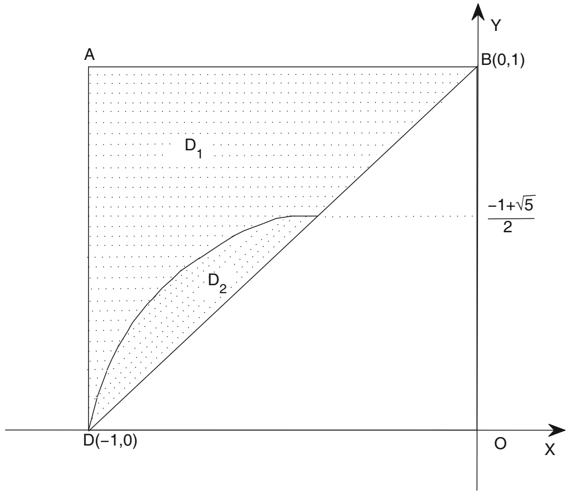

Figure 3.2: *The domains of D_1 and D_2.*

and

$$\mathscr{D}_2 = \left\{(x,y) : \frac{y^3+2y^2+3y-6}{(2-y)(y+3)} \leqslant x \leqslant y-1,\ 0 \leqslant y \leqslant \frac{-1+\sqrt{5}}{2}\right\}. \tag{3.19}$$

In fact, from (3.15), we have that

$$F'\Big(\frac{-x_0+y_0-1}{y_0}\Big) = \frac{\pi^2}{9y_0^2}\,G(x_0,y_0), \tag{3.20}$$

where

$$G(x_0,y_0) = x_0^2(2-y_0)(y_0+3) - x_0(y_0^3+3y_0^2+4y_0-12) - (y_0+2)(y_0^3+3y_0-3).$$

Noting that $G(x_0,y_0)$ is a quadratic function of the variable x_0 defined on $[-1,y_0-1]$ and $0 \leqslant y_0 \leqslant 1$, the graph of the quadratic function is a parabola opening upwards.

When $x_0 = -1$, we obtain

$$G(-1,y_0) = -y_0^2(y_0^2+y_0+1) < 0.$$

When $x_0 = y_0 - 1$, we have

$$G(y_0-1,y_0) = -3y_0^2(y_0^2+y_0-1).$$

Then we have

$$G(y_0-1,y_0) \leqslant 0 \ \text{ for } \frac{-1+\sqrt{5}}{2} \leqslant y_0 \leqslant 1$$

and

$$G(y_0-1,y_0) \geqslant 0 \ \text{ for } 0 \leqslant y_0 < \frac{-1+\sqrt{5}}{2}.$$

When

$$x_0 = \frac{y_0^3 + 2y_0^2 + 3y_0 - 6}{(2 - y_0)(y_0 + 3)} \in [-1, y_0 - 1],$$

we have

$$G\Big(\frac{y_0^3 + 2y_0^2 + 3y_0 - 6}{(2 - y_0)(y_0 + 3)}, y_0\Big) = G(-1, y_0) < 0.$$

Hence,

$$G(x_0, y_0) \leqslant 0, \text{ for } (x_0, y_0) \in \mathscr{D}_1.$$

From (3.20), we have

$$F'\Big(\frac{-x_0 + y_0 - 1}{y_0}\Big) \leqslant 0 \text{ for } (x_0, y_0) \in \mathscr{D}_1. \tag{3.21}$$

By (3.17), we get

$$F''\Big(\frac{-x_0 + y_0 - 1}{y_0}\Big) = \frac{\pi^2}{9} \frac{1}{y_0(1 - y_0)} H(x_0, y_0), \tag{3.22}$$

where

$$H(x_0, y_0) = 12x_0^2 - x_0(4y_0^3 + 2y_0 - 12) - 2y_0^3(y_0 + 2). \tag{3.23}$$

Noting that $H(x_0, y_0)$ is a quadratic function of the variable x_0 defined on $[-1, y_0 - 1]$ and the coefficient of the quadratic term is positive, the graph of the quadratic function is a parabola opening upwards.

Let $x_0 = y_0 - 1$, we have

$$H(y_0 - 1, y_0) = -6y_0^4 - 10y_0(1 - y_0) \leqslant 0. \tag{3.24}$$

Let

$$x_0 = \frac{y_0^3 + 2y_0^2 + 3y_0 - 6}{(2 - y_0)(y_0 + 3)},$$

we have

$$\begin{aligned} &H\Big(\frac{y_0^3 + 2y_0^2 + 3y_0 - 6}{(2 - y_0)(y_0 + 3)}, y_0\Big) \\ &= \frac{2y_0^8 + 4y_0^7 + 24y_0^6 + 50y_0^5 - 38y_0^4 - 18y_0^3 - 48y_0^2 - 72y_0}{(2 - y_0)^2(y_0 + 3)^2} \\ &\leqslant 0. \end{aligned} \tag{3.25}$$

From (3.24) and (3.25), we have

$$H(x_0, y_0) \leqslant 0 \text{ for } (x_0, y_0) \in \mathscr{D}_2.$$

Therefore, from (3.22) and $0 < y_0 < 1$, we have

$$F''\Big(\frac{-x_0 + y_0 - 1}{y_0}\Big) \leqslant 0 \text{ for } (x_0, y_0) \in \mathscr{D}_2. \tag{3.26}$$

Third step. *We prove* $\mathscr{P}(R) \geqslant \min\{\mathscr{P}(R_1), \mathscr{P}(R_2)\}$.

By (3.17), we have

$$F''(t) = \frac{\pi^2}{9} \frac{(y_0-1)^2}{(ty_0+x_0)^4} I(t), \tag{3.27}$$

where

$$\begin{aligned} I(t) = {} & [-2x_0(y_0+2)(y_0^2-2y_0+3)+2y_0^3(y_0+2)]t \\ & +[x_0^2(-2y_0^2-10)+x_0y_0^2(8y_0-2)] \end{aligned} \tag{3.28}$$

and

$$0 \leqslant t \leqslant \frac{-x_0+y_0-1}{y_0}. \tag{3.29}$$

Since

$$-2x_0(y_0+2)(y_0^2-2y_0+3)+2y_0^3(y_0+2) > 0,$$

$I(t)$ is an increasing function of the variable t.

By (3.26), for any

$$(x_0, y_0) \in \mathscr{D}_2,$$

we have

$$F''\Big(\frac{-x_0+y_0-1}{y_0}\Big) \leqslant 0.$$

From (3.27), we have

$$I\Big(\frac{-x_0+y_0-1}{y_0}\Big) \leqslant 0,$$

which implies that $I(t) \leqslant 0$ for any

$$0 \leqslant t \leqslant \frac{-x_0+y_0-1}{y_0}.$$

Therefore $F''(t) \leqslant 0$ for any

$$0 \leqslant t \leqslant \frac{-x_0+y_0-1}{y_0}.$$

It follows that the function $F(t)$ is concave on the interval

$$\Big[0, \frac{-x_0+y_0-1}{y_0}\Big],$$

which implies

$$F(t) \geqslant \min\Big\{F(0), F\Big(\frac{-x_0+y_0-1}{y_0}\Big)\Big\}.$$

Therefore, we have

$$\mathscr{P}(R) \geqslant \min\{\mathscr{P}(R_1), \mathscr{P}(R_2)\}.$$

By (3.21), for any $(x_0,y_0)\in\mathscr{D}_1$, we have

$$F'\Big(\frac{-x_0+y_0-1}{y_0}\Big)\leqslant 0.$$

Now we prove that the inequality (3.4) holds in each of the following situations:

$$\begin{aligned}&\text{(i)} && I\Big(\frac{-x_0+y_0-1}{y_0}\Big)\leqslant 0;\\ &\text{(ii)} && I\Big(\frac{-x_0+y_0-1}{y_0}\Big)>0 \text{ and } I(0)<0;\\ &\text{(iii)} && I(0)\geqslant 0.\end{aligned}$$

We have proved (3.4) in the case (i), and now we prove (3.4) in cases (ii) and (iii).

For the case (ii), since $I(t)$ is increasing and by (3.27), there exists a real number

$$t_0\in\Big(0,\frac{-x_0+y_0-1}{y_0}\Big)$$

satisfying

$$F''(t)\leqslant 0 \text{ for } t\in[0,t_0]$$

and

$$F''(t)>0 \text{ for } t\in\Big(t_0,\frac{-x_0+y_0-1}{y_0}\Big].$$

It follows that $F'(t)$ is decreasing on the interval $[0,t_0]$ and increasing on the interval

$$\Big(t_0,\frac{-x_0+y_0-1}{y_0}\Big].$$

If $F'(0)\leqslant 0$, and since

$$F'\Big(\frac{-x_0+y_0-1}{y_0}\Big)\leqslant 0,$$

we have

$$F'(t)\leqslant 0 \text{ for any } t\in\Big[0,\frac{-x_0+y_0-1}{y_0}\Big],$$

which implies that the function $F(t)$ is decreasing and

$$F(t)\geqslant F\Big(\frac{-x_0+y_0-1}{y_0}\Big) \text{ for any } t\in\Big[0,\frac{-x_0+y_0-1}{y_0}\Big].$$

Therefore we have

$$\mathscr{P}(R)\geqslant\min\{\mathscr{P}(R_1),\mathscr{P}(R_2)\}=\mathscr{P}(R_2).$$

If $F'(0)>0$, there exists a real number

$$t_1\in\Big(0,\frac{-x_0+y_0-1}{y_0}\Big)$$

satisfying

$$F'(t) > 0 \text{ for any } t \in [0, t_1)$$

and

$$F'(t) \leqslant 0 \text{ for any } t \in \left[t_1, \frac{-x_0 + y_0 - 1}{y_0}\right],$$

which implies that the function $F(t)$ is increasing on the interval $[0, t_1)$ and decreasing on the interval

$$\left[t_1, \frac{-x_0 + y_0 - 1}{y_0}\right].$$

It follows that

$$F(t) \geqslant \min\left\{F(0), F\left(\frac{-x_0 + y_0 - 1}{y_0}\right)\right\} \text{ for any } t \in \left[0, \frac{-x_0 + y_0 - 1}{y_0}\right].$$

We then have

$$\mathscr{P}(R) \geqslant \min\{\mathscr{P}(R_1), \mathscr{P}(R_2)\}.$$

For the case (iii), since the function $I(t)$ is increasing, we have

$$I(t) \geqslant 0 \text{ for any } t \in \left[0, \frac{-x_0 + y_0 - 1}{y_0}\right].$$

Hence, from (3.27), we have

$$F''(t) \geqslant 0 \text{ for any } t \in \left[0, \frac{-x_0 + y_0 - 1}{y_0}\right].$$

Therefore, the function $F'(t)$ is increasing on the interval

$$\left[0, \frac{-x_0 + y_0 - 1}{y_0}\right],$$

and since

$$F'\left(\frac{-x_0 + y_0 - 1}{y_0}\right) \leqslant 0,$$

we have

$$F'(t) \leqslant 0 \text{ for any } t \in \left[0, \frac{-x_0 + y_0 - 1}{y_0}\right],$$

which implies that the function $F(t)$ is decreasing on the interval

$$\left[0, \frac{-x_0 + y_0 - 1}{y_0}\right].$$

Therefore, we have

$$F(t) \geqslant F\left(\frac{-x_0 + y_0 - 1}{y_0}\right) \text{ for any } t \in \left[0, \frac{-x_0 + y_0 - 1}{y_0}\right],$$

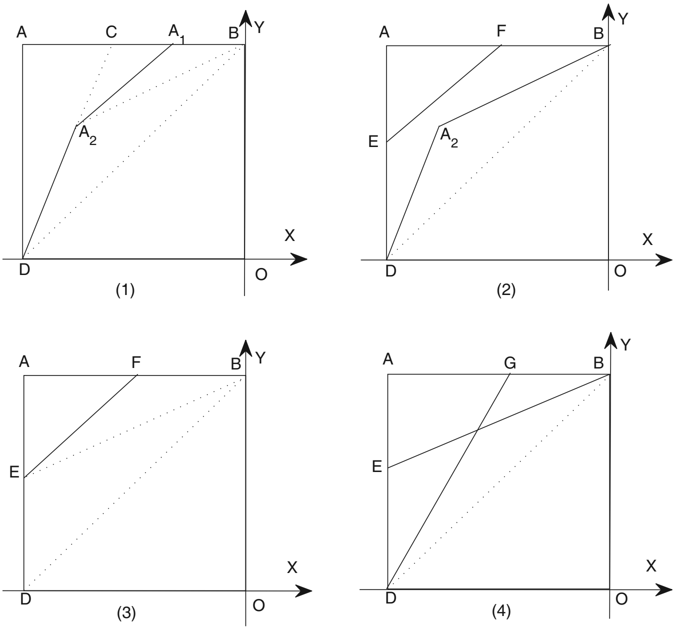

Figure 3.3: *P, P_1, P_{DEB}, and their polar bodies in the second quadrant.*

which implies that

$$\mathscr{P}(R) \geqslant \min\{\mathscr{P}(R_1), \mathscr{P}(R_2)\} = \mathscr{P}(R_2).$$

Secondly, we prove (3.5).

In (3.4), if

$$\min\{\mathscr{P}(R_1), \mathscr{P}(R_2)\} = \mathscr{P}(R_2).$$

Let

$$T = \{(x,y) : |x| + |y| \leqslant 1\}$$

and

$$Q = \{(x,y) : \max\{|x|, |y|\} \leqslant 1\}.$$

Let R_T and R_Q be the origin-symmetric bodies of revolution generated by T and Q, respectively. In (3.4), replacing R, R_1, and R_2, by R_2, R_T, and R_Q, respectively (see (1) of Figure 3.3), we obtain

$$\mathscr{P}(R_2) \geqslant \min\{\mathscr{P}(R_T), \mathscr{P}(R_Q)\} = \frac{4\pi^2}{3}. \tag{3.30}$$

It follows that

$$\mathscr{P}(R) \geqslant \mathscr{P}(R_2) \geqslant \frac{4\pi^2}{3}.$$

In (3.4), if

$$\min\{\mathscr{P}(R_1), \mathscr{P}(R_2)\} = \mathscr{P}(R_1),$$

let E, F be the vertices of P_1^* in the second quadrant, where E, F lie on line segments AD and AB, respectively (see (2) of Figure 3.3). Let P_{DEB} be a 1-unconditional polygon satisfying

$$P_{DEB} \cap \{(x,y) : x \leqslant 0, y \geqslant 0\} = \operatorname{conv}\{E,D,O,B\},$$

and let R_{DEB} be an origin-symmetric body of revolution generated by P_{DEB}. In (3.4), replacing R, R_1, and R_2, by $R_1{}^*$, R_{DEB}, and R_Q, respectively (see (3) of Figure 3.3), we have

$$\mathscr{P}(R) \geqslant \mathscr{P}(R_1) = \mathscr{P}(R_1{}^*) \geqslant \min\{\mathscr{P}(R_{DEB}), \mathscr{P}(R_Q)\}. \tag{3.31}$$

In (3.31), if

$$\min\{\mathscr{P}(R_{DEB}), \mathscr{P}(R_Q)\} = \mathscr{P}(R_Q),$$

we have proved (3.5); if

$$\min\{\mathscr{P}(R_{DEB}), \mathscr{P}(R_Q)\} = \mathscr{P}(R_{DEB}),$$

let

$$P_{DEB}{}^* \cap \{(x,y) : x \leqslant 0,\ y \geqslant 0\} = \operatorname{conv}\{G,D,O,B\},$$

where G lies on the line segment AB, which is a vertex of $P_{DEB}{}^*$ (see (4) of Figure 3.3). In (3.4), replacing R, R_1, and R_2, by $R_{DEB}{}^*$, R_T, and R_Q, respectively, we obtain

$$\mathscr{P}(R_{DEB}{}^*) \geqslant \min\{\mathscr{P}(R_T), \mathscr{P}(R_Q)\} = \frac{4\pi^2}{3}. \tag{3.32}$$

Hence, we have

$$\mathscr{P}(R) \geqslant \mathscr{P}(R_1) \geqslant \mathscr{P}(R_{DEB}) \geqslant \frac{4\pi^2}{3}. \quad \square$$

LEMMA 3.5. *Let P be a 1-unconditional polygon in the coordinate plane XOY satisfying*

$$P \cap \{(x,y) : x \leqslant 0,\ y \geqslant 0\} = \operatorname{conv}\{A_1, A_2, \cdots, A_{n-1}, D, O, B\},$$

where A_1 lies on the line segment AB, $A_2, \cdots, A_{n-1} \in \operatorname{int}\triangle ABD$, and the slopes of lines OA_i $(i = 1, \cdots, n-1)$ are increasing on i, R the origin-symmetric body of revolution generated by P. Then

$$\mathscr{P}(R) \geqslant \min\{\mathscr{P}(R_1), \mathscr{P}(R_2)\}, \tag{3.33}$$

where R_1 and R_2 are origin-symmetric bodies of revolution generated by 1-unconditional polygons P_1 and P_2 satisfying

$$P_1 \cap \{(x,y) : x \leqslant 0,\ y \geqslant 0\} = \operatorname{conv}\{A_2, A_3, \cdots, A_{n-1}, D, O, B\}$$

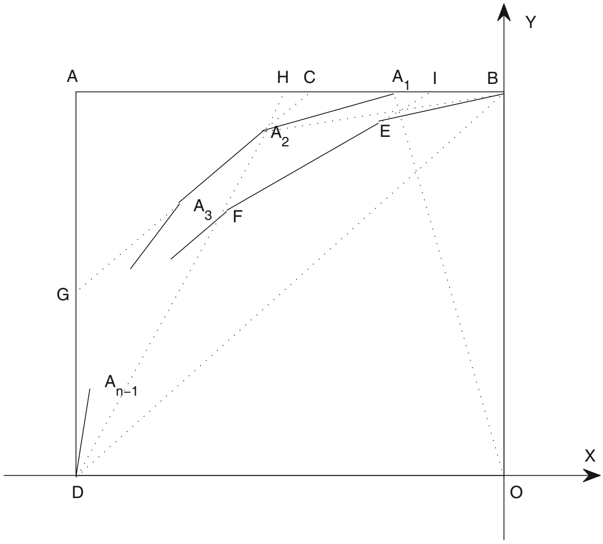

Figure 3.4: *P and P^* in the second quadrant.*

and

$$P_2 \cap \{(x,y) : x \leqslant 0,\ y \geqslant 0\} = \operatorname{conv}\{C, A_3, \cdots, A_{n-1}, D, O, B\},$$

respectively, where C is the point of intersection between two lines A_2A_3 and AB.

Proof. In Figure 3.4, let $A_1 = (-t, 1)$ and $A_2 = (x_0, y_0)$. Let the slope of the line A_3A_2 be k, then

$$\frac{1-y_0}{-x_0} < k < \frac{y_0}{x_0+1} \tag{3.34}$$

and the equation of the line A_3A_2 is

$$y - y_0 = k(x - x_0). \tag{3.35}$$

In (3.35), let $y = 1$, we get the abscissa of C

$$x_C = x_0 + \frac{1-y_0}{k}.$$

Let E, F and B be the vertices of P^* satisfying $BE \perp OA_1$ and $EF \perp OA_2$. Let I be the point of intersection between two lines EF and AB. We have

$$BE :\ y = tx + 1$$

and

$$EF :\ y = -\frac{x_0}{y_0}x + \frac{1}{y_0}.$$

Then, we get

$$I=\Big(\frac{1-y_0}{x_0},1\Big)$$

and

$$E=\Big(\frac{1-y_0}{ty_0+x_0},\frac{t+x_0}{ty_0+x_0}\Big). \tag{3.36}$$

Let

$$F(t)=\frac{1}{2}V(R)\frac{1}{2}V(R^*)=\frac{1}{4}\mathscr{P}(R), \tag{3.37}$$

which is a function of the variable t, where

$$0\leqslant t\leqslant -x_C=\frac{-x_0k+y_0-1}{k}.$$

Our proof has three steps.

First step. *Calculate* $F'(t)$ *and* $F''(t)$.
Let $V=\frac{1}{2}V(R_1)$ and $V^0=\frac{1}{2}V(R_1{}^*)$, then we obtain

$$\begin{aligned}F(t)\,=\,&\Big(V+\frac{\pi}{3}(2-y_0-y_0^2)t\Big)\\&\times\left(V^0-\frac{\pi}{3}\frac{y_0-1}{x_0}\left(2-\frac{t+x_0}{ty_0+x_0}-\left(\frac{t+x_0}{ty_0+x_0}\right)^2\right)\right).\end{aligned} \tag{3.38}$$

Therefore, we have

$$F'(t)\,=\,\frac{\pi}{3}\frac{(2-y_0-y_0^2)(\Phi_1t^3+\Phi_2t^2+\Phi_3t+\Phi_4)}{(y_0t+x_0)^3}, \tag{3.39}$$

where

$$\begin{aligned}\Phi_1&=y_0\Big[-\frac{\pi}{3}\frac{(1-y_0)^2(2y_0+1)}{x_0}+V^0y_0^2\Big],\\\Phi_2&=-\pi(1-y_0)^2(2y_0+1)+3V^0x_0y_0^2,\\\Phi_3&=-2\pi(1-y_0)^2x_0+3V^0x_0^2y_0+(y_0-1)V,\\\Phi_4&=V^0x_0^3-\frac{3x_0(1-y_0)V}{y_0+2}.\end{aligned} \tag{3.40}$$

Thus, we have

$$F''(t)\,=\,\frac{2\pi}{3}\frac{(1-y_0)^2}{(ty_0+x_0)^4}J(t), \tag{3.41}$$

where

$$
\begin{aligned}
J(t) = &(y_0+2)[Vy_0+\pi x_0(y_0-1)]t \\
&+x_0[V(4y_0-1)+\pi x_0(y_0^2+y_0-2)].
\end{aligned} \tag{3.42}
$$

Second step. *We prove that*

$$
(i)\ F'\Big(\frac{-x_0k+y_0-1}{k}\Big)\leqslant 0\ \ or\ \ F''\Big(\frac{-x_0k+y_0-1}{k}\Big)\leqslant 0\ for\ (x_0,y_0)\in\mathscr{D}_1
$$

and

$$
(ii)\ F''\Big(\frac{-x_0k+y_0-1}{k}\Big)\leqslant 0\ for\ (x_0,y_0)\in\mathscr{D}_2,
$$

where $\mathscr{D}_1$ *and* $\mathscr{D}_2$ *have been given in* (*3.18*) *and* (*3.19*)*.*

By (3.39) and (3.40), let

$$
t_0=\frac{-x_0k+y_0-1}{k},
$$

we have

$$
F'(t_0)=\frac{\pi}{3}(\Upsilon_1V^0+\Upsilon_2V+\Upsilon_3), \tag{3.43}
$$

where

$$
\begin{aligned}
\Upsilon_1 &= (1-y_0)(y_0+2),\\
\Upsilon_2 &= \frac{k^2(-x_0k+y_0+2)}{(x_0k-y_0)^3},\\
\Upsilon_3 &= -\frac{\pi}{3}\frac{y_0+2}{x_0(x_0k-y_0)^3}[k^3x_0^3(y_0-1)(-2y_0+3)+3k^2x_0^2y_0(y_0-1)(2y_0-3)\\
&\quad +3kx_0(1-y_0)^3(2y_0+1)+y_0(2y_0+1)(y_0-1)^3].
\end{aligned} \tag{3.44}
$$

Since $k>0$, $x_0<0$ and $0<y_0<1$, we have that $\Upsilon_1\geqslant 0$ and $\Upsilon_2\leqslant 0$, thus, as V increases and V^0 decreases, $F'(t_0)$ decreases.

Let P_0 be a 1-unconditional polygon satisfying

$$
P_0\cap\{(x,y): x\leqslant 0,\ y\geqslant 0\}=\mathrm{conv}\{A_2,D,O,B\}
$$

and R_0 be an origin-symmetric body of revolution generated by P_0. Let $V_0=\frac{1}{2}V(R_0)$ and $V_0{}^*=\frac{1}{2}V(R_0^*)$. In (3.38), let $V=V_0$ and $V^0=V_0{}^*$, we get a function $F_0(t)$, which is the same function as $F(t)$ in Lemma 3.4.

Since $V\geqslant V_0$ and $V^0\leqslant V_0^*$, we have

$$
F'\Big(\frac{-x_0k+y_0-1}{k}\Big)\leqslant F_0'\Big(\frac{-x_0k+y_0-1}{k}\Big). \tag{3.45}
$$

Since

$$\frac{1-y_0}{-x_0} \leqslant k \leqslant \frac{y_0}{x_0+1},$$

we have

$$0 \leqslant \frac{-x_0k+y_0-1}{k} \leqslant \frac{-x_0+y_0-1}{y_0}.$$

In (3.45), let

$$k = \frac{y_0}{x_0+1},$$

we have

$$F'\Big(\frac{-x_0+y_0-1}{y_0}\Big) \leqslant F_0'\Big(\frac{-x_0+y_0-1}{y_0}\Big). \tag{3.46}$$

From Lemma 3.4, we have

$$F_0'\Big(\frac{-x_0+y_0-1}{y_0}\Big) \leqslant 0 \ \text{ for any } \ (x_0,y_0) \in \mathscr{D}_1, \tag{3.47}$$

hence

$$F'\Big(\frac{-x_0+y_0-1}{y_0}\Big) \leqslant 0 \ \text{ for any } \ (x_0,y_0) \in \mathscr{D}_1. \tag{3.48}$$

If $F''(t_0) > 0$, by (3.41), $J(t_0) > 0$, since $x_0 < 0$ and $0 \leqslant y_0 \leqslant 1$, $J(t)$ is an increasing linear function, thus $J(t) > 0$ for $t \geqslant t_0$, which implies $F''(t) > 0$ for $t \geqslant t_0$. Thus $F'(t)$ is increasing for $t \geqslant t_0$. Since

$$F'\Big(\frac{-x_0+y_0-1}{y_0}\Big) \leqslant 0,$$

we have $F'(t_0) \leqslant 0$. Therefore we have proved (i).

Next we prove (ii).

Let G be the point of intersection between two lines AD and A_2A_3, then $G = (-1, y_0 - k(x_0+1))$. Let P_M be a 1-unconditional polygon satisfying

$$P_M \cap \{(x,y) : x \leqslant 0,\ y \geqslant 0\} = \operatorname{conv}\{A_2, G, D, O, B\}$$

and R_M an origin-symmetric body of revolution generated by P_M. From Lemma 2.1, we have that

$$\begin{aligned}\frac{1}{2}V(R_M) = {} & \frac{\pi}{3}(x_0+1)[(y_0-k(x_0+1))^2 + (y_0-k(x_0+1))y_0 + y_0^2] \\ & + \frac{\pi}{3}(-x_0)(y_0^2+y_0+1).\end{aligned} \tag{3.49}$$

In (3.42), let

$$V = \frac{1}{2}V(R_M)$$

and

$$t = \frac{-x_0 k + y_0 - 1}{k},$$

we get a function of the variable k

$$L(k) = \frac{\Theta_1 k^3 + \Theta_2 k^2 + \Theta_3 k + \Theta_4}{k}, \tag{3.50}$$

where

$$\begin{aligned}
\Theta_1 &= -\frac{\pi}{3} x_0 (x_0+1)^3 (y_0-1)^2,\\
\Theta_2 &= \frac{\pi}{3}(x_0+1)^2 y_0 (y_0-1)(4x_0y_0 - x_0 + y_0 + 2),\\
\Theta_3 &= \frac{\pi}{3}(y_0-1)(-5x_0^2y_0^3 - 9x_0y_0^3 - 3x_0^2y_0^2 - 9x_0y_0^2 - x_0^2 - 3y_0^3 - 6y_0^2),\\
\Theta_4 &= \frac{\pi}{3}(y_0-1)(y_0+2)(2x_0y_0^3 + 3y_0^3 - x_0y_0^2 + 2x_0y_0 - 3x_0).
\end{aligned} \tag{3.51}$$

Let

$$L_1(k) = \Theta_1 k^3 + \Theta_2 k^2 + \Theta_3 k + \Theta_4. \tag{3.52}$$

Since $k > 0$, to prove $L(k) \leqslant 0$, it suffices to prove $L_1(k) \leqslant 0$. In the following, we prove $L_1(k) \leqslant 0$ for

$$\frac{1-y_0}{-x_0} \leqslant k \leqslant \frac{y_0}{x_0+1}.$$

By (3.52), we have

$$L_1''(k) = 6\Theta_1 k + 2\Theta_2. \tag{3.53}$$

Since

$$L_1''\Big(\frac{y_0}{x_0+1}\Big) = \frac{2\pi}{3}(x_0+1)^3 y_0 (y_0-1)(y_0+2) \leqslant 0$$

and

$$\Theta_1 = -\frac{\pi}{3} x_0 (x_0+1)^3 (y_0-1)^2 > 0,$$

then

$$L_1''(k) \leqslant 0 \ \text{ for any } \ \frac{1-y_0}{-x_0} \leqslant k \leqslant \frac{y_0}{x_0+1}.$$

Hence, the function $L_1'(k)$ is decreasing on the interval

$$\Big[\frac{1-y_0}{-x_0}, \frac{y_0}{x_0+1}\Big].$$

By (3.52), we have

$$L_1'(k) = 3\Theta_1 k^2 + 2\Theta_2 k + \Theta_3. \tag{3.54}$$

From (3.54), we have that

$$\begin{aligned} L_1'\left(\frac{y_0}{x_0+1}\right) &= \frac{\pi}{3}(1-y_0)[x_0^2(2y_0^2+1)+x_0(2y_0^3+4y_0^2)+y_0^3+2y_0^2] \\ &= \frac{\pi}{3}(1-y_0)\left[(2y_0^2+1)\left(x_0+\frac{y_0^3+2y_0^2}{2y_0^2+1}\right)^2+\frac{y_0^2(y_0+2)(1-y_0^3)}{2y_0^2+1}\right] \\ &\geqslant 0. \end{aligned} \tag{3.55}$$

Therefore

$$L_1'(k)\geqslant 0 \text{ for any } \frac{1-y_0}{-x_0}\leqslant k\leqslant \frac{y_0}{x_0+1}.$$

It follows that the function $L_1(k)$ is increasing on the interval

$$\left[\frac{1-y_0}{-x_0},\frac{y_0}{x_0+1}\right].$$

When

$$k=\frac{y_0}{x_0+1},$$

we have $R_M=R_0$ and

$$\frac{-x_0k+y_0-1}{k}=\frac{-x_0+y_0-1}{y_0}.$$

In Lemma 3.4, for $R=R_0$, we had proved

$$F''\left(\frac{-x_0+y_0-1}{y_0}\right)\leqslant 0 \text{ for } (x_0,y_0)\in\mathscr{D}_2.$$

Hence,

$$L_1\left(\frac{y_0}{x_0+1}\right)\leqslant 0 \text{ for } (x_0,y_0)\in\mathscr{D}_2,$$

which implies that $L_1(k)\leqslant 0$ for any

$$\frac{1-y_0}{-x_0}\leqslant k\leqslant \frac{y_0}{x_0+1} \text{ when } (x_0,y_0)\in\mathscr{D}_2.$$

It follows that, for $R=R_M$,

$$F''\left(\frac{-x_0k+y_0-1}{k}\right)\leqslant 0 \text{ for any } \frac{1-y_0}{-x_0}\leqslant k\leqslant \frac{y_0}{x_0+1}$$

when $(x_0,y_0)\in\mathscr{D}_2$.

In Lemma 3.4, for $R=R_0$, we know that

$$F''\left(\frac{-x_0+y_0-1}{y_0}\right)\leqslant 0 \text{ for } (x_0,y_0)\in\mathscr{D}_2,$$

from (3.41), which implies that

$$J\left(\frac{-x_0+y_0-1}{y_0}\right)\leqslant 0 \text{ for } (x_0,y_0)\in\mathscr{D}_2.$$

Since $J(t)$ is an increasing linear function and

$$\frac{-x_0k+y_0-1}{k} \leqslant \frac{-x_0+y_0-1}{y_0} \quad \text{for } k < \frac{y_0}{x_0+1},$$

we have

$$J\Big(\frac{-x_0k+y_0-1}{k}\Big) \leqslant 0 \text{ for } (x_0,y_0) \in \mathscr{D}_2,$$

which implies, for $R = R_0$, that

$$F''\Big(\frac{-x_0k+y_0-1}{k}\Big) \leqslant 0$$

for any

$$\frac{1-y_0}{-x_0} \leqslant k \leqslant \frac{y_0}{x_0+1} \text{ and } (x_0,y_0) \in \mathscr{D}_2.$$

Therefore, for

$$V = V(R_0) \text{ or } V = V(R_M),$$

we have

$$J\Big(\frac{-x_0k+y_0-1}{k}\Big) \leqslant 0 \text{ for } (x_0,y_0) \in \mathscr{D}_2.$$

Since

$$J(t) = [(y_0+2)y_0t + x_0(4y_0-1)]V + [\pi x_0(y_0-1)(y_0+2)t - \pi x_0^2(2-y_0-y_0^2)], \quad (3.56)$$

which can be considered as a linear function of the variable V, and

$$V(R_0) < V(R) < V(R_M),$$

we have, for any $V = V(R)$, that

$$J\Big(\frac{-x_0k+y_0-1}{k}\Big) \leqslant 0 \text{ for } (x_0,y_0) \in \mathscr{D}_2. \quad (3.57)$$

It follows that

$$F''\Big(\frac{-x_0k+y_0-1}{k}\Big) \leqslant 0 \text{ for } (x_0,y_0) \in \mathscr{D}_2. \quad (3.58)$$

Third step. *We prove*

$$\mathscr{P}(R) \geqslant \min\{\mathscr{P}(R_1), \mathscr{P}(R_2)\}.$$

We omit the proof of this step which is similar to the proof of third step in Lemma 3.4. □

LEMMA 3.6. *For any a 1-unconditional polygon $P \subset [-1,1]^2$ in the coordinate plane XOY satisfying $B, D \in P$, let R be an origin-symmetric body of revolution generated by P. Then*

$$\mathscr{P}(R) \geqslant \frac{4\pi^2}{3}, \quad (3.59)$$

with equality if and only if R is a cylinder or bicone.

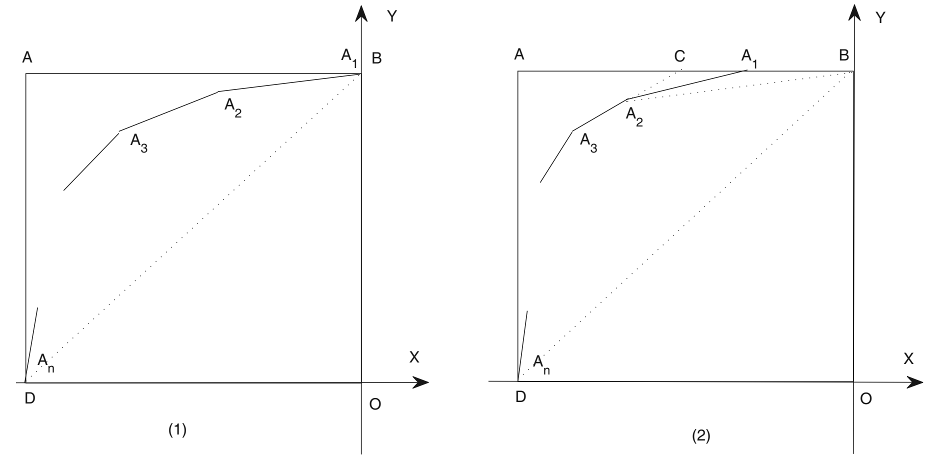

Figure 3.5: *The two cases of 1-unconditional polygon P.*

Proof. Let $A_1, A_2, \cdots, A_n$ be the vertices of P contained in the domain $\{(x,y): x \leqslant 0, y \geqslant 0\}$ and the slopes of lines OA_i $(i = 1, \cdots n)$ are increasing on i. Without loss of generality, suppose that the vertex A_n coincides with point D. The vertex A_1 satisfies the following two cases:

(i) A_1 coincides with the point B;

(ii) A_1 does not coincide with the point B, but lies on the line segment BC (C is the point of intersection between two lines A_2A_3 and AB).

If R satisfies the case (ii), from the Lemma 3.5, we obtain an origin-symmetric body of revolution R_1 with smaller Mahler volume than R and its generating domain P_1 has fewer vertices than P.

If R satisfies the case (i), then its polar body R^* satisfies the case (ii). Since $\mathscr{P}(R) = \mathscr{P}(R^*)$ and P has the same number of vertices as P^*, from the Lemma 3.5, we can also obtain an origin-symmetric body of revolution R_1 with smaller Mahler volume than R and its generating domain P_1 has fewer vertices than P.

From the above discuss and the proof of (3.5), let $R_0 = R$, we can get a sequence of origin-symmetric bodies of revolution

$$\{R_0, R_1, R_2 \cdots, R_N\},$$

where N is a natural number depending on the number of vertices of P, satisfying $\mathscr{P}(R_{i+1}) \leqslant \mathscr{P}(R_i)$ $(i = 0, 1, \cdots, N-1)$ and R_N is a cylinder or bicone. Therefore, we have

$$\mathscr{P}(R) \geqslant \frac{4\pi^2}{3},$$

with equality if and only if R is a cylinder or bicone. $\square$

THEOREM 3.7. *For any origin-symmetric body of revolution K in $\mathbb{R}^3$, we have*

$$\mathscr{P}(K) \geqslant \frac{4\pi^2}{3}, \tag{3.60}$$

with equality if and only if K is a cylinder or bicone.

Proof. By Remark 3, without loss of generality, suppose that the generating domain P of K is contained in the square $[-1,1]^2$ and $B,D \in P$.

Since a convex body can be approximated by a polytope in the sense of the Hausdorff metric (see Theorem 1.8.13 in [29]), hence, for P and any $\varepsilon > 0$, there is a 1-unconditional polygon P_ε with $\delta(P,P_\varepsilon) \leqslant \varepsilon$. Let R_ε be an origin-symmetric body of revolution generated by P_ε, then $\delta(K,R_\varepsilon) \leqslant \varepsilon$. Thus, there exists a sequence of origin-symmetric bodies of revolution $(R_i)_{i\in\mathbb{N}}$ satisfying

$$\lim_{i\to\infty} \delta(R_i,K) = 0.$$

Since $\mathscr{P}(K)$ is continuous in the sense of the Hausdorff metric, applying Lemma 3.6, we have

$$\mathscr{P}(K) \geqslant \frac{4\pi^2}{3}, \tag{3.61}$$

with equality if and only if K is a cylinder or bicone. □

In the following, we will restate and prove Theorem 1.2 and 1.3.

THEOREM 3.8. *Let $f(x)$ be a concave, even and nonnegative function defined on $[-a,a]$, $a > 0$, and for $x' \in [-\frac{1}{a},\frac{1}{a}]$ define*

$$f^*(x') = \inf_{x\in[-a,a]} \frac{1-x'x}{f(x)}. \tag{3.62}$$

Then

$$\left(\int_{-a}^{a} (f(x))^2 dx\right)\left(\int_{-\frac{1}{a}}^{\frac{1}{a}} (f^*(x'))^2 dx'\right) \geqslant \frac{4}{3}, \tag{3.63}$$

with equality if and if $f(x) = f(0)$ or $f^*(x') = 1/f(0)$.

Proof. Let R and R' be origin-symmetric bodies of revolution generated by $f(x)$ and $f^*(x')$, respectively, then their generating domains are

$$D = \{(x,y) : -a \leqslant x \leqslant a, |y| \leqslant f(x)\}$$

and

$$D' = \{(x',y') : -\frac{1}{a} \leqslant x' \leqslant \frac{1}{a}, |y'| \leqslant f^*(x')\},$$

respectively.

Next, we prove $D' = D^*$. For $(x',y') \in D'$ and $(x,y) \in D$, we have

$$(x',y') \cdot (x,y) = x'x + y'y \leqslant x'x + f^*(x')f(x) \leqslant x'x + \frac{1-x'x}{f(x)} f(x) = 1,$$

which implies $(x',y') \in D^*$. If $(x',y') \notin D'$, then either $|x'| > \frac{1}{a}$ or $|x'| \leqslant \frac{1}{a}$ and $|y'| > f^*(x')$. If $x' > \frac{1}{a}$ (or $x' < -\frac{1}{a}$), then for $(a,0) \in D$ (or $(-a,0) \in D$), we have

$$(x',y') \cdot (a,0) > 1 \ \ (\text{or} \ \ (x',y') \cdot (-a,0) > 1),$$

which implies $(x',y') \notin D^*$. If $|x'| \leqslant \frac{1}{a}$ and $y' > f^*(x')$ (or $y' < -f^*(x')$), let

$$f^*(x') = \frac{1-x'x_0}{f(x_0)},$$

then for $(x_0, f(x_0)) \in D$ (or $(x_0, -f(x_0)) \in D$), we have

$$(x',y') \cdot (x_0, f(x_0)) > x'x_0 + f^*(x')f(x_0) = 1$$

$$(\text{or} \ \ (x',y') \cdot (x_0, -f(x_0)) > x'x_0 + f^*(x')f(x_0) = 1),$$

which implies $(x',y') \notin D^*$. Hence, we have $D' = D^*$. By Lemma 3.2, we get $R' = R^*$. By Theorem 3.7, we have

$$\int_{-a}^{a} (f(x))^2 dx \int_{-\frac{1}{a}}^{\frac{1}{a}} (f^*(x'))^2 dx' = \frac{1}{\pi^2} V(R)V(R') = \frac{1}{\pi^2} \mathscr{P}(R) \geqslant \frac{4}{3},$$

with equality if and if $f(x) = f(0)$ or $f^*(x') = 1/f(0)$. $\square$

By Theorem 3.8, we prove that among parallel sections homothety bodies in $\mathbb{R}^3$, 3-cubes have the minimal Mahler volume.

THEOREM 3.9. *For any parallel sections homothety body K in $\mathbb{R}^3$, we have*

$$\mathscr{P}(K) \geqslant \frac{4^3}{3!}, \tag{3.64}$$

with equality if and only if K is a 3-cube or octahedron.

Proof. Let

$$K = \bigcup_{x \in [-a,a]} \{f(x)C + xv\},$$

where $f(x)$ is its generating function and C is homothetic section. Next, for

$$K' = \bigcup_{x' \in [-\frac{1}{a}, \frac{1}{a}]} \{f^*(x')C^* + x'v\},$$

where $f^*(x')$ is given in (3.62), we prove $K' = K^*$. For any

$$(x',y',z') \in K' \text{ and } (x,y,z) \in K,$$

we have

$$(0,y',z') \in f^*(x')C^* \text{ and } (0,y,z) \in f(x)C.$$

Hence, we have

$$(0,y',z')\cdot(0,y,z) \leqslant f^*(x')f(x) \leqslant \frac{1-x'x}{f(x)}f(x) = 1-x'x.$$

It follows that

$$(x',y',z')\cdot(x,y,z) = x'x + (0,y',z')\cdot(0,y,z) \leqslant 1,$$

which implies that $(x',y',z') \in K^*$.

If $(x',y',z') \notin K'$, then either $|x'| > \frac{1}{a}$ or $|x'| \leqslant \frac{1}{a}$ and $(0,y',z') \notin f^*(x')C^*$. If $x > \frac{1}{a}$ (or $x < -\frac{1}{a}$), then for $(a,0,0) \in K$ (or $(-a,0,0) \in K$), we have

$$(x',y',z')\cdot(a,0,0) > 1 \;\; (\text{or } (x',y',z')\cdot(-a,0,0) > 1),$$

which implies that $(x',y',z') \notin K^*$. If $|x'| \leqslant \frac{1}{a}$ and $(0,y',z') \notin f^*(x')C^*$, there exists $(0,y,z) \in C$ such that

$$(0,y,z)\cdot(0,y',z') > f^*(x').$$

Let

$$f^*(x') = \frac{1-x'x_0}{f(x_0)}.$$

For

$$(x_0, f(x_0)y, f(x_0)z) \in K$$

we have

$$\begin{aligned}
&(x',y',z')\cdot(x_0,f(x_0)y,f(x_0)z)\\
&= x'x_0 + f(x_0)(0,y,z)\cdot(0,y',z')\\
&> x'x_0 + f(x_0)f^*(x')\\
&= x'x_0 + f(x_0)\frac{1-x'x_0}{f(x_0)}\\
&= 1,
\end{aligned} \tag{3.65}$$

which implies that $(x',y',z') \notin K^*$. Hence, we have $K' = K^*$.

Therefore, we obtain

$$\begin{aligned}
\mathscr{P}(K) &= V(K)V(K')\\
&= \mathscr{P}(C)\int_{-a}^{a}(f(x))^2dx\int_{-\frac{1}{a}}^{\frac{1}{a}}(f^*(x'))^2dx'\\
&\geqslant \frac{4^2}{2!}\frac{4}{3} = \frac{4^3}{3!},
\end{aligned} \tag{3.66}$$

with equality if and only if K is a 3-cube or octahedron. □

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