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A SUPPLEMENT TO THE STRONG LAWS FOR WEIGHTED SUMS OF φ–MIXING RANDOM VARIABLES

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Abstract. Some complete convergence theorems for linear statistics that are weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ are established, where $\{X_n; n \geqslant 1\}$ is a sequence of φ - mixing random variables and $\{a_{ni}; 1 \leqslant i \leqslant n, n \geqslant 1\}$ is an array of constants. As an application, the Marcinkiewicz-Zygmund strong law of large numbers for weighted sums of φ - mixing random variables is obtained.

1. Introduction

As Bai and Cheng (2000) remarked, many useful linear statistics based on a random sample are weighted sums of independent and identically distributed (i.i.d.) random variables. Examples include least-squares estimators, nonparametric regression function estimators, jackknife estimates, and so on. In this respect, studies of strong laws for these weighted sums have demonstrated significant progress in probability theory with applications in mathematical statistics. In many stochastic models, the assumption of independence among random variables is not plausible. So it is necessary to extend the concept of independence to dependence cases, one of these dependence structures is φ-mixing. So we want to know if the results obtained for i.i.d. random variables are still true for φ-mixing sequences of random variables.

Let $\{X_n; n \geqslant 1\}$ be a sequence of random variables defined on a fixed probability space (Ω, F, P) and $S_n = \sum_{i=1}^{n} X_i$ for each $n \geqslant 1$. Let n and m be positive integers. Write $F_n^m = \sigma\ (X_i; n \leqslant i \leqslant m)$. Given two σ-algebras ψ, ζ in F, define that

$$\varphi(\psi, \zeta) = \sup\{|P(B|A) - P(B)|; A \in \psi, P(A) > 0, B \in \zeta\}. \tag{1.1}$$

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and define the φ-mixing coefficients by

$$\varphi(n) = \sup_{k\geqslant 1} \varphi(F_1^k, F_{k+n}^\infty), \quad n \geqslant 0. \tag{1.2}$$

Obviously, $0 \leqslant \varphi(n+1) \leqslant \varphi(n) \leqslant 1$, $n \geqslant 0$ and $\varphi(0) = 1$.

DEFINITION 1.1. A sequence of random variables $\{X_n; n \geqslant 1\}$ is said to be a φ-mixing sequence if $\varphi(n) \downarrow 0$ as $n \to \infty$.

Note that if $\{X_n; n \geqslant 1\}$ is a sequence of independent random variables, then $\varphi(n) = 0$ for all $n \geqslant 1$.

The concept of φ-mixing random variables was introduced by Dobrushin (1956) and many applications have been found. We can refer to Dobrushin (1956), Utev (1990) and Chen (1991) for central limit theorem, Herrndorf (1983) and Peligrad (1985) for weak invariance principle, Sen (1971, 1974) for weak convergence of empirical processes, Iosifescu (1977) for limit theorem, Peligrad (1990) for Ibragimov-Iosifescu conjecture, Shao (1993) for almost sure invariance principles, Hu and Wang (2008) for large deviations, Wang et al (2009) for strong law of large numbers, Wang et al (2010) for complete convergence for weighted sums, and so forth. When these results are compared with the corresponding results for sequences of independent random variables, there still remains much to be desired.

Let $\{X_n; n \geqslant 1\}$ be a sequence of random variables and let $\{a_{ni}; 1 \leqslant i \leqslant n, n \geqslant 1\}$ is an array of constants. If there exists some α with $0 < \alpha < 2$ such that $\sum_{i=1}^n |a_{ni}|^\alpha = O(n)$, Bai and Cheng (2000) established Marcinkiewicz-Zygmund strong laws for linear statistics of i.i.d. random variables. Recently, Cai (2006) generalized and improved the result of Bai and Cheng (2000) to the case of ρ^*-mixing sequences of random variables under the same condition. But, they did not study the condition "there exists some δ with $0 < \delta < 1$ and some α with $0 < \alpha < 2$ such that $\sum_{i=1}^n |a_{ni}|^\alpha = O(n^\delta)$". We are inspired by the result of Cai (2006), and will further study the condition "there exists some δ with $0 < \delta < 1$ and some α with $0 < \alpha < 2$ such that $\sum_{i=1}^n |a_{ni}|^\alpha = O(n^\delta)$". The results obtained complement and improve the corresponding results for i.i.d. random variable sequences to the case of φ-mixing random variables. The techniques used in the paper are inspired by Cai (2006).

2. Main results and Proofs

Throughout this paper, C will represent a generic positive constant whose value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \leqslant C(b_n)$.

To prove our results, we need the following lemmas.

LEMMA 2.1. (Lu and Lin (1997)) *Let $\{X_n; n \geqslant 1\}$ be a sequence of φ-mixing random variables. Let $X \in L_p(F_1^k)$, $Y \in L_q(F_{k+n}^\infty)$, $p \geqslant 1$, $q \geqslant 1$ and $1/p + 1/q = 1$. Then*

$$|EXY - EXEY| \leqslant 2(\varphi(n))^{1/p} \left(E|X|^p\right)^{1/p} \left(E|Y|^q\right)^{1/q}. \tag{2.1}$$

LEMMA 2.2. (Shao (1993)) *Let $\{X_n;n \geqslant 1\}$ be a sequence of φ-mixing random variables. Put $T_a(n) = \sum_{i=a+1}^{a+n} X_i$ for $\forall a \geqslant 0$. Suppose that there exists an array $\{C_{a,n}\}$ of positive numbers such that*

$$ET_a^2(n) \leqslant C_{a,n} \quad \text{for any } a \geqslant 0,\ n \geqslant 1. \tag{2.2}$$

Then for each $p \geqslant 2$, there exists a constant C depending only on p and $\varphi(\cdot)$ such that

$$E(\max_{1\leqslant j\leqslant n} |T_a(j)|^p) \leqslant C[C_{a,n}^{p/2} + E(\max_{a+1\leqslant i\leqslant a+n} |X_i|^p)]. \tag{2.3}$$

LEMMA 2.3. (Wang et al (2010)) *Let $\{X_n;n \geqslant 1\}$ be a sequence of φ-mixing random variables satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$. Assume that $EX_n = 0$ and $E|X_n|^p < \infty$ for some $p \geqslant 2$ and each $n \geqslant 1$. Then there exists a constant $C = C(p,\varphi(\cdot))$ depending only on p and $\varphi(\cdot)$ such that for any $n \geqslant 1$ and $a \geqslant 0$,*

$$E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=a+1}^{a+j} X_i\right|^p\right) \leqslant C\left[\sum_{i=a+1}^{a+n} E|X_i|^p + \left(\sum_{i=a+1}^{a+n} (EX_i^2)\right)^{p/2}\right]. \tag{2.4}$$

In particular, for any $n \geqslant 1$,

$$E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \leqslant C\left[\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} (EX_i^2)\right)^{p/2}\right]. \tag{2.5}$$

Now, we state and prove the main results of this paper.

THEOREM 2.1. *Let $\{X,X_n;n \geqslant 1\}$ be a sequence of φ-mixing random variables with identically distributed satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$, let $T_n = \sum_{i=1}^{n} a_{ni}X_i$, $n \geqslant 1$, where the weights $\{a_{ni}; 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of constants such that $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n^{\delta})$ for some δ with $0 < \delta < 1$ and some α with $0 < \alpha < 2$. If $1 < \alpha < 2$, assume additionally that $EX_n = 0$. Suppose that for some $h > 0$, $\gamma > 0$,*

$$E[\exp(h|X|^{\gamma})] < \infty. \tag{2.6}$$

Then, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{s\alpha-2} P(\max_{1\leqslant j\leqslant n} |T_j| > \varepsilon b_n) < \infty, \tag{2.7}$$

where $s \geqslant \frac{1}{\alpha}$ and $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$.

Proof of Theorem 2.1. For $\forall i \geqslant 1$, define

$$X_i^{(n)} = X_i I(|X_i| \leqslant b_n);$$

$$T_j^{(n)} = \sum_{i=1}^{j} (a_{ni}X_i^{(n)} - Ea_{ni}X_i^{(n)}).$$

Then for $\forall \varepsilon > 0$, we get that

$$\begin{aligned}
&P\left(\max_{1\leqslant j\leqslant n} |T_j| > \varepsilon b_n\right) \\
&= P\left(\max_{1\leqslant j\leqslant n} |T_j| > \varepsilon b_n, \max_{1\leqslant j\leqslant n} |X_j| > b_n\right) + P\left(\max_{1\leqslant j\leqslant n} |T_j| > \varepsilon b_n, \max_{1\leqslant j\leqslant n} |X_j| \leqslant b_n\right) \\
&\leqslant P\left(\max_{1\leqslant j\leqslant n} |X_j| > b_n\right) + P\left(\max_{1\leqslant j\leqslant n} \left|T_j^{(n)} + \sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\right| > \varepsilon b_n\right) \\
&\leqslant P\left(\max_{1\leqslant j\leqslant n} |X_j| > b_n\right) + P\left(\max_{1\leqslant j\leqslant n} \left|T_j^{(n)}\right| > \varepsilon b_n - \max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\right|\right). \qquad (2.8)
\end{aligned}$$

Firstly, we will prove that

$$b_n^{-1} \max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\right| \to 0, \text{ as } n \to \infty. \qquad (2.9)$$

It follows from $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n^{\delta})$ and Hölder inequality that

$$\sum_{i=1}^{n} |a_{ni}|^k \leqslant \left(\sum_{i=1}^{n} |a_{ni}|^{k\times\frac{\alpha}{k}}\right)^{\frac{k}{\alpha}} \left(\sum_{i=1}^{n} 1\right)^{\frac{\alpha-k}{\alpha}} \leqslant Cn \quad \text{for } \forall 1 \leqslant k \leqslant \alpha. \qquad (2.10)$$

When $1 < \alpha < 2$, from $EX_n = 0$, (2.10), Markov inequality and (2.6), as $n \to \infty$, we can obtain that

$$\begin{aligned}
b_n^{-1} \max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\right| &\leqslant b_n^{-1} \sum_{i=1}^{n} \left|Ea_{ni}X_i^{(n)}\right| \\
&= b_n^{-1} \sum_{i=1}^{n} E|a_{ni}X_i| I(|X_i| > b_n) \\
&\leqslant Cb_n^{-1} nE|X| I(|X| > b_n) \\
&= Cb_n^{-1} n \sum_{k=n}^{\infty} E|X| I(b_k < |X| \leqslant b_{k+1}) \\
&\leqslant Cb_n^{-1} n \sum_{k=n}^{\infty} b_{k+1} EI(b_k < |X| \leqslant b_{k+1}) \\
&\leqslant Cb_n^{-1} n \sum_{k=n}^{\infty} b_{k+1} P(|X| > b_k) \\
&\leqslant Cb_n^{-1} n \sum_{k=n}^{\infty} b_{k+1} \frac{E[\exp(h|X|^{\gamma})]}{\exp(hb_k^{\gamma})}
\end{aligned}$$

$$\leqslant Cb_n^{-1} n \sum_{k=n}^{\infty} (k+1)^{1/\alpha} [\log(k+1)]^{1/\gamma} k^{-hk^{\gamma/\alpha}}$$
$$\leqslant Cb_n^{-1} n \sum_{k=n}^{\infty} k^{-2}$$
$$\leqslant Cn^{-1/\alpha} (\log n)^{-1/\gamma} nn^{-1} \to 0. \tag{2.11}$$

Note that for any $0 < p < 1$, then,

$$\left(\sum_{i=1}^{n} |b_i| \right)^p \leqslant \sum_{i=1}^{n} |b_i|^p.$$

By using the above C_r inequality, for any $0 < s < t$, we get that

$$\left(\sum_{i=1}^{n} (|a_{ni}|^s)^{t/s} \right)^{s/t} \leqslant \sum_{i=1}^{n} |a_{ni}|^s.$$

Hence,

$$\left(\sum_{i=1}^{n} |a_{ni}|^t \right)^{1/t} \leqslant \left(\sum_{i=1}^{n} |a_{ni}|^s \right)^{1/s}. \tag{2.12}$$

Therefore, when $0 < \alpha \leqslant 1$, it follows from (2.12), Markov inequality and (2.6) that

$$b_n^{-1} \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} E a_{ni} X_i^{(n)} \right| \leqslant b_n^{-1} \sum_{i=1}^{n} \left| E a_{ni} X_i^{(n)} \right|$$
$$\leqslant b_n^{-1} \sum_{i=1}^{n} E |a_{ni} X_i| I(|X_i| \leqslant b_n)$$
$$= b_n^{-1} \sum_{i=1}^{n} |a_{ni}| E |X| I(|X| \leqslant b_n)$$
$$\leqslant Cb_n^{-1} n^{\delta/\alpha} E |X| I(|X| \leqslant b_n)$$
$$= Cb_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} E |X| I(b_{k-1} < |X| \leqslant b_k)$$
$$\leqslant Cb_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} b_k EI(b_{k-1} < |X| \leqslant b_k)$$
$$\leqslant Cb_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} b_k P(|X| > b_{k-1})$$
$$\leqslant Cb_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} b_k \frac{E[\exp(h|X|^\gamma)]}{\exp(hb_{k-1}^\gamma)}$$
$$\leqslant Cb_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} k^{1/\alpha} (\log k)^{1/\gamma} (k-1)^{-h(k-1)^{\gamma/\alpha}}$$

$$\begin{aligned}&\leqslant Cn^{-1/\alpha}(\log n)^{-1/\gamma}n^{\delta/\alpha}\\&=C(\log n)^{-1/\gamma}n^{\delta/\alpha-1/\alpha}\to 0,\end{aligned}\tag{2.13}$$

as $n\to\infty$. From (2.11) and (2.13), hence (2.9) holds true.

It follows from (2.8) and (2.9) that

$$P(\max_{1\leqslant j\leqslant n}|T_j|>\varepsilon b_n)\leqslant\sum_{j=1}^{n}P(|X_j|>b_n)+P(\max_{1\leqslant j\leqslant n}\left|T_j^{(n)}\right|>\frac{\varepsilon b_n}{2}),\tag{2.14}$$

for n large enough.

Next, we need only to prove that

$$I\triangleq\sum_{n=1}^{\infty}n^{s\alpha-2}\sum_{j=1}^{n}P(|X_j|>b_n)<\infty;\tag{2.15}$$

$$II\triangleq\sum_{n=1}^{\infty}n^{s\alpha-2}P(\max_{1\leqslant j\leqslant n}\left|T_j^{(n)}\right|>\frac{\varepsilon b_n}{2})<\infty.\tag{2.16}$$

By the fact that $E[\exp(h|X|^{\gamma})]<\infty$, we easily obtain that

$$\begin{aligned}I&\triangleq\sum_{n=1}^{\infty}n^{s\alpha-2}\sum_{j=1}^{n}P(|X_j|>b_n)\\&=C\sum_{n=1}^{\infty}n^{s\alpha-2}\sum_{j=1}^{n}P(|X|>b_n)\\&\leqslant C\sum_{n=1}^{\infty}n^{s\alpha-1}\frac{E[\exp(h|X|^{\gamma})]}{\exp(hb_n^{\gamma})}\\&\leqslant C\sum_{n=1}^{\infty}\frac{n^{s\alpha-1}}{n^{hn^{\frac{\gamma}{\alpha}}}}<\infty.\end{aligned}\tag{2.17}$$

It follows from Lemma 2.3 and Markov's inequality that for $q\geqslant 2$

$$\begin{aligned}II&\triangleq\sum_{n=1}^{\infty}n^{s\alpha-2}P(\max_{1\leqslant j\leqslant n}\left|T_j^{(n)}\right|>\frac{\varepsilon b_n}{2})\\&\leqslant C\sum_{n=2}^{\infty}n^{s\alpha-2}b_n^{-q}E(\max_{1\leqslant j\leqslant n}\left|T_j^{(n)}\right|^q)\\&\leqslant C\sum_{n=2}^{\infty}n^{s\alpha-2}b_n^{-q}[\sum_{j=1}^{n}E\left|a_{nj}X_j^{(n)}\right|^q+(\sum_{j=1}^{n}E\left|a_{nj}X_j^{(n)}\right|^2)^{q/2}]\\&\triangleq II_1+II_2.\end{aligned}\tag{2.18}$$

Let $q>\max\{2,\alpha,\alpha(s\alpha-1)/(1-\delta)\}$. Then it follows from (2.12) that

$$\begin{aligned}II_1&=C\sum_{n=2}^{\infty}n^{s\alpha-2}b_n^{-q}\sum_{i=1}^{n}E\left|a_{ni}X_i^{(n)}\right|^q\\&=C\sum_{n=2}^{\infty}n^{s\alpha-2}b_n^{-q}\sum_{i=1}^{n}|a_{ni}|^qE|X_i|^qI(|X_i|\leqslant b_n)\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{n=2}^{\infty} n^{s\alpha-2} b_n^{-q} \sum_{i=1}^{n} |a_{ni}|^q E|X|^q I(|X| \leqslant b_n) \\
&\leqslant C\sum_{n=2}^{\infty} n^{s\alpha-2} b_n^{-q} n^{q\delta/\alpha} E|X|^q I(|X| \leqslant b_n) \\
&= C\sum_{n=2}^{\infty} n^{s\alpha-2} b_n^{-q} n^{q\delta/\alpha} \sum_{k=2}^{n} E|X|^q I(b_{k-1} < |X| \leqslant b_k) \\
&\leqslant C\sum_{k=2}^{\infty}\sum_{n=k}^{\infty} n^{s\alpha-2+(q\delta/\alpha)} n^{-q/\alpha} (\log n)^{-q/\gamma} b_k^q P(|X| > b_{k-1}) \\
&\leqslant C\sum_{k=2}^{\infty} b_k^q \frac{E[\exp(h|X|^\gamma)]}{\exp(hb_{k-1}^\gamma)} \\
&\leqslant C\sum_{k=2}^{\infty} k^{\frac{q}{\alpha}} (\log k)^{\frac{q}{\gamma}} (k-1)^{-h(k-1)^{\gamma/\alpha}} < \infty.
\end{aligned} \tag{2.19}
$$

By $0 < \alpha < 2$ and (2.12), we have that

$$
\begin{aligned}
II_2 &= C\sum_{n=2}^{\infty} n^{s\alpha-2} b_n^{-q} \Big(\sum_{i=1}^{n} E\Big|a_{ni}X_i^{(n)}\Big|^2\Big)^{q/2} \\
&= C\sum_{n=2}^{\infty} n^{s\alpha-2} b_n^{-q} \Big[\sum_{i=1}^{n} |a_{ni}|^2 (E|X|^2 I(|X| \leqslant b_n))\Big]^{q/2} \\
&\leqslant C\sum_{n=2}^{\infty} n^{s\alpha-2} b_n^{-q} n^{q\delta/\alpha} [E|X|^2 I(|X| \leqslant b_n)]^{q/2} \\
&= C\sum_{n=2}^{\infty} n^{s\alpha-2+(q\delta/\alpha)} b_n^{-q} \Big[\sum_{k=2}^{n} E|X|^2 I(b_{k-1} < |X| \leqslant b_k)\Big]^{q/2} \\
&\leqslant C\sum_{n=2}^{\infty} n^{s\alpha-2+(q\delta/\alpha)} b_n^{-q} \Big[\sum_{k=2}^{n} b_k^2 P(|X| > b_{k-1})\Big]^{q/2} \\
&\leqslant C\sum_{n=2}^{\infty} n^{s\alpha-2+(q\delta/\alpha)} b_n^{-q} \Big[\sum_{k=2}^{n} b_k^2 \frac{E\exp(h|X|^\gamma)}{\exp(hb_{k-1}^\gamma)}\Big]^{q/2} \\
&\leqslant C\sum_{n=2}^{\infty} n^{s\alpha-2+(q\delta/\alpha)} b_n^{-q} \Big[\sum_{k=2}^{n} \frac{k^{2/\alpha} (\log k)^{2/\gamma}}{\exp(h(k-1)^{\gamma/\alpha} \log(k-1))}\Big]^{q/2} \\
&= C\sum_{n=2}^{\infty} n^{s\alpha-2+(q\delta/\alpha)} b_n^{-q} \Big[\sum_{k=2}^{n} k^{2/\alpha} (\log k)^{2/\gamma} (k-1)^{-h(k-1)^{\gamma/\alpha}}\Big]^{q/2} \\
&\leqslant C\sum_{n=2}^{\infty} n^{s\alpha-2+(q\delta/\alpha)} b_n^{-q} \Big[\sum_{k=2}^{n} k^{-2}\Big]^{q/2} \\
&\leqslant C\sum_{n=2}^{\infty} n^{s\alpha-2+(q\delta/\alpha)} b_n^{-q} < \infty.
\end{aligned} \tag{2.20}
$$

Putting (2.19) and (2.20) into (2.18), we get that $II < \infty$. The proof of Theorem 2.1 is complete. $\square$

COROLLARY 2.1. *Under the conditions of Theorem 2.1, then*

$$\lim_{n\to\infty} |T_n|/b_n = 0 \quad a.s. \tag{2.21}$$

The proof of above Corollary 2.1 is analogous to that the proof of Corollary 2.1 in Cai (2006), so we omit it here.

REMARK 2.1. The result in Theorem 2.1: $\sum_{n=1}^{\infty} n^{s\alpha-2} P(\max_{1\leqslant j\leqslant n} |T_j| > \varepsilon b_n) < \infty$ (where $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$) is obtained under the weights $\{a_{ni}; 1\leqslant i\leqslant n, n\geqslant 1\}$ such that $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n^{\delta})$ for some δ with $0<\delta<1$ and some α with $0<\alpha<2$, while Theorem 2.4 in Wang et al. (2010): $\sum_{n=1}^{\infty} n^{\alpha p-2} P(\max_{1\leqslant j\leqslant n} |T_j| \geqslant \varepsilon n^{\alpha}) < \infty$ is obtained under the weights of $\sum_{i=1}^{n} |a_{ni}|^{p} = O(n^{\delta})$ for some p with $1\leqslant p\leqslant 2$ and some δ with $0<\delta\leqslant \frac{2}{q} \quad (q\geqslant 2)$.

THEOREM 2.2. *Let $\{X, X_n; n\geqslant 1\}$ be a sequence of φ-mixing random variables with identically distributed satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$, let $T_n = \sum_{i=1}^{n} a_{ni}X_i$, $n\geqslant 1$, where the weights $\{a_{ni}; 1\leqslant i\leqslant n, n\geqslant 1\}$ be an array of constants such that $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ for some $0<\alpha<2$. Furthermore, if $1<\alpha<2$, assume additionally that $EX_n = 0$. Suppose that for some $h>0$, $\gamma>0$ such that (2.6) satisfies. Then,*

$$\forall\varepsilon>0, \sum_{n=1}^{\infty} n^{-1} P(\max_{1\leqslant j\leqslant n} |T_j| > \varepsilon b_n) < \infty. \tag{2.22}$$

where $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$.

Proof of Theorem 2.2. Theorem 2.2 is in fact a special case of Theorem 2.1 for $s=\frac{1}{\alpha}$, $\delta=1$ and $q>\max\{2,\alpha,2\gamma\}$, since then the series

$$\sum_{n=3}^{\infty} n^{s\alpha-2+(q\delta/\alpha)} b_n^{-q} = \sum_{n=3}^{\infty} \frac{1}{n(\log n)^{q/\gamma}} \leqslant \sum_{n=3}^{\infty} \frac{1}{n(\log n)^{2}} < \infty$$

converges, and this is enough to conclude that (2.19) and (2.20) hold in this case. The rest of the proof is similar to that of the above Theorem 2.1 and is omitted. □

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