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FUNDAMENTAL INEQUALITIES FOR FRACTIONAL HYBRID DIFFERENTIAL EQUATIONS OF DISTRIBUTED ORDER AND APPLICATIONS

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Abstract. In this article, we establish some fundamental strict and non-strict differential inequalities for the fractional hybrid differential equations of distributed order (DOFHDEs). We derive these inequalities with respect to a nonnegative density function in the Riemann-Liouville derivative of order $0<q<1$. As an application of these inequalities, we prove the existence results for extremal solution of DOFHDEs and state the comparison principle.

1. Introduction

Differential inequalities are important in qualitative study of the nonlinear differential equations. An extensive literature of the differential inequalities along with some applications may be found in the works of many researchers. For example see [18]. In the scope of the fractional differential equations [17], [21], other researchers (e.g. [19]) established strict and non-strict fractional differential inequalities for the following fractional differential equation involving the Riemann-Liouville derivative of order $0<q<1$,

$$D_t^q x(t) = f(t,x), \qquad x(0) = x_0. \tag{1}$$

Also, the quadratic perturbations of the nonlinear differential equations and the first order ordinary functional differential equations in Banach algebras, have attracted much attention to researchers. These type equations have been called the hybrid differential equations (HDE), [7–14]. In this sense, the differential inequalities for implicit perturbations of the first order ordinary differential equations have been studied in Dhage [15]. One of the important first order hybrid differential equations is defined as

$$\begin{cases} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t\in J, \\ x(t_0) = x_0, \end{cases} \tag{2}$$

where $f\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $g\in\mathscr{C}(J\times\mathbb{R})$. For the above hybrid differential equation, Dhage and Lakshmikantham [10] established some fundamental hybrid differential inequalities which are useful for the existence of extremal solution and comparison

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theorems. Later, Zhao et al. [22] developed the following fractional hybrid differential equations involving the Riemann-Liouville derivative of order $0 < q < 1$,

$$\begin{cases} D^q\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J, \\ x(0) = 0, \end{cases} \tag{3}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathscr{C}(J \times \mathbb{R})$. They established some basic fractional hybrid differential inequalities for the existence of extremal solutions. Also, they considered necessary tools under the mixed Lipschitz and Caratheodory conditions to prove the comparison principle.

Now, in this article in view of the distributed order fractional derivative [1–3], we develop the distributed order fractional hybrid differential equations (DOFHDEs) with respect to a nonnegative density function and establish some differential inequalities for DOFHDEs which are useful for the existence of extremal solutions and comparison theorems.

In this regard, we introduce the fractional hybrid differential equation of distributed order and state some basic strict and non-strict fractional hybrid differential inequalities of distributed order. Next, we prove the existence theorem for this class and express the existence of extremal solution theorem. Finally, we conclude some comparison theorem for distributed order fractional hybrid differential equations.

2. The fractional hybrid differential equation of distributed order

The distributed order fractional hybrid differential equation with respect to the nonnegative density function $b(q) > 0$, is defined in the sense of Riemann-Liouville derivative of order $0 < q < 1$, as follows [20]

$$\begin{cases} \int_0^1 b(q) D^q\left[\frac{x(t)}{f(t,x(t))}\right] dq = g(t,x(t)), \quad t \in J, \quad \int_0^1 b(q)dq = 1, \\ x(0) = 0, \end{cases} \tag{1}$$

where the function $t \mapsto \frac{x}{f(t,x)}$ is continuous for each $x \in \mathbb{R}$ and $J = [0,T]$ is closed in $\mathbb{R}$ for some $T \in \mathbb{R}$. Also, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathscr{C}(J \times \mathbb{R})$, such that $C(J \times \mathbb{R}, \mathbb{R})$ is the class of continuous functions and $\mathscr{C}(J \times \mathbb{R})$ is called the Caratheodory class of bounded functions $g : J \times \mathbb{R} \to \mathbb{R}$ which are Lebesgue integrable on J. Furthermore

(i) the map $t \mapsto g(t,x)$ is measurable for each $x \in \mathbb{R}$,

(ii) the map $x \mapsto g(t,x)$ is continuous for each $t \in J$.

Also, for the above DOFHDE we consider some hypotheses as follows:

(A_0) The function $x \mapsto \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ for each $t \in J$.

(A_1) There exists a constant $L > 0$, such that

$$| f(t,x) - f(t,y) | \leqslant L \, | x - y |, \tag{2}$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

(A_2) There exists a function $h \in L^1(J,\mathbb{R})$, such that

$$| g(t,x) | \leqslant h(t), \tag{3}$$

for all $t \in J$ and $x \in \mathbb{R}$.

3. The fractional hybrid differential inequalities of distributed order

In this section, we prove the fundamental results related to strict and non-strict inequalities for the DOFHDE (1). We begin with a result of strict inequalities. The following lemma may be useful in next sections.

LEMMA 3.1. (Lakshmikantham and Vatsala [19]) *Let $m : \mathbb{R}^+ \to \mathbb{R}$ be locally Hölder continuous such that for any $t_1 \in (0,\infty)$, we have*

$$m(t_1) = 0, \quad m(t) \leqslant 0, \quad 0 \leqslant t \leqslant t_1, \tag{1}$$

then

$$D^q m(t_1) \geqslant 0. \tag{2}$$

THEOREM 3.2. *Suppose that the hypothesis (A_0) holds and there exist two functions $u,v : [0,T] \to \mathbb{R}$, which are locally Hölder continuous such that*

$$\int_0^1 b(q) D^q \Big[\frac{u(t)}{f(t,u(t))}\Big] dq \leqslant g(t,u(t)), \qquad \int_0^1 b(q) dq = 1, \tag{3}$$

$$\int_0^1 b(q) D^q \Big[\frac{v(t)}{f(t,v(t))}\Big] dq \geqslant g(t,v(t)), \qquad \int_0^1 b(q) dq = 1, \tag{4}$$

where $b(q) > 0$ is the density function. Then

$$u(0) < v(0), \tag{5}$$

and for all $t \in J$, we have

$$u(t) < v(t). \tag{6}$$

Proof. Assume that the inequality (4) is strict and the inequality (6) is false. Then the set Z^* defined by

$$Z^* = \{t \in J \ : \ u(t) \geqslant v(t), t \in J\}, \tag{7}$$

is non-empty. By denoting $t_1 = \inf Z^*$ and without loss of generality, we may suppose that $u(t_1) = v(t_1)$ and $u(t) < v(t)$ for all $t < t_1$. Define the function U and V on J as

$$U(t) = \frac{u(t)}{f(t,u(t))}, \qquad V(t) = \frac{v(t)}{f(t,v(t))},$$

then, we have

$$U(t_1) = V(t_1), \tag{8}$$

and in view of the hypothesis (A_0) for all $t < t_1$, we get

$$U(t_1) = V(t_1). \tag{9}$$

Now, by setting

$$m(t) = U(t) - V(t), \quad 0 \leqslant t \leqslant t_1, \tag{10}$$

we have

$$m(t) \leqslant 0, \;\; 0 \leqslant t \leqslant t_1, \;\; m(t_1) = 0. \tag{11}$$

which by Lemma 3.1 we obtain $D^q m(t_1) \geqslant 0$ and for $b(q) > 0$, we get

$$\int_0^1 b(q) D^q[m(t_1)]dq \geqslant 0.$$

Also, by the inequalities (3) and (4), we find that

$$g(t_1, u(t_1)) \geqslant \int_0^1 b(q) D^q[U(t_1)]dq \geqslant \int_0^1 b(q) D^q[V(t_1)] > g(t_1, v(t_1)). \tag{12}$$

This is a contradiction with $u(t_1) = v(t_1)$ and hence the set Z^* is empty. Finally, the inequality (6) holds for all $t \in J$. □

THEOREM 3.3. *Suppose that the conditions of Theorem 3.2 and the inequalities (3) and (4) hold. Also, for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \geqslant x_2$, assume that there exists a real number $M > 0$, such that*

$$g(t, x_1) - g(t, x_2) \leqslant \frac{M}{1 + t^q} \left(\frac{x_1}{f(t, x_1)} - \frac{x_2}{f(t, x_2)} \right), \qquad t \in J, \tag{13}$$

and

$$M \leqslant \int_0^1 \frac{b(q)}{T^q \Gamma(1-q)} dq, \qquad \int_0^1 b(q) dq = 1.$$

Then

$$u(0) \leqslant v(0), \tag{14}$$

which implies for all $t \in J$,

$$u(t) \leqslant v(t). \tag{15}$$

Proof. Let $\varepsilon > 0$ be given. Setting

$$\frac{v_\varepsilon(t)}{f(t, v_\varepsilon(t))} = \frac{v(t)}{f(t, v(t))} + \varepsilon(1 + t^q), \tag{16}$$

we find that

$$\frac{v_\varepsilon(t)}{f(t, v_\varepsilon(t))} > \frac{v(t)}{f(t, v(t))},$$

and by hypothesis (A_0), we get

$$v_\varepsilon(t) > v(t). \tag{17}$$

Now, for all $t \in J$ we define

$$V_\varepsilon(t) = \frac{v_\varepsilon(t)}{f(t, v_\varepsilon(t))}, \quad V(t) = \frac{v(t)}{f(t, v(t))},$$

which by the relation (13), we get

$$g(t,v) \geqslant g(t,v_\varepsilon) - \frac{M}{1+t^q}(V_\varepsilon - V).$$

Since

$$V_\varepsilon - V = \varepsilon(1+t^q),$$

we obtain

$$g(t,v) \geqslant g(t,v_\varepsilon) - \varepsilon M. \tag{18}$$

Applying the fractional differential of distributed operator $\int_0^1 b(q)D^q dq$, on the both sides of equation (16), we have

$$\int_0^1 b(q)D^q[V_\varepsilon(t)]dq = \int_0^1 b(q)D^q[V(t)]dq + \varepsilon \int_0^1 b(q)D^q[1+t^q]dq. \tag{19}$$

Hence by using the relations (4) and (19) and $M \leqslant \int_0^1 \frac{b(q)}{T^q\Gamma(1-q)}dq$, we find that

$$\begin{aligned}
\int_0^1 b(q)D^q[V_\varepsilon(t)]dq &\geqslant g(t,v(t)) + \varepsilon \int_0^1 b(q)\Big(\frac{1}{t^q\Gamma(1-q)} + \Gamma(1+q)\Big)dq, \\
&> g(t,v_\varepsilon(t)) - M\varepsilon + \varepsilon \int_0^1 \frac{b(q)}{t^q\Gamma(1-q)}dq \\
&> g(t,v_\varepsilon(t)) - M\varepsilon + M\varepsilon = g(t,v_\varepsilon(t)).
\end{aligned} \tag{20}$$

Also, we get $v_\varepsilon(0) > v(0) \geqslant u(0)$ which by setting $v = v_\varepsilon$ for all $t \in J$, we obtain $u(t) < v_\varepsilon(t)$. Since $\varepsilon > 0$ is arbitrary, by taking the limit as $\varepsilon \to 0$, we deduce that $u(t) \leqslant v(t)$. $\square$

4. Existence of extremal solutions

In this section, we apply the inequalities expressed in previous section and prove the maximal and minimal solutions for the DOFHDE (1) on $J = [0,T]$. For small real number $\varepsilon > 0$, we consider the following DOFHDE of order $0 < q < 1$, with the density function $b(q) > 0$,

$$\begin{cases} \int_0^1 b(q)D^q[\frac{x(t)}{f(t,x(t))}]dq = g(t,x(t)) + \varepsilon, \quad t \in J, \quad \int_0^1 b(q)dq = 1, \\ x(0) = 0, \end{cases} \tag{1}$$

where $J = [0,T]$, $f \in C(J \times \mathbb{R}, \mathbb{R} - \{0\})$ and $g \in \mathscr{C}(J \times \mathbb{R}, \mathbb{R})$. We define a supremum norm of $\|.\|$ in $C(J,\mathbb{R})$ as

$$\|x\| = \sup_{t \in J}|x(t)|, \tag{2}$$

and for $x,y \in C(J,\mathbb{R})$, we set

$$(xy)(t) = x(t)y(t). \tag{3}$$

It is easy to verify that $C(J,\mathbb{R})$ is a Banach algebra with respect to norm $\|.\|$ and multiplication (3). Moreover the norm $\|.\|_{L^1}$ for $x \in C(J,\mathbb{R})$ is defined by

$$\|x\|_{L^1} = \int_0^T |x(s)|\, ds. \tag{4}$$

Now, for expressing the existence theorem for the DOFHDE (1), we state a fixed point theorem in the Banach algebra and recall the Titchmarsh theorem for the inverse Laplace transform of a function with branch point.

THEOREM 4.1. (Dhage [6]) *Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and suppose that $A : X \to X$ and $B : S \to X$ are two operators such that*

(a) *A is Lipschitz constant α,*

(b) *B is completely continuous,*

(c) *$x = AxBy$ for all $y \in S$ implies that $x \in S$,*

(d) *$\alpha M_1 < 1$, where $M_1 = \| B(S) \| = \sup\{\| B(x) \| : x \in S\}$.*

Then, the operator equation $AxBx = x$ has a solution in S.

THEOREM 4.2. (Titchmarsh Theorem [4]) *Let $F(s)$ be an analytic function which has a branch cut on the real negative semiaxis. Furthermore, $F(s)$ has the following properties*

$$F(s) = O(1), \quad |s| \to \infty, \tag{5}$$

$$F(s) = O\left(\frac{1}{|s|}\right), \quad |s| \to 0, \tag{6}$$

for any sector $|arg(s)| < \pi - \eta$, where $0 < \eta < \pi$. Then, the inverse Laplace transform $f(t)$, can be written as the Laplace transform of the imaginary part of the function $F(re^{-i\pi})$ as follows:

$$f(t) = \mathscr{L}^{-1}\{F(s);t\} = \frac{1}{\pi}\int_0^\infty e^{-rt}\Im(F(re^{-i\pi}))dr. \tag{7}$$

LEMMA 4.3. *Assume that the hypothesis (A_0) holds, then for any $h \in L^1(J,\mathbb{R})$ and $0 < q < 1$, the function $x \in C(J,\mathbb{R})$ is a solution of the DOFHDE (1) if and only if x satisfies the following equation*

$$x(t) = \frac{f(t,x(t))}{\pi}\int_0^t \mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t-\tau\Big\}(g(\tau,x(\tau))+\varepsilon)d\tau, \tag{8}$$

such that $0 \leqslant \tau \leqslant t \leqslant T$ and

$$B(s) = \int_0^1 b(q)s^q dq. \tag{9}$$

Proof. Applying the Laplace transform on the both sides of (1) and setting

$$H(t) = \frac{x(t)}{f(t,x(t))}, \tag{10}$$

we have

$$\begin{aligned}\mathscr{L}\Big\{\int_0^1 b(q)D^q[H(t)]dq;s\Big\} &= \mathscr{L}\{g(t,x(t))+\varepsilon;s\}\\ &= \int_0^1 b(q)[s^q H(s) - D_t^{q-1}H(0)]dq\\ &= G(s)+\frac{\varepsilon}{s}.\end{aligned} \tag{11}$$

Since $H(0)=0$, we have

$$H(s)\Big(\int_0^1 b(q)s^q dq\Big) = G(s)+\frac{\varepsilon}{s},$$

and hence

$$H(s) = \frac{1}{B(s)}\Big(G(s)+\frac{\varepsilon}{s}\Big), \tag{12}$$

such that

$$B(s) = \int_0^1 b(q)s^q dq. \tag{13}$$

Now, using the inverse Laplace transform on the both sides of (12) and applying the convolution product, we get

$$\begin{aligned}\mathscr{L}^{-1}\{H(s);t\} = \frac{x(t)}{f(t,x(t))} &= \mathscr{L}^{-1}\Big\{\frac{1}{B(s)}(G(s)+\frac{\varepsilon}{s});t\Big\}\\ &= \int_0^t \mathscr{L}^{-1}\Big\{\frac{1}{B(s)};t-\tau\Big\}(g(\tau,x(\tau))+\varepsilon)d\tau,\end{aligned}$$

or equivalently

$$x(t) = f(t,x(t))\int_0^t \mathscr{L}^{-1}\Big\{\frac{1}{B(s)};t-\tau\Big\}(g(\tau,x(\tau))+\varepsilon)d\tau. \tag{14}$$

Since $B(s)$ is an analytic function which has a branch cut on the real negative semiaxis, according to the Theorem 4.2 we get

$$x(t) = \frac{f(t,x(t))}{\pi}\int_0^t\int_0^\infty e^{-r(t-\tau)}\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\}(g(\tau,x(\tau))+\varepsilon)drd\tau, \tag{15}$$

which by the Laplace transform definition, the equation (8) is held. Conversely, let x satisfies the equation (8), therefore x satisfies the equivalent equation (14) and by setting $t=0$ in the equation (8), we obtain

$$\frac{x(0)}{f(0,x(0))} = 0 = \frac{0}{f(0,0)}.$$

According to the hypothesis (A_0), the map $x \mapsto \frac{x}{f(0,x)}$ is injective in $\mathbb{R}$ and hence $x(0) = 0$. Now, by using the fact that $H(0) = 0$, and applying the inverse Laplace transform on (11), the relation (1) is held and the proof is completed. $\square$

At this point, the main existence theorem for the DOFHDE (1) is stated.

THEOREM 4.4. *Suppose that the hypotheses* (A_0)–(A_2) *hold and for small real number* $\varepsilon > 0$, *we have*

$$\frac{LM(\|h\|_{L^1} + \varepsilon T)}{\pi} < 1, \quad M > 0, \tag{16}$$

then, the DOFHDE (1) *has a solution on* $J = [0,T]$.

Proof. We set $X = C(J,\mathbb{R})$ as a Banach algebra and define a subset S of X by

$$S = \{x \in X \,|\, \|x\| \leqslant N\}, \tag{17}$$

such that for $\varepsilon > 0$, we have

$$N = \frac{F_0 M(\|h\|_{L^1} + \varepsilon T)}{\pi - LM(\|h\|_{L^1} + \varepsilon T)}, \quad F_0 = \sup_{t\in J} |f(t,0)|\,. \tag{18}$$

It is obvious that S is closed and if $x_1, x_2 \in S$, then $\|x_1\| \leqslant N$ and $\|x_2\| \leqslant N$. Also, by the properties of the norm, we get

$$\|\lambda x_1 + (1-\lambda)x_2\| \leqslant \lambda \|x_1\| + (1-\lambda)\|x_2\| \leqslant \lambda N + (1-\lambda)N = N,$$

which implies that S is convex and bounded. Now, by applying Lemma 4.3, DOFHDE (1) is equivalent to the equation (8). Also, we define the operators $A : X \longrightarrow X$ and $B : S \longrightarrow X$ by

$$Ax(t) = f(t,x(t)), \quad t \in J, \tag{19}$$

and

$$Bx(t) = \frac{1}{\pi}\int_0^t \mathscr{L}\Big\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t-\tau\Big\}(g(\tau,x(\tau)) + \varepsilon)d\tau, \quad t \in J. \tag{20}$$

Thus, from the equation (8) we obtain the operator equation as follows:

$$Ax(t)Bx(t) = x(t), \quad t \in J. \tag{21}$$

If the operators A and B satisfy all the conditions of Theorem 4.1, then the operator equation (21) has a solution in S. To see this, let $x,y \in X$, then by the hypothesis (A_1) we have

$$|Ax(t) - Ay(t)| = |f(t,x(t)) - f(t,y(t))| \leqslant L|x(t) - y(t)| \leqslant L\|x-y\|, \quad t \in J,$$

and if for all $x,y \in X$ take a supremum over t, we get

$$\|Ax - Ay\| \leqslant L\|x-y\|. \tag{22}$$

Therefore, A is a Lipschitz operator on X with the Lipschitz constant $L > 0$, and the condition (a) from Theorem 4.1 is held. Now, for checking the condition (b) of this theorem, let $\{x_n\}$ be a sequence in S such that

$$\lim_{n\to\infty} x_n = x, \tag{23}$$

with $x \in S$. Now,

$$\begin{aligned}
\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty} \frac{1}{\pi}\int_0^t \mathscr{L}\Big\{\Im\{\frac{1}{B(re^{-i\pi})}\};t-\tau\Big\}(g(\tau,x_n(\tau))+\varepsilon)d\tau \\
&= \frac{1}{\pi}\int_0^t \mathscr{L}\Big\{\Im\{\frac{1}{B(re^{-i\pi})}\Big\};t-\tau\} \lim_{n\to\infty}(g(\tau,x_n(\tau))+\varepsilon)d\tau \\
&= \frac{1}{\pi}\int_0^t \mathscr{L}\Big\{\Im\{\frac{1}{B(re^{-i\pi})}\Big\};t-\tau\}(g(\tau,x(\tau))+\varepsilon)d\tau \\
&= Bx(t).
\end{aligned} \tag{24}$$

This shows that B is pointwise continuous on J. It can be shown as in the following part that the sequence $\{Bx_n\}$ is an equicontinuous set in $C(J,\mathbb{R})$. So the convergence $Bx_n \to Bx$ is uniform. As a result, B is continuous on $C(J,\mathbb{R})$. In this stage, we shall show that B is a compact operator on S. To see this, we shall show that $B(s)$ is a uniformly bounded and eqicontinuous set in X. Let $x \in S$, then by hypothesis (A_2) for all $t \in J$, we have

$$\begin{aligned}
|Bx(t)| &= \left|\frac{1}{\pi}\int_0^t \mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t-\tau\Big\}(g(\tau,x(\tau))+\varepsilon)d\tau\right| \\
&\leqslant \frac{1}{\pi}\int_0^t \left|\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t-\tau\Big\}\right| |h(\tau)+\varepsilon|\,d\tau.
\end{aligned} \tag{25}$$

Let $s = t-\tau$ be such that $0 \leqslant \tau \leqslant t \leqslant T$. Then by the existence theorem of the Laplace transform [5], there exists a constant $M' > 0$ such that for $s > c$,

$$\left|\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\}\right| \leqslant M'e^{cr}. \tag{26}$$

Hence, we find an upper bound for the integrand of (25) as follows.

$$\begin{aligned}
\left|\mathscr{L}\Big\{\Im\{\frac{1}{B(re^{-i\pi})}\Big\};t-\tau\}\right| &= \left|\int_0^\infty e^{-sr}\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\}dr\right| \\
&\leqslant \int_0^\infty e^{-sr}\left|\Im\{\frac{1}{B(re^{-i\pi})}\}\right| dr \\
&\leqslant \int_0^\infty M'e^{(c-s)r}dr \leqslant \frac{M'}{|s-c|} \leqslant M,
\end{aligned} \tag{27}$$

such that

$$M = \sup_{0\leqslant\tau\leqslant t\leqslant T} \frac{M'}{|t-\tau-c|}. \tag{28}$$

Finally, with respect to the inequality (25), we obtain

$$|Bx(t)| \leqslant \frac{M(\|h\|_{L^1}+\varepsilon T)}{\pi},$$

and by applying supremum over t, we get for all $x\in S$

$$\|Bx\| \leqslant \frac{M}{\pi}(\|h\|_{L^1}+\varepsilon T). \tag{29}$$

Thus, B is uniformly bounded on S.

In this stage, we show that $B(S)$ is an equicontinuous set in X. Let $t_1,t_2\in J$, with $t_1<t_2$, then for all $x\in S$, we have

$$\begin{aligned}|Bx(t_1)-Bx(t_2)| &= \Big|\frac{1}{\pi}\int_0^{t_1}\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t_1-\tau\Big\}(g(\tau,x(\tau))+\varepsilon)d\tau \\ &\quad -\frac{1}{\pi}\int_0^{t_2}\mathscr{L}\Big\{\Im\{\frac{1}{B(re^{-i\pi})}\Big\};t_2-\tau\}(g(\tau,x(\tau))+\varepsilon)d\tau\Big| \\ &\leqslant \Big|\frac{1}{\pi}\int_0^{t_1}\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t_1-\tau\Big\}(g(\tau,x(\tau))+\varepsilon)d\tau \\ &\quad -\frac{1}{\pi}\int_0^{t_1}\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t_2-\tau\Big\}(g(\tau,x(\tau))+\varepsilon)d\tau\Big| \\ &\quad +\Big|\frac{1}{\pi}\int_0^{t_1}\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t_2-\tau\Big\}(g(\tau,x(\tau))+\varepsilon)d\tau \\ &\quad -\frac{1}{\pi}\int_0^{t_2}\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t_2-\tau\Big\}(g(\tau,x(\tau))+\varepsilon)d\tau\Big|. \end{aligned} \tag{30}$$

If we set $s_1=t_1-\tau$ and $s_2=t_2-\tau$, then by definition of the Laplace transform and the equation (4–19), for $s_1>c$ and $s_2>c$ we can write

$$\begin{aligned} &\Big|\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};s_1\Big\}-\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};s_2\Big\}\Big| \\ &=\Big|\int_0^{\infty}e^{-s_1r}\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\})dr-\int_0^{\infty}e^{-s_2r}\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\}dr\Big| \\ &\leqslant\int_0^{\infty}|e^{-s_1r}-e^{-s_2r}|\Big|\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\}\Big|dr \\ &\leqslant M'\int_0^{\infty}(e^{(c-s_1)r}-e^{(c-s_2)r})dr=M'\Big(\frac{1}{s_1-c}-\frac{1}{s_2-c}\Big). \end{aligned} \tag{31}$$

Therefore, we have

$$\begin{aligned} &\Big|\frac{1}{\pi}\int_0^{t_1}\Big(\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};s_1\Big\}-\mathscr{L}\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};s_2\}\Big)(g(\tau,x(\tau))+\varepsilon)d\tau\Big| \\ &\leqslant\frac{(\|h\|_{L^1}+\varepsilon T)}{\pi}\int_0^{t_1}M'\Big(\frac{1}{t_1-\tau-c}-\frac{1}{t_2-\tau-c}\Big)d\tau \\ &=\frac{M'\Big(\|h\|_{L^1}+\varepsilon T\Big)}{\pi}\ln\Big(\frac{(c+t_1-t_2)(c-t_1)}{c(c-t_2)}\Big). \end{aligned} \tag{32}$$

Also, by equation (27) we have

$$\begin{aligned}
&\left|\frac{1}{\pi}\int_{t_2}^{t_1}\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t_2-\tau\Big\}(g(\tau,x(\tau))+\varepsilon)d\tau\right| \\
&\leqslant \frac{(\|h\|_{L^1}+\varepsilon T)}{\pi}\int_{t_2}^{t_1}\frac{M'}{t_2-\tau-c}d\tau \\
&= \frac{M'(\|h\|_{L^1}+\varepsilon T)}{\pi}\ln\Big(\frac{c}{c+t_1-t_2}\Big). \qquad (33)
\end{aligned}$$

Finally with respect to relations (30), (32) and (33), we obtain

$$\begin{aligned}
|Bx(t_1)-Bx(t_2)| &\leqslant \frac{M'(\|h\|_{L^1}+\varepsilon T)}{\pi}\Big(\ln\Big(\frac{(c+t_1-t_2)(c-t_1)}{c(c-t_2)}\Big)+\ln\Big(\frac{c}{c+t_1-t_2}\Big)\Big) \\
&= \frac{M'(\|h\|_{L^1}+\varepsilon T)}{\pi}\ln\Big(\frac{c-t_1}{c-t_2}\Big). \qquad (34)
\end{aligned}$$

Hence, for $\varepsilon>0$, there exists $\delta>0$ such that if $|t_1-t_2|<\delta$, then for all $t_1,t_2\in J$ and all $x\in S$ we have

$$|Bx(t_1)-Bx(t_2)|<\varepsilon, \qquad (35)$$

which implies that $B(S)$ is an equicontinuous set in X and according to the Arzelá-Ascoli theorem, B is compact. Therefore B is continuous and compact operator on S into X and B is a completely continuous operator on S and the condition (b) from the Theorem 4.1 is held. □

For checking the condition (c) of Theorem 4.1, let $x\in X$ and $y\in S$ be arbitrary such that $x=AxBy$. Then, by hypothesis (A_1) we get

$$\begin{aligned}
|x(t)| &= |Ax(t)|\,|By(t)| \\
&= |f(t,x(t))|\left|\frac{1}{\pi}\int_0^t\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t-\tau\Big\}(g(\tau,x(\tau))+\varepsilon)d\tau\right| \\
&\leqslant (|f(t,x(t))-f(t,0)|+|f(t,0)|) \\
&\quad\times\Big(\frac{1}{\pi}\int_0^t\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t-\tau\Big\}\,|(g(\tau,x(\tau))+\varepsilon)|\,d\tau\Big) \\
&\leqslant (L|x(t)|+F_0)\Big(\frac{1}{\pi}\int_0^t\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t-\tau\Big\}(h(\tau)+\varepsilon)d\tau\Big) \\
&\leqslant (L|x(t)|+F_0)\Big(\frac{M(\|h\|_{L^1}+\varepsilon T)}{\pi}\Big). \qquad (36)
\end{aligned}$$

Therefore,

$$|x(t)|\leqslant\frac{F_0M(\|h\|_{L^1}+\varepsilon T)}{\pi-LM(\|h\|_{L^1}+\varepsilon T)},$$

which by taking a supremum over t, we obtain

$$\|x(t)\|\leqslant\frac{F_0M(\|h\|_{L^1}+\varepsilon T)}{\pi-LM(\|h\|_{L^1}+\varepsilon T)}=N. \qquad (37)$$

Thus, the condition (c) of Theorem 4.1 is satisfied. If we consider

$$M_1 = \|B(s)\| = \sup\{\|Bx\| : x \in S\} \leqslant \frac{M}{\pi}(\|h\|_{L^1} + \varepsilon T), \tag{38}$$

and

$$\alpha M_1 \leqslant L\Big(\frac{M}{\pi}(\|h\|_{L^1} + \varepsilon T)\Big) < 1, \tag{39}$$

the hypothesis (d) of Theorem 4.1 is satisfied. Hence, all the conditions of Theorem 4.1 are satisfied and therefore the operator equation $AxBx = x$ has a solution in S. As a result, the DOFHDE (1) has a solution on J and proof is completed.

DEFINITION 4.5. A solution y of DOFHDE (1) is maximal if for all $t \in J$ and solution x of this system, $x(t) \leqslant y(t)$. Similarly, a solution z of the DOFHDE (1) is minimal if for all $t \in J$, one has $z(t) \leqslant x(t)$, such that x is the solution of the DOFHDE (1).

Now, we ready to expressing the main existence theorem about the maximal solution for the DOFHDE (1). Also, the case of minimal solution is similar and can be obtained with the same arguments with appropriate modifications.

THEOREM 4.6. *Suppose that the hypotheses* (A_0)–(A_2) *and the condition* (16) *hold, then the DOFHDE* (1) *has a maximal solution on* $J = [0,T]$.

Proof. We set $\{\varepsilon_n\}_0^\infty$ as a decreasing sequence of positive real numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$. Also, in view of the inequality (16), there exists a positive real number ε_0 such that

$$\frac{LM(\|h\|_{L^1} + \varepsilon_0 T)}{\pi} < 1. \tag{40}$$

If we apply the Theorem 4.4, then for the DOFHDE

$$\begin{cases} \int_0^1 b(q)D^q[\frac{x(t)}{f(t,x(t))}]dq = g(t,x(t)) + \varepsilon_n, \quad t \in J, \quad \int_0^1 b(q)dq = 1, \\ x(0) = 0, \end{cases} \tag{41}$$

where $b(q)$ is a nonnegative density function, we have a solution $y(t,\varepsilon_n)$ such that

$$\int_0^1 b(q)D^q\Big[\frac{y(t,\varepsilon_n)}{f(t,y(t,\varepsilon_n))}\Big]dq = g(t,y(t,\varepsilon_n)) + \varepsilon_n > g(t,y(t,\varepsilon_n)). \tag{42}$$

Also, for any solution w of the DOFHDE (1) we get

$$\int_0^1 b(q)D^q\Big[\frac{w(t)}{f(t,w(t))}\Big]dq \leqslant g(t,w(t)), \tag{43}$$

such that $w(0) = 0 \leqslant y(0,\varepsilon_n) = \varepsilon_n$. Thus, by applying the Theorem 3.3, we have

$$w(t) \leqslant y(t,\varepsilon_n), \qquad t \in J, \qquad n = 0,1,2,... \tag{44}$$

Also, since $\varepsilon_2 = y(0,\varepsilon_2) \leqslant y(0,\varepsilon_1) = \varepsilon_1$, in view of the Theorem 3.3, we obtain

$$y(t,\varepsilon_2) \leqslant y(t,\varepsilon_1).$$

Then, $\{y(t,\varepsilon_n)\}$ is decreasing sequence of positive real numbers and the limit

$$y(t) = \lim_{n\to\infty} y(t,\varepsilon_n), \tag{45}$$

exists. We shall show that the limit (45) is uniform on $J=[0,T]$. To see this, we prove the sequence $y(t,\varepsilon_n)$ is equicontinuous. Suppose that $t_1,t_2 \in J$ such that $t_1 < t_2$. Since $y(t,\varepsilon_n)$ is the solution of DOFHDE (41), then by Lemma 4.3 $y(t,\varepsilon_n)$ satisfies in equation

$$y(t,\varepsilon_n) = (f(t,y(t,\varepsilon_n)))\Big(\frac{1}{\pi}\int_0^t \mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t-\tau\Big\}(g(\tau,y(\tau,\varepsilon_n))+\varepsilon_n)d\tau\Big). \tag{46}$$

Therefore, by the relations (32) and (33) we have

$$\begin{aligned}
&|y(t_1,\varepsilon_n)-y(t_2,\varepsilon_n)|\\
&= \Big|(f(t_1,y(t_1,\varepsilon_n)))\Big(\frac{1}{\pi}\int_0^{t_1}\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t_1-\tau\Big\}(g(\tau,y(\tau,\varepsilon_n))+\varepsilon_n)d\tau\Big)\\
&\quad -(f(t_2,y(t_2,\varepsilon_n)))\Big(\frac{1}{\pi}\int_0^{t_2}\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t_2-\tau\Big\}(g(\tau,y(\tau,\varepsilon_n))+\varepsilon_n)d\tau\Big)\Big|\\
&\leqslant \Big|(f(t_1,y(t_1,\varepsilon_n)))\Big(\frac{1}{\pi}\int_0^{t_1}\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t_1-\tau\Big\}(g(\tau,y(\tau,\varepsilon_n))+\varepsilon_n)d\tau\Big)\\
&\quad -(f(t_2,y(t_2,\varepsilon_n)))\Big(\frac{1}{\pi}\int_0^{t_1}\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t_2-\tau\Big\}(g(\tau,y(\tau,\varepsilon_n))+\varepsilon_n)d\tau\Big)\Big|\\
&\quad +\Big|(f(t_2,y(t_2,\varepsilon_n)))\Big(\frac{1}{\pi}\int_0^{t_1}\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t_2-\tau\Big\}(g(\tau,y(\tau,\varepsilon_n))+\varepsilon_n)d\tau\Big)\\
&\quad -(f(t_2,y(t_2,\varepsilon_n)))\Big(\frac{1}{\pi}\int_0^{t_2}\mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t_2-\tau\Big\}(g(\tau,y(\tau,\varepsilon_n))+\varepsilon_n)d\tau\Big)\Big|.\\
&\leqslant |f(t_1,y(t_1,\varepsilon_n))-f(t_2,y(t_2,\varepsilon_n))|\,\frac{M'(\|h\|_{L^1}+\varepsilon_n)}{\pi}\ln\Big(\frac{(c+t_1-t_2)(c-t_1)}{c(c-t_2)}\Big)\\
&\quad +F\frac{M'(\|h\|_{L^1}+\varepsilon_n)}{\pi}\ln\Big(\frac{c}{c+t_1-t_2}\Big),
\end{aligned} \tag{47}$$

such that $F=\sup_{(t,x)\in J\times[-\mathbb{N},\mathbb{N}]}|f(t,x)|$. Hence, for $\varepsilon>0$ there exists $\delta>0$ such that for $|t_1-t_2|<\delta$, we have

$$|y(t_1,\varepsilon_n)-y(t_2,\varepsilon_n)|<\varepsilon, \quad n\in\mathbb{N},$$

which implies that for all $t\in J$, $y(t,\varepsilon_n)\to y(t)$. Now, taking the limits from equation (46) when $n\to\infty$, we get

$$y(t) = [f(t,y(t))]\Big(\frac{1}{\pi}\int_0^t \mathscr{L}\Big\{\Im\Big\{\frac{1}{B(re^{-i\pi})}\Big\};t-\tau\Big\}(g(\tau,r(\tau))d\tau\Big), \qquad t\in J.$$

Therefore, y is a solution of the DOFHDE (1) on J and from inequality (44), we deduce $w(t)\leqslant y(t)$. Hence, the DOFHDE (1) has a maximal solution on $J=[0,T]$. $\square$

5. Comparison Theorems

In this section, we estimate a bound for the solution set of the differential inequality related to DOFHDE (1). Also, we prove that the extremal solutions are bounds for the solutions of this differential inequality.

THEOREM 5.1. *Suppose that the hypotheses* (A_0)–(A_2) *and the condition* (16) *hold. Also, assume that there exists a real number* $M > 0$, *such that for all* $t \in J$

$$g(t,x_1) - g(t,x_2) \leqslant \frac{M}{1+t^q}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right), \quad x_1,x_2 \in \mathbb{R}, \quad x_1 \geqslant x_2, \tag{1}$$

where

$$M \leqslant \int_0^1 \frac{b(q)}{T^q\Gamma(1-q)}dq, \qquad \int_0^1 b(q)dq = 1. \tag{2}$$

Furthermore, if there exists a function $w \in C(J,\mathbb{R})$, *such that*

$$\begin{cases} \int_0^1 b(q)D^q\left[\frac{w(t)}{f(t,w(t))}\right]dq \leqslant g(t,w(t)), \quad t \in J, \\ w(0) \leqslant 0, \end{cases} \tag{3}$$

then, for all $t \in J$

$$w(t) \leqslant y(t), \tag{4}$$

where y *is a maximal solution of the DOFHDE* (1).

Proof. Letting $\varepsilon > 0$ and using the Theorem 4.6, $y(t,\varepsilon)$ is a maximal solution of the DOFHDE (1) such that

$$y(t) = \lim_{\varepsilon\to 0} y(t,\varepsilon), \tag{5}$$

is uniform on $J = [0,T]$. Therefore, for nonnegative density function $b(q)$, we have

$$\begin{cases} \int_0^1 b(q)D^q\left[\frac{y(t,\varepsilon)}{f(t,y(t,\varepsilon))}\right]dq = g(t,y(t,\varepsilon)) + \varepsilon, \quad t \in J, \quad \int_0^1 b(q)dq = 1, \\ y(0,\varepsilon) = 0. \end{cases} \tag{6}$$

Hence

$$\begin{cases} \int_0^1 b(q)D^q[\frac{y(t,\varepsilon)}{f(t,y(t,\varepsilon))}]dq > g(t,y(t,\varepsilon)), \quad t \in J, \quad \int_0^1 b(q)dq = 1, \\ y(0,\varepsilon) = 0. \end{cases} \tag{7}$$

Now, by the Theorem 3.3 for the inequalities (3) and (7) we obtain $w(t) < y(t,\varepsilon)$. Finally, the limit (5) implies that $w(t) \leqslant y(t)$. □

COROLLARY 5.2. *Suppose that the hypotheses* (A_0)–(A_2) *and the condition* (16) *hold. Also, assume that there exists a real number* $M > 0$, *such that for all* $t \in J$

$$g(t,x_1) - g(t,x_2) \leqslant \frac{M}{1+t^q}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right), \quad x_1,x_2 \in \mathbb{R}, \quad x_1 \geqslant x_2,$$

where

$$M \leqslant \int_0^1 \frac{b(q)}{T^q\Gamma(1-q)}dq, \qquad \int_0^1 b(q)dq = 1.$$

Furthermore, if there exists a function $u \in C(J,\mathbb{R})$ *such that*

$$\begin{cases} \int_0^1 b(q)D^q\left[\frac{u(t)}{f(t,u(t))}\right]dq \geqslant g(t,u(t)), \quad t \in J, \\ u(0) > 0, \end{cases}$$

then

$$z(t) \leqslant u(t),$$

where z *is a minimal solution of the DOFHDE* (1).

Next theorem is a result about the uniqueness of solutions of DOFHDE (1).

THEOREM 5.3. *Suppose that the hypotheses* (A_0)–(A_2) *and the condition* (16) *hold. Also, assume that there exists a real number* $M > 0$, *such that for all* $t \in J$

$$g(t,x_1) - g(t,x_2) \leqslant \frac{M}{1+t^q}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right), \quad x_1,x_2 \in \mathbb{R}, \quad x_1 \geqslant x_2,$$

where

$$M \leqslant \int_0^1 \frac{b(q)}{T^q\Gamma(1-q)}dq, \qquad \int_0^1 b(q)dq = 1.$$

If identically zero function is the only solution of the differential equation

$$\int_0^1 b(q)D^q[p(t)]dq = \frac{M}{1+t^q}p(t), \qquad p(0) = 0, \qquad \int_0^1 b(q)dq = 1, \tag{8}$$

then, the DOFHDE (1) *has a unique solution on* $J = [0,T]$.

Proof. According to the Theorem 4.4, the DOFHDE (1) has a solution on $J = [0,T]$. Let v_1 and v_2 be two solution of the DOFHDE (1) with $v_1 > v_2$ and set the function $p : J \to \mathbb{R}$

$$p(t) = \frac{v_1(t)}{f(t,v_1(t))} - \frac{v_2(t)}{f(t,v_2(t))}. \tag{9}$$

Since $v_1 > v_2$, by the hypothesis (A_0) we obtain $p(t) > 0$. Therefore for the nonnegative density function $b(q)$, we get

$$\begin{aligned} \int_0^1 b(q)D^q[p(t)]dq &\leqslant \int_0^1 b(q)D^q\Big[\frac{v_1(t)}{f(t,v_1(t))}\Big]dq - \int_0^1 b(q)D^q\Big[\frac{v_2(t)}{f(t,v_2(t))}\Big]dq \\ &\leqslant g(t,v_1) - g(t,v_2) \\ &\leqslant \frac{M}{1+t^q}\Big(\frac{v_1}{f(t,v_1)} - \frac{v_2}{f(t,v_2)}\Big) \\ &= \frac{M}{1+t^q}p(t), \qquad t \in J, \qquad p(0) = 0. \end{aligned}$$

Since identically zero function is the only solution of the differential equation (8), applying the Theorem 5.1 with $f(t,x) \equiv 1$, implies that $p(t) \leqslant 0$, which is a contradiction with $p(t) > 0$. Finally, $v_1 = v_2$. $\square$

6. Conclusions

In this paper, we introduced a new class of the fractional hybrid differential inequalities of distributed order with respect to a nonnegative density function. By these inequalities, we established the existence of extremal solution and proved some comparison theorems for this class. These results enable us to find the extremal solutions of many fractional differential equations with respect to the various order density function.

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