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SOME INEQUALITIES FOR EDGE LENGTHS AND CIRCUM–RADIUS OF A SIMPLEX IN HYPERBOLIC SPACE

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Abstract. For an n-dimensional simplex in hyperbolic space $H_n(-1)$ and spherical space $S_n(1)$, we establish some inequalities for its edge lengths and circum-radius.

1. Introduction

Let R_1^{n+1} be $(n+1)$-dimensional vector space in which the scalar product of two vectors $x = (x_0, x_1, \cdots, x_n)$ and $y = (y_0, y_1, \cdots, y_n)$ is given by

$$\langle x, y\rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i.$$

The unit pseudo-sphere with index one S_1^n in R_1^{n+1} is given by $\{x \in R_1^{n+1} | \langle x, x\rangle = 1\}$. The unit pseudo-hyperbolic space is defined as

$$H_0^n = \{x \in R_1^{n+1} | \langle x, x\rangle = -1\},$$

Which has two connected components $H_{0,+}^n$ and $H_{0,-}^n$. Each of them can be taken as a model for the n-dimensional hyperbolic space of curvature -1, and denote it by H_{-n}^n (see [7, 8]). Another model for the n-dimensional hyperbolic space is given in [10].

The n-dimensional spherical space of curvature 1 is defined as follows [10].

The elements of the space are all the ordered $(n+1)$-tuples $x = (x_1, x_2, \cdots, x_{n+1})$ of real numbers with $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$.

Distance is defined for each pair of elements x, y to be the smallest non-negative number $\widehat{xy}$ such that

$$\cos\widehat{xy} = x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1}.$$

Using $S_n(1)$ denote the n-dimensional spherical space of curvature 1. Actually, $S_n(1)$ is the boundary of an n-dimensional sphere of radius 1 in the $(n+1)$-dimensional

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Euclidean space E_{n+1} with geodesic (that is, shorter arc) metric. We suppose that the circum-center of $S_n(1)$ is the origin O of the $(n+1)$-dimensional Euclidean space E_{n+1}.

Recently, relations between edge lengths and volume of 3-dimensional simplex in hyperbolic and spherical space are discussed in [1–6], and generalized law of sines for an n-dimensional simplex in hyperbolic and spherical space are established in [7, 8, 9].

Let Δ_n be an n-dimensional simplex in the n-dimensional Euclidean space E_n, a_{ij} $(0 \leqslant i < j \leqslant n)$ and R be its edge lengths and circum-radius, respectively. Some inequalities for edge lengths and circum-radius of Δ_n were obtained in [11], closely related research can also see [12, 13].

In this paper, we discuss the problems for inequalities for edge lengths and circum-radius of a simplex in the hyperbolic space $H_n(-1)$ and the spherical space $S_n(1)$, and obtain some inequalities involving edge lengths and circum-radius of a simplex in $H_n(-1)$ and $S_n(1)$.

2. Inequalities for a simplex in hyperbolic space $H_n(-1)$

Let Ω_n be an n-simplex with vertices $P_0, P_1, \cdots, P_n$ in hyperbolic space $H_n(-1)$. The hyperbolic distance $\phi_{ij} = \operatorname{arccosh}(-\langle P_i, P_j\rangle)$ between any two vertices P_i, P_j $(i \neq j)$ are called the edge length of Ω_n. For an n-simplex Ω_n in $H_n(-1)$, we know that either there is its circum-scribed sphere or there is not its circumscribed sphere. If there exists a circum-scribed sphere for an n-simplex Ω_n in $H_n(-1)$, we use R and C denote circum-radius and circum-center of Ω_n respectively. In hyperbolic space $H_n(-1)$, let α_{ij} be angle formed by rays CP_i and CP_j. Let $\lambda_i \neq 0$ $(i = 0, 1, \cdots, n)$ be real numbers, we put

$$G = (\lambda_i \lambda_j \cos \alpha_{ij})_{i,j=0}^{n}.$$

We obtain some inequalities for an n-simplex in $H_n(-1)$ as follows.

THEOREM 1. *Let Ω_n be an n-simplex in $H_n(-1)$, and ϕ_{ij} $(0 \leqslant i < j \leqslant n)$ be its edge lengths. If there exists a circumscribed sphere of Ω_n, let R be its circum-radius and $\lambda_i \neq 0$ $(i = 0, 1, \cdots, n)$ real numbers, then we have*

$$\sum_{0 \leqslant i < j \leqslant n} \lambda_i \lambda_j \sinh^2 \frac{\phi_{ij}}{2} \leqslant \frac{1}{4} \left(\sum_{i=0}^{n} \lambda_i\right)^2 \sinh^2 R, \tag{1}$$

equality holds if $\lambda_0 = \lambda_1 = \cdots = \lambda_n$ and Ω_n is regular.

Put $\lambda_0 = \lambda_1 = \cdots = \lambda_n$ in inequality (1), we get

COROLLARY 1. *If there exists a circumscribed sphere of n-simplex Ω_n in $H_n(-1)$, then we have*

$$\sum_{0 \leqslant i < j \leqslant n} \sinh^2 \frac{\phi_{ij}}{2} \leqslant \frac{1}{4}(n+1)^2 \sinh^2 R, \tag{2}$$

equality holds if Ω_n is regular.

THEOREM 2. *Let Ω_n be an n-simplex in $H_n(-1)$, and ϕ_{ij} $(0 \leqslant i < j \leqslant n)$ be its edge lengths. If there exists a circumscribed sphere of Ω_n, let R be its circum-radius and $\lambda_i \neq 0$ $(i = 0, 1, \cdots, n)$ real numbers, then we have*

$$\sum_{0\leqslant i<j\leqslant n} \lambda_i^2\lambda_j^2(\cosh^2 R - \cosh\phi_{ij})^2 \geqslant \frac{1}{n}\Big(\sum_{0\leqslant i<j\leqslant n} \lambda_i^2\lambda_j^2 - \frac{n-1}{2}\sum_{i=0}^{n}\lambda_i^4\Big)\sinh^4 R, \qquad (3)$$

equality holds if and only if the nonzero eigenvalues of matrix G are all equal.

If take $\lambda_0 = \lambda_1 = \cdots = \lambda_n$ in inequality (3), we get an inequality as follow.

COROLLARY 2. *If there exists a circumscribed sphere of n-simplex Ω_n in $H_n(-1)$, then we have*

$$\sum_{0\leqslant i<j\leqslant n} (\cosh^2 R - \cosh\phi_{ij})^2 \geqslant \frac{n+1}{2n}\sinh^4 R, \qquad (4)$$

equality holds if and only if the nonzero eigenvalues of matrix $G' = (\cos\alpha_{ij})_{i,j=0}^{n}$ are all equal.

To prove Theorem 1 and Theorem 2, we need some lemmas as follows.

LEMMA 1. *Let $S_k(1)$ be k-dimensional sphere of radius 1 in the s-dimensional Euclidean space E_s $(s > k)$, and the origin O be its circum-center, points $A_i \in S_k(1)$ $(i = 0, 1, \cdots, m)$ and $\lambda_i \neq 0$ $(i = 0, 1, \cdots, m)$ be real numbers, then we have*

$$\sum_{0\leqslant i<j\leqslant m} \lambda_i\lambda_j a_{ij}^2 \leqslant \Big(\sum_{i=0}^{m}\lambda_i\Big)^2, \qquad (5)$$

equality if and only if $\lambda_0\overrightarrow{OA_0} + \lambda_1\overrightarrow{OA_1} + \cdots + \lambda_m\overrightarrow{OA_m} = 0$, that is equality holds in (5) if and only if the barycenter of the mass-points system $\{A_i(\lambda_i) | i = 0, 1, \cdots, m\}$ in E_s is the origin O of E_s. Where $a_{ij} = |A_iA_j|$ be the Euclidean distance between two points A_i and A_j.

Proof. We put

$$\overrightarrow{OA} = \lambda_0\overrightarrow{OA_0} + \lambda_1\overrightarrow{OA_1} + \cdots + \lambda_m\overrightarrow{OA_m},$$

then

$$\begin{aligned}
0 \leqslant |\overrightarrow{OA}|^2 &= |\lambda_0\overrightarrow{OA_0} + \lambda_1\overrightarrow{OA_1} + \cdots + \lambda_m\overrightarrow{OA_m}|^2 \\
&= \sum_{i=0}^{m}\lambda_i^2|\overrightarrow{OA_0}|^2 + 2\sum_{0\leqslant i<j\leqslant m}\lambda_i\lambda_j(\overrightarrow{OA_i}\cdot\overrightarrow{OA_j}) \\
&= \sum_{i=0}^{m}\lambda_i^2 + \sum_{0\leqslant i<j\leqslant m}\lambda_i\lambda_j(|\overrightarrow{OA_i}|^2 + |\overrightarrow{OA_j}|^2 - |\overrightarrow{A_iA_j}|^2) \\
&= \sum_{i=0}^{m}\lambda_i^2 + 2\sum_{0\leqslant i<j\leqslant m}\lambda_i\lambda_j - \sum_{0\leqslant i<j\leqslant m}\lambda_i\lambda_j a_{ij}^2.
\end{aligned}$$

From this we obtain inequality (5)holds.It is easy to see that equality holds if and only if $A \equiv O$. □

LEMMA 2. *In $H_n(-1)$, let Ω_n be an n-dimensional simplex with vertices P_i $(i = 0,1,\cdots,n)$ and Ω_n be inscribed a sphere with center C. Let α_{ij} be the angle formed by rays CP_i and CP_j $(i \neq j,\ i,j = 0,1,\cdots,n)$ in $H_n(-1)$, then there exist rays OA_i $(i = 0,1,\cdots,n)$ in the n-dimensional Euclidean space E_n such that the angle formed by rays OA_i and OA_j is equal to α_{ij} for $i \neq j,\ i,j = 0,1,\cdots,n$.*

Proof. Lemma 2 can see the exercise 2 in [10, P_{273}]. We now give the following proof.

There are several hyperbolic geometric models. By the theory of differential geometry, E. Beltrami constructs one hyperbolic geometric model. He gives a describe in detail as follows: In the $n+1$-dimensional Euclidean space, let $S_n(-1)$ is the pseudo-sphere with Gaussian curvature $K = -1$, the hyperbolic geometry in the n-dimensional hyperbolic space $H_n(-1)$ is regarded as the geometry on $S_n(-1)$ whose geodesics are straight lines [14, 15].

By the condition of Lemma 2, there exist $n+1$ rays (i.e. geodesics) $\widehat{DB_i}$ $(i = 0,1,\cdots,n)$ passed through the same point D on $S_n(-1)$, such that the angle between any two rays $\widehat{DB_i}$ and $\widehat{DB_j}$ is α_{ij} (for $i \neq j,\ i,j = 0,1,\cdots,n$). Let δ_i $(i = 0,1,\cdots,n)$ denote the unit tangent vector at the point D of the the geodesics $\widehat{DB_i}$. Then α_{ij} is the angle formed by vectors δ_i and δ_j $(i \neq j, i,j = 0,1,\cdots,n)$. Suppose that $T_n(D)$ is the tangent plane of the pseudo-sphere $S_n(-1)$ at the point D, then $\delta_i \in T_n(D)$ $(i = 0,1,\cdots,n)$. Therefore, there exist $n+1$ rays OA_i $(i = 0,1,\cdots,n)$ passed through the same point O on the n-dimensional tangent plane $T_n(D)$ such that the angle between any two rays OA_i and OA_j is α_{ij} $(i \neq j,\ i,j = 0,1,\cdots,n)$. □

Proofs of Theorem 1 and Theorem 2. In $H_n(-1)$, let point C be the circum-center of the simplex Ω_n and α_{ij} be angle formed by rays CP_i and CP_j for $i,j \in \{0,1,\cdots,n\}$ and $i \neq j$. By Lemma 2 we know that there are $n+1$ unit vectors $\overrightarrow{OA_i}$ in E_n such that α_{ij} is the angle formed by vectors $\overrightarrow{OA_i}$ and $\overrightarrow{OA_j}$ for $i,j \in \{0,1,\cdots,n\}$ and $i \neq j$. Let $a_{ij} = |A_iA_j|$ be Euclidean distance between points A_i and A_j, by Lemma 1 we have

$$\sum_{0 \leqslant i<j \leqslant m} \lambda_i\lambda_j a_{ij}^2 \leqslant \Big(\sum_{i=0}^{n} \lambda_i\Big)^2. \tag{6}$$

Using the cosine formula of hyperbolic simplex (see the equality (56) in [16]) for 2-dimensional hyperbolic simplex CP_iP_j, we have

$$\cos\alpha_{ij} = \frac{\cosh^2 R - \cosh\phi_{ij}}{\sinh^2 R}. \tag{7}$$

Using the cosine theorem for triangle OA_iA_j in E_n and equality (7), we get

$$a_{ij}^2 = |A_iA_j|^2 = 2 - 2\cos\alpha_{ij} = 2 - \frac{2(\cosh^2 R - \cosh\phi_{ij})}{\sinh^2 R} = \frac{4\sinh^2\frac{\phi_{ij}}{2}}{\sinh^2 R}. \tag{8}$$

Substituting (8) into (6), we get inequality (1). It is easy to prove that equality holds in (1) if $\lambda_0 = \lambda_1 = \cdots = \lambda_n$ and the simplex Ω_n is regular. The proof of Theorem 1 is complete.

From the proof of Theorem 1 above and Ω_n is an n-dimensional simplex in hyperbolic space $H_n(-1)$, we know that the rank of unit vectors system $\overrightarrow{OA_0}, \overrightarrow{OA_1}, \cdots, \overrightarrow{OA_n}$ in E_n is n. Because $\lambda_i \neq 0$ $(i=0,1,\cdots,n)$, thus the rank of vectors system $\lambda_0\overrightarrow{OA_0}, \lambda_1\overrightarrow{OA_1}, \cdots, \lambda_n\overrightarrow{OA_n}$ is also n. From this, we know that Gram matrix $G=(\lambda_i\lambda_j\overrightarrow{OA_i}\cdot\overrightarrow{OA_j})_{i,j=0}^n = (\lambda_i\lambda_j\cos\alpha_{ij})_{i,j=0}^n$ of vectors system $\lambda_i\overrightarrow{OA_i}$ $(i=0,1,\cdots,n)$ is semi-positive definite symmetric matrix and its rank is n. Let $x_i>0$ $(i=1,2,\cdots,n)$ and $x_0=0$ be the eigenvalues of the matrix G, we put

$$\sigma_1=\sum_{i=0}^n x_i=\sum_{i=1}^n x_i, \sigma_2=\sum_{0\leqslant i<j\leqslant n} x_ix_j=\sum_{1\leqslant i<j\leqslant n} x_ix_j. \tag{9}$$

Using Maclaurin's inequality (see [11]), we have

$$(\frac{1}{n}\sigma_1)^2 \geqslant \frac{2\sigma_2}{n(n-1)}, \tag{10}$$

with equality if and only if $x_1=x_2=\cdots=x_n$.

By the relation between roots and determinants of principal sub-matrices of matrix G, we have

$$\sigma_1=\sum_{i=0}^n\lambda_i^2, \sigma_2=\sum_{0\leqslant i<j\leqslant n}\lambda_i^2\lambda_j^2(1-\cos^2\alpha_{ij}). \tag{11}$$

Substituting (11) into (10), we get

$$\sum_{0\leqslant i<j\leqslant n}\lambda_i^2\lambda_j^2(1-\cos^2\alpha_{ij}) \leqslant \frac{n-1}{2n}(\sum_{i=0}^n\lambda_i^2)^2. \tag{12}$$

Equality holds in (12) if and only if the nonzero eigenvalues of matrix G are all equal.

Substituting (7) into (12), we get inequality (3). The proof of Theorem 2 is complete. □

In fact, let $x_i=\frac{\lambda_i}{\lambda_0+\cdots+\lambda_n}$ $(i=0,1,\cdots,n)$ in inequality (1), because $\cosh^2 R-\sinh^2 R=1$, $2\sinh^2(\frac{\phi_{ij}}{2})=\cosh\phi_{ij}-1$, we have

$$\cosh^2 R \geqslant \sum_{i=0}^n\sum_{i=0}^n x_ix_j\cosh\phi_{ij}$$

So our result implies one of main results of [13], moreover, this inequality more concise and elegant.

3. Inequalities for a simplex and any point in spherical space $S_n(1)$

In this section, let Ω_n be an n-dimensional simplex with vertices P_i $(i=0,1,\cdots,n)$ in spherical space $S_n(1)$. The spherical distance $\phi_{ij}=\widehat{P_iP_j}$ between two vertices P_i and P_j $(i\neq j)$ are called the edge length of Ω_n. $S_n(1)$ can be regard as n-dimensional unite sphere in $(n+1)$-dimensional Euclidean space E_{n+1} and its center is the origin O of E_{n+1}, and $\phi_{ij} = \angle P_i O P_j$ is the spherical distance between points P_i and P_j for $i \neq j$, $i, j = 0, 1, \cdots, n$. Let R and point C denote the circum-radius and the circum-center of spherical simplex Ω_n in $S_n(1)$ $(C \in S_n(1))$, we obtain an inequality as follows.

THEOREM 3. *Let Ω_n be an n-dimensional simplex in $S_n(1)$ and ϕ_{ij} $(0 \leqslant i < j \leqslant n)$ its edge lengths, $\lambda_i \neq 0$ $(i = 0, 1, \cdots, n)$ be real numbers, then we have*

$$\sum_{0 \leqslant i<j \leqslant n} \lambda_i \lambda_j \sin^2 \frac{\phi_{ij}}{2} \leqslant \left[\frac{1}{4}\Big(\sum_{i=0}^{n} \lambda_i + 1\Big)^2 - \sum_{i=0}^{n} \lambda_i\right] + \Big(\sum_{i=0}^{n} \lambda_i\Big) \cos^2 \frac{R}{2}, \tag{13}$$

equality holds if and only if the barycenter of the mass-points system $\{P_0(\lambda_0), P_1(\lambda_1), \cdots, P_n(\lambda_n), C(1)\}$ in E_{n+1} is the origin O of E_{n+1}.

If take $\lambda_0 = \lambda_1 = \cdots = \lambda_n$ in (13), we get an inequality as follows.

COROLLARY 3. *For an n-simplex Ω_n in spherical space $S_n(1)$, then*

$$\sum_{0 \leqslant i<j \leqslant n} \sin^2 \frac{\phi_{ij}}{2} \leqslant \frac{n^2}{4} + (n+1) \cos^2 \frac{R}{2}. \tag{14}$$

with equality if and only if the barycenter of the points system $\{P_0, P_1, \cdots, P_n, C\}$ in E_{n+1} is the origin O of E_{n+1}.

Proof of Theorem 3. The points $P_0, P_1, \cdots, P_n, C$ (the point C is the circum-center of spherical simplex Ω_n) are on the n-dimensional unit sphere $S_n(1)$ in E_{n+1}. By Lemma 1 we have

$$\sum_{0 \leqslant i<j \leqslant n} \lambda_i \lambda_j |P_i P_j|^2 + \sum_{i=0}^{n} \lambda_i |CP_i|^2 \leqslant \Big(\sum_{i=0}^{n} \lambda_i + 1\Big)^2, \tag{15}$$

with equality if and only if the barycenter of the mass points system $\{P_0(\lambda_0), P_1(\lambda_1), \cdots, P_n(\lambda_n), C(1)\}$ in E_{n+1} is the origin O of E_{n+1}. Where $|P_i P_j|$ denotes the Euclidean distance between points P_i and P_j. It is easy to see that

$$\phi_{ij} = \widehat{P_i P_j} = \angle P_i O P_j \quad (i, j = 0, 1, \cdots, n), \qquad R = \widehat{CP_i} = \angle COP_i \quad (i = 0, 1, \cdots, n).$$

For triangles COP_i and $P_i O P_j$ in Euclidean space E_{n+1}, by cosine theorem for a triangle we have

$$|CP_i|^2 = 2 - 2\cos R = 4 \sin^2 \frac{R}{2} \quad (i = 0, 1, \cdots, n), \tag{16}$$

$$|P_i P_j|^2 = 2 - 2\cos \phi_{ij} = 4 \sin^2 \frac{\phi_{ij}}{2} \quad (i, j = 0, 1, \cdots, n), \tag{17}$$

Substituting (16) and (17) into (15), we get inequality (13). The proof of Theorem 3 is complete. □

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