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UNCERTAINTY PRINCIPLE FOR THE SPHERICAL MEAN OPERATOR

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Abstract. The $L^p - L^q$ version of Miyachi's theorem is proved for the Fourier transform associated with the spherical mean operator.

1. Introduction

In the classical case, Miyachi's uncertainty principle [4] states that if f is a measurable function on $\mathbb{R}$ satisfying

$$e^{ax^2} f \in L^1(d\mu) + L^\infty(d\mu),$$

and

$$\int_{\mathbb{R}} \log^+ \frac{|\widehat{f}(x)| e^{bx^2}}{\delta} dx < +\infty,$$

for some positive constants a, b, δ such that $ab = \frac{1}{4}$, where $L^1(d\mu)$ and $L^\infty(d\mu)$ denote the standard Lebesgue spaces, then f is a multiple constant of a gaussian function. Recently, Miyachi's theorem has been proved by Daher, for Jacobi-Dunkl transform [1]. The spherical mean operator $\mathscr{R}$ is defined, for a function f on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable, by [5]

$$\mathscr{R}(f)(r,x) = \int_{S^n} f(r\eta, x + r\xi) d\sigma_n(\eta, \xi), \quad (r,x) \in \mathbb{R} \times \mathbb{R}^n,$$

where S^n is the unit sphere of $\mathbb{R}^{n+1}$ and $d\sigma_n$ is the surface measure on S^n normalized to have total measure one. The operator $\mathscr{R}$ has many important physical applications, namely in image processing of so-called synthetic aperture radar (SAR) data [3], or in the linearized inverse scattering problem in acoustics [2]. Many harmonic analysis result related to the spherical mean operator and its Fourier transform have already been proved notably by Nessibi, Rachdi, Omri and Trimèche [5, 6]. Our purpose in this work is to establish an $L^p - L^q$ version of Miyachi's theorem for the Fourier transform associated with the spherical mean operator. This paper is organized as follows, in

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the second section we prove some useful properties relates to the Bessel operator and the Hankel transform, the third section is devoted to recall and show some harmonic analysis results related to the sphercial mean operator $\mathscr{R}$ and its Fourier transform $\mathscr{F}$, and finally in the last section we prove the main result of this paper that is, the $L^p - L^q$ version of Miyachi's theorem for the Fourier transform $\mathscr{F}$.

2. The Bessel operator and The Hankel transform

In this section we shall show some useful properties related to the Bessel operator. For every $\alpha > -\frac{1}{2}$, the Bessel operator is defined on $]0,+\infty[$, by $\ell_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r}$, and it is well known that for every nonnegative real number s, the function $j_\alpha(s.)$, is the unique infinitely differentiable function on $[0,+\infty[$ satisfying the following Cauchy problem

$$\begin{cases} \ell_\alpha u = -s^2 u, \\ u'(0) = 0, \\ u(0) = 1, \end{cases},$$

where j_α is the modified Bessel function defined by [8]

$$\forall z \in \mathbb{C},\ j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{+\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{\Gamma(\alpha+n+1)n!}. \tag{2.1}$$

The modified Bessel function j_α has the following integral representation

$$\forall z \in \mathbb{C},\ j_\alpha(z) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}}\cos(zt)dt.$$

In particular, we have

$$\forall z \in \mathbb{C},\ p \in \mathbb{N}, \quad \left| j_\alpha^{(p)}(z) \right| \leqslant e^{|Im(z)|}. \tag{2.2}$$

In the following we denote by

- $d\tau_\alpha$ the measure defined on $[0,+\infty[$, by $d\tau_\alpha(r) = \frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}dr$.
- $L^p(d\tau_\alpha)$ the Lebesgue spaces of measurable functions on $[0,+\infty[$ satisfying
- $\|f\|_{p,\tau_\alpha} = \left(\int_0^{+\infty} |f(r)|^p d\tau_\alpha(r)\right)^{\frac{1}{p}} < +\infty$, if $p \in [1,+\infty[$.
- $\|f\|_{\infty,\tau_\alpha} \operatorname*{ess\ sup}_{r\in[0,+\infty[} |f(r)| < +\infty$, If $p = +\infty$.
- $\mathscr{E}_e(\mathbb{R})$ the space of even functions, infinitely differentiable on $\mathbb{R}$.
- $S_e(\mathbb{R})$ the subspace of the schwartz class $S(\mathbb{R})$, formed by the even functions.
- $\mathscr{C}_{0,e}(\mathbb{R})$ the space of continuous even functions on $\mathbb{R}$ satisfying $\lim_{\pm\infty} f(r) = 0$

DEFINITION 2.1. The Hankel transform $\mathscr{H}_\alpha$ is defined on $L^1(d\tau_\alpha)$ according to Schwartz [7], by

$$\forall s \in [0,+\infty[,\ \mathscr{H}_\alpha(f)(s) = \int_0^{+\infty} f(r) j_\alpha(rs) d\tau_\alpha(r).$$

Then it is known that the Hankel transform is a bounded linear operator from $L^1(d\tau_\alpha)$ into $\mathscr{C}_{0,e}(\mathbb{R})$. Moreover, for every $f \in S_e(\mathbb{R})$, we have the following inversion formula

$$f = \mathscr{H}_\alpha\left(\mathscr{H}_\alpha(f)\right), \tag{2.3}$$

and for every $m \in \mathbb{N}$ we have

$$\forall s \in [0,+\infty[,\ \mathscr{H}_\alpha\left((Id-\ell_\alpha)^m(f)\right)(s) = (1+s^2)^m \mathscr{H}_\alpha(f)(s). \tag{2.4}$$

PROPOSITION 2.2. *For every nonnegative integer k, we have*

$$\ell_\alpha^k\left(e^{-t.^2}\right)(s) = (-4t)^k k! L_k^\alpha(ts^2) e^{-ts^2}, \tag{2.5}$$

where L_k^α denote the classical Laguerre polynomial [8].

Proof. By a standard calculus we know that for every positive real number t, we have

$$\mathscr{H}_\alpha(e^{-t.^2})(s) = \frac{e^{-\frac{s^2}{4t}}}{(2t)^{\alpha+1}}. \tag{2.6}$$

Let m be an integer, then by relations (2.3) and (2.4), we have

$$\begin{aligned}\mathscr{H}_\alpha\left((1+.^2)^m \mathscr{H}_\alpha(e^{-t.^2})\right)(s) &= (Id-\ell_\alpha)^m(e^{-t.^2})(s)\\ &= \sum_{k=0}^m (-1)^k C_m^k \ell_\alpha^k(e^{-t.^2})(s)\end{aligned} \tag{2.7}$$

On the other hand by relation (2.6), we get

$$\mathscr{H}_\alpha\left((1+.^2)^m \mathscr{H}_\alpha(e^{-t.^2})\right)(s) = \frac{1}{2^{2\alpha+1} t^{\alpha+1} \Gamma(\alpha+1)} \sum_{k=0}^m C_m^k \int_0^{+\infty} r^{2k+2\alpha+1} e^{-\frac{r^2}{4t}} j_\alpha(rs) dr,$$

and by change of variables $u = \dfrac{r}{2\sqrt{t}}$, we deduce that

$$\begin{aligned}\mathscr{H}_\alpha\left((1+.^2)^m \mathscr{H}_\alpha(e^{-t.^2})\right)(s) &= \frac{2}{\Gamma(\alpha+1)} \sum_{k=0}^m C_m^k 2^{2k} t^k \int_0^{+\infty} u^{2k+2\alpha+1} e^{-u^2} j_\alpha(2\sqrt{t}us) du\\ &= \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^m C_m^k 2^{2k} t^k \int_0^{+\infty} v^{k+\alpha} e^{-v} j_\alpha(2\sqrt{vts^2}) dv\end{aligned}$$

Using now the integral representation of Laguerre polynomials [8], we obtain

$$\mathscr{H}_\alpha\left((1+.^2)^m\mathscr{H}_\alpha(e^{-t.^2})\right)(s)=\sum_{k=0}^{m}C_m^k 2^{2k}t^kL_k^\alpha(ts^2)k!e^{-ts^2} \tag{2.8}$$

Combining relations (2.7) and (2.8), we deduce that

$$\sum_{k=0}^{m}C_m^k(-1)^k\ell_\alpha^k(e^{-t.^2})(s)=\sum_{k=0}^{m}C_m^k 2^{2k}t^kL_k^\alpha(ts^2)k!e^{-ts^2},$$

and by the classical Pascal's inversion formula, we conclude finally that for every integer k, and for every positive real number t, we have

$$\forall s\in[0,+\infty[,\ \ell_\alpha^k\left(e^{-t.^2}\right)(s)=(-4t)^kk!L_k^\alpha(ts^2)e^{-ts^2}. \quad \square$$

3. The spherical mean operator

Nessibi et al. [5] showed that for every $(s,y)\in\mathbb{C}\times\mathbb{C}^n$, the function $\varphi_{(s,y)}$ defined on $\mathbb{R}\times\mathbb{R}^n$ by

$$\varphi_{(s,y)}(r,x)=\mathscr{R}\left(\cos(s.)e^{-i\langle y|.\rangle}\right)(r,x), \tag{3.1}$$

is the unique infinitely differentiable function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, satisfying the following system

$$\begin{cases}\dfrac{\partial u}{\partial x_j}(r,x_1,...,x_n)=-iy_ju(r,x_1,...,x_n), & 1\leqslant j\leqslant n,\\ \Delta_{\frac{n-1}{2}}u(r,x_1,...,x_n)=-s^2u(r,x_1,...,x_n), & \\ u(0,...,0)=1, & \\ \dfrac{\partial u}{\partial r}(0,x_1,...,x_n)=0, & (x_1,\dots,x_n)\in\mathbb{R}^n,\end{cases} \tag{3.2}$$

where $\Delta_{\frac{n-1}{2}}=\ell_{\frac{n-1}{2}}-\Delta$ and Δ denotes as usual the Laplacian operator given by $\Delta=\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}$. Now we consider the following notations.

For every $\lambda=(\lambda_1,\ \dots,\ \lambda_n)$, $\mu=(\mu_1,\ \dots,\ \mu_n)\in\mathbb{C}^n$, and $\beta=(\beta_1,\ \dots,\ \beta_n)\in\mathbb{N}^n$, we denote by

- $\langle\lambda|\mu\rangle=\sum_{i=1}^{n}\lambda_i\overline{\mu_i}$, $|\lambda|=\left(\sum_{i=1}^{n}|\lambda_i|^2\right)^{\frac{1}{2}}$ and $\lambda^\beta=\prod_{i=1}^{n}\lambda_i^{\beta_i}$.
- $|\beta|=\sum_{i=1}^{n}\beta_i$, $Im\lambda=(Im\lambda_1,\ \dots,\ Im\lambda_n)$ and $Re\,\lambda=(Re\,\lambda_1,\ \dots,\ Re\,\lambda_n)$.

In [5], the authors proved that the eigenfunction $\varphi_{(s,y)}$ defined by relation (3.1), is explicitly given by

$$\forall (r,x) \in \mathbb{R}\times\mathbb{R}^n,\ (s,y) \in \mathbb{C}\times\mathbb{C}^n,\ \varphi_{(s,y)}(r,x) = j_{\frac{n-1}{2}}(r\sqrt{s^2+|y|^2})e^{-i\langle y|x\rangle}. \tag{3.3}$$

From relations (2.2) and (3.3), it follows that the function $\varphi_{(s,y)}$ satisfies

$$\forall (s,y) \in \mathbb{C}\times\mathbb{C}^n,\ (r,x) \in \mathbb{R}\times\mathbb{R}^n,\ \left|\varphi_{(s,y)}(r,x)\right| \leqslant e^{(|r|+|x|)(|Ims|+|Imy|)},$$

and therefore, $\varphi_{(s,y)}$ is bounded on $\mathbb{R}\times\mathbb{R}^n$, if and only if (s,y) belongs to the set Γ_{n+1} defined by

$$\Gamma_{n+1} = \mathbb{R}\times\mathbb{R}^n \cup \left\{(ir,x),\ (r,x)\in\mathbb{R}\times\mathbb{R}^n,\ |r|\leqslant|x|\right\}, \tag{3.4}$$

and in this case

$$\sup_{(r,x)\in[0,+\infty[\times\mathbb{R}^n} \left|\varphi_{(s,y)}(r,x)\right| = 1. \tag{3.5}$$

In the following we shall recall some properties related to the spherical mean operator. For this we denote by

- dm_{n+1} the measure defined on $[0,+\infty[\times\mathbb{R}^n$, by $dm_{n+1}(r,x) = \sqrt{\frac{2}{\pi}}\frac{dr\,dx}{(2\pi)^{\frac{n}{2}}}$.
- $L^p(dm_{n+1})$ the space of measurable functions f on $[0,+\infty[\times\mathbb{R}^n$, such that

$$\|f\|_{p,m_{n+1}} = \left(\int_0^{+\infty}\int_{\mathbb{R}^n}|f(r,x)|^p\,dm_{n+1}(r,x)\right)^{\frac{1}{p}} < +\infty, \text{ if } p\in[1,+\infty[,$$
$$\|f\|_{\infty,m_{n+1}} = \operatorname*{ess\,sup}_{(r,x)\in[0,+\infty[\times\mathbb{R}^n}|f(r,x)| < +\infty, \qquad \text{if } p=+\infty.$$

- $d\nu_{n+1}$ the measure defined on $[0,+\infty[\times\mathbb{R}^n$ by $d\nu_{n+1}(r,x) = \dfrac{r^n dr dx}{(\pi)^{\frac{n}{2}}2^{n-\frac{1}{2}}\Gamma(\frac{n+1}{2})}$.
- $L^p(d\nu_{n+1})$, $p\in[1,+\infty]$ the Lebesgue space of measurable functions f on $[0,+\infty[\times\mathbb{R}^n$, satisfying

$$\|f\|_{p,\nu_{n+1}} = \left(\int_0^{+\infty}\int_{\mathbb{R}^n}|f(r,x)|^p\,d\nu_{n+1}(r,x)\right)^{\frac{1}{p}} < +\infty, \text{ if } p\in[1,+\infty[;$$
$$\|f\|_{\infty,\nu_{n+1}} = \operatorname*{ess\,sup}_{(r,x)\in[0,+\infty[\times\mathbb{R}^n}|f(r,x)| < +\infty, \qquad \text{if } p=+\infty.$$

- $\Gamma_{n+1,+}$ the subset of Γ_{n+1} defined by

$$\Gamma_{n+1,+} = [0,+\infty[\times\mathbb{R}^n \cup \left\{(is,y)\ ;\ (s,y)\in[0,+\infty[\times\mathbb{R}^n;\ s\leqslant|y|\right\}.$$

- $\mathscr{B}_{\Gamma_{n+1,+}}$ the σ-algebra defined on $\Gamma_{n+1,+}$ by

$$\mathscr{B}_{\Gamma_{n+1,+}} = \left\{\theta^{-1}(B)\ ,\ B\in\mathscr{B}_{Bor}([0,+\infty[\times\mathbb{R}^n)\right\},$$

where θ is the bijective function defined on the set $\Gamma_{n+1,+}$ by

$$\theta(s,y) = (\sqrt{s^2+|y|^2},y).$$

• $d\gamma_{n+1}$ the measure defined on $\mathscr{B}_{\Gamma_{n+1,+}}$ by

$$\forall\, B \in \mathscr{B}_{\Gamma_{n+1,+}}, \quad \gamma_{n+1}(B) = \nu_{n+1}(\theta(B)).$$

• $L^p(d\gamma_{n+1})$, $p \in [1,+\infty]$ the space of measurable functions f on $\Gamma_{n+1,+}$, satisfying

$$\begin{aligned} \|f\|_{p,\gamma_{n+1}} &= \left(\int_{\Gamma_{n+1,+}} |f(s,y)|^p \, d\gamma_{n+1}(s,y)\right)^{\frac{1}{p}} < +\infty, \text{ if } p \in [1,+\infty[; \\ \|f\|_{\infty,\gamma_{n+1}} &= \operatorname*{ess\,sup}_{(s,y)\in\Gamma_{n+1,+}} |f(s,y)| < +\infty, \qquad \text{if } p = +\infty. \end{aligned}$$

• $\mathscr{C}_e(\mathbb{R}\times\mathbb{R}^n)$, the space of continuous functions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable.

• $S_e(\mathbb{R}\times\mathbb{R}^n)$ the space of infinitely differentiable functions on $\mathbb{R}\times\mathbb{R}^n$, rapidly decreasing together with all their derivatives and even with respect to the first variable.

• $\mathbb{C}_e[X^{n+1}]$, the vector space of polynomials functions of $n+1$ variables, even with respect to the first variable.

The dual of the spherical mean operator ${}^t\mathscr{R}$ is defined according to [5] by

$$ {}^t\mathscr{R}(g)(r,x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(\sqrt{r^2+|x-y|^2},y)dy, \tag{3.6}$$

whenever the integral of the right-hand side is well defined, where dy denotes the Lebesgue measure on $\mathbb{R}^n$. As shown in [5] the spherical mean operator $\mathscr{R}$ and its dual ${}^t\mathscr{R}$ satisfy the following harmonic analysis results.

LEMMA 3.1. *i) For every function $f \in L^1(d\nu_{n+1})$, the function ${}^t\mathscr{R}(f)$ belongs to $L^1(dm_{n+1})$ and we have*

$$\|{}^t\mathscr{R}(f)\|_{1,m_{n+1}} \leqslant \|f\|_{1,\nu_{n+1}}. \tag{3.7}$$

ii) For every bounded function $f \in \mathscr{C}_e(\mathbb{R}\times\mathbb{R}^n)$, and for every function $g \in L^1(d\nu_{n+1})$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}^n} \mathscr{R}(f)(r,x)g(r,x)d\nu_{n+1}(r,x) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)^t\mathscr{R}(g)(r,x)dm_{n+1}(r,x), \tag{3.8}$$

The Fourier transform $\mathscr{F}$ associated with the spherical mean operator is defined for every measurable function $f \in L^1(d\nu_{n+1})$, by [5]

$$\forall (s,y) \in \Gamma_{n+1},\ \mathscr{F}(f)(s,y) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\varphi_{(s,y)}(r,x)d\nu_{n+1}(r,x), \tag{3.9}$$

where Γ_{n+1} is the subset of $\mathbb{C}\times\mathbb{C}^n$ given by relation (3.4).

PROPOSITION 3.2. *i) For every nonnegative measurable function g on $\Gamma_{n+1,+}$, we have*

$$\begin{aligned} \int_{\Gamma_{n+1,+}} g(\mu,\lambda)\, d\gamma_{n+1}(\mu,\lambda) = \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})(2\pi)^{\frac{n}{2}}} \Big(\int_0^{+\infty}\int_{\mathbb{R}^n} g(\mu,\lambda)(\mu^2+|\lambda|^2)^{\frac{n-1}{2}}\mu\, d\mu\, d\lambda \\ + \int_{\mathbb{R}^n}\int_0^{|\lambda|} g(i\mu,\lambda)(|\lambda|^2-\mu^2)^{\frac{n-1}{2}}\mu\, d\mu\, d\lambda \Big). \end{aligned}$$

ii) For every nonnegative measurable function f on $[0,+\infty[\times\mathbb{R}^n$ (respectively integrable on $[0,+\infty[\times\mathbb{R}^n$ with respect to the measure $d\nu_{n+1}$), $fo\theta$ is a measurable nonnegative function on $\Gamma_{n+1,+}$, (respectively integrable on $\Gamma_{n+1,+}$ with respect to the measure $d\gamma_{n+1}$) and we have

$$\int_{\Gamma_{n+1,+}}(f\circ\theta)(\mu,\lambda)\,d\gamma_{n+1}(\mu,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}^n}f(r,x)\,d\nu_{n+1}(r,x). \tag{3.10}$$

From relation (3.10), we deduce that the Fourier transform $\mathscr{F}$ defined by relation (3.9) satisfies the following relation

$$\forall(s,y)\in\Gamma_{n+1},\quad \mathscr{F}(f)(s,y)=\widetilde{\mathscr{F}}(f)\circ\theta(s,y), \tag{3.11}$$

where $\widetilde{\mathscr{F}}$ is the integral transform defined on $L^1(d\nu_{n+1})$ by

$$\forall(s,y)\in\mathbb{R}\times\mathbb{R}^n,\ \widetilde{\mathscr{F}}(f)(s,y)=\int_0^{+\infty}\int_{\mathbb{R}^n}f(r,x)\,j_{\frac{n-1}{2}}(rs)e^{-i\langle y|x\rangle}d\nu_{n+1}(r,x).$$

On the other hand it is known by using in particular relation (3.5), that the Fourier transform $\mathscr{F}$ associated with the spherical mean operator, is a linear bounded operator from $L^1(d\nu_{n+1})$ into $L^\infty(d\gamma_{n+1})$ and that for every $f\in L^1(d\nu_{n+1})$, we have

$$\|\mathscr{F}(f)\|_{\infty,\gamma_{n+1}}\leqslant\|f\|_{1,\nu_{n+1}}.$$

Furthermore, the Fourier transform $\mathscr{F}$ satisfies the following inversion formula and Plancherel theorem:

THEOREM 3.3. (Inversion formula) *Let $f\in L^1(d\nu_{n+1})$ such that $\mathscr{F}(f)\in L^1(d\gamma_{n+1})$, then for almost every $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, we have*

$$f(r,x)=\int_{\Gamma_{n+1,+}}\mathscr{F}(f)(s,y)\overline{\varphi_{(s,y)}(r,x)}d\gamma_{n+1}(s,y).$$

REMARK 3.4. The same inversion formula holds also for the transform $\widetilde{\mathscr{F}}$, indeed for every $f\in L^1(d\nu_{n+1})$, such that $\widetilde{\mathscr{F}}$ belongs to $L^1(d\nu_{n+1})$, we have for almost every $(r,x)\in[0,+\infty[\times\mathbb{R}^n$,

$$f(r,x)=\int_0^{\infty}\int_{\mathbb{R}^n}\widetilde{\mathscr{F}}(f)(s,y)\,j_{\frac{n-1}{2}}(rs)e^{i\langle y|x\rangle}d\nu_{n+1}(s,y).$$

THEOREM 3.5. (Plancherel) *The Fourier transform $\mathscr{F}$ can be extended to an isometric isomorphism from $L^2(d\nu_{n+1})$ onto $L^2(d\gamma_{n+1})$.*

DEFINITION 3.6. For every $t>0$, the Gauss kernel g_t, associated with the spherical mean operator, is defined on $\mathbb{R}\times\mathbb{R}^n$, by

$$\begin{aligned}g_t(r,x)=\frac{e^{-\frac{r^2+|x|^2}{4t}}}{(2t)^{n+\frac{1}{2}}}&=\int_{\Gamma_{n+1,+}}e^{-t(s^2+2|y|^2)}\overline{\varphi_{(s,y)}(r,x)}d\gamma_{n+1}(s,y)\\&=\int_0^{+\infty}\int_{\mathbb{R}^n}e^{-t(s^2+|y|^2)}j_{\frac{n-1}{2}}(rs)e^{i\langle y|x\rangle}d\nu_{n+1}(s,y).\end{aligned} \tag{3.12}$$

PROPOSITION 3.7. *Let t be a positive real number, and let R be a polynomial function even with respect to the first variable, given by* $R(r,x)=\sum_{\substack{(k,p)\in\mathbb{N}\times\mathbb{N}^n\\ k+|p|\leqslant m}} a_{kp}r^{2k}x^p$.

Then, for every $(s,y)\in\mathbb{R}\times\mathbb{R}^n$, we have

$$\mathscr{F}(Rg_t)(s,y)=\sum_{\substack{(k,p)\in\mathbb{N}\times\mathbb{N}^n\\ k+|p|\leqslant m}} a_{kp}i^{|p|}4^k k!\sqrt{t}^{2k+|p|}L_k^{\frac{n-1}{2}}\left(t(s^2+|y|^2)\right)H_p(\sqrt{t}y)e^{-t(s^2+2|y|^2)},$$

where $L_k^{\frac{n-1}{2}}$ is the Laguerre polynomial and H_p is the Hermite polynomial of n variables.

Proof. We know that the function Rg_t belongs to the space $S_e(\mathbb{R}\times\mathbb{R}^n)$, hence $\mathscr{F}(Rg_t)$ is well defined. Let $(k,p)\in\mathbb{N}\times\mathbb{N}^n$, then by relations (2.5) and (3.2), we have for every $(s,y)\in\mathbb{R}\times\mathbb{R}^n$,

$$\begin{aligned}\widetilde{\mathscr{F}}\left(r^{2k}x^p g_t\right)(s,y)&=(-1)^k(-i)^{|p|}\ell^k_{\frac{n-1}{2}}\left(\frac{\partial^p\widetilde{\mathscr{F}}(g_t)}{\partial y^p}\right)(s,y)\\&=(-1)^k(-i)^{|p|}\ell^k_{\frac{n-1}{2}}\left(e^{-t.^2}\right)(s)\left(\frac{\partial^p e^{-t|.|^2}}{\partial y^p}\right)(y)\\&=i^{|p|}4^k k!\sqrt{t}^{2k+|p|}L_k^{\frac{n-1}{2}}(ts^2)H_p(\sqrt{t}y)e^{-t(s^2+|y|^2)}.\end{aligned}$$

Therefore by relation (3.11), we get

$$\mathscr{F}(Rg_t)(s,y)=\sum_{\substack{(k,p)\in\mathbb{N}\times\mathbb{N}^n\\ k+|p|\leqslant m}} a_{kp}i^{|p|}4^k k!\sqrt{t}^{2k+|p|}L_k^{\frac{n-1}{2}}\left(t(s^2+|y|^2)\right)H_p(\sqrt{t}y)e^{-t(s^2+2|y|^2)}.$$

□

4. Miyachi's theorem associated with the spherical mean operator

According to relation (3.6), we establish the following lemma

LEMMA 4.1. *For every $b>0$, we have*

$$\forall(r,x)\in\mathbb{R}\times\mathbb{R}^n,\ {}^t\mathscr{R}\left(e^{-b|(.,.)|^2}\right)(r,x)=\frac{e^{-b(r^2+\frac{|x|^2}{2})}}{\left(2\sqrt{b}\right)^n}.\tag{4.1}$$

LEMMA 4.2. *Let h be an entire function on $\mathbb{C}\times\mathbb{C}^n$. If f satisfies*

$$|h(z_0,z)|\leqslant Ce^{B((Re\,z_0)^2+2(Re\,z)^2)},\quad \text{and}\quad \int_{\mathbb{R}\times\mathbb{R}^n}\log^+|h(r,x)|drdx<+\infty,$$

for some positive constants C and B, then h is constant.

Proof. The proof is identical to the proof given in [4]. □

LEMMA 4.3. *Let $p \in [1,+\infty]$ and let a be a positive real number. Then for every function $g \in L^p(d\nu_{n+1})$, the function $e^{a(.^2+\frac{|.|^2}{2})}\,{}^t\mathscr{R}\left(e^{-a|(.,.)|^2}g\right)$ belongs to the space $L^p(dm_{n+1})$, and there is a positive constant C, such that for every $g \in L^p(d\nu_{n+1})$, we have*

$$\left\| e^{a(.^2+\frac{|.|^2}{2})}\,{}^t\mathscr{R}\left(e^{-a|(.,.)|^2}g\right)\right\|_{p,m_{n+1}} \leqslant C\|g\|_{p,\nu_{n+1}}. \tag{4.2}$$

Proof. • If $p \in]1,+\infty[$. Let q be the conjugate component of p, then by relations (3.6), (3.7), (3.8) and (4.1) and by using Hölder's inequality we get

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}^n} e^{ap(r^2+\frac{|x|^2}{2})}\left|{}^t\mathscr{R}(e^{-a|(.,.)|^2}g)(r,x)\right|^p dm_{n+1}(r,x)\\
&= \frac{1}{(2\pi)^{\frac{np}{2}}}\int_0^{+\infty}\int_{\mathbb{R}^n} e^{ap(r^2+\frac{|x|^2}{2})}\left|\int_{\mathbb{R}^n} e^{-a(r^2+|x-y|^2+|y|^2)}g(\sqrt{r^2+|x-y|^2},y)dy\right|^p dm_{n+1}(r,x)\\
&\leqslant \frac{1}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\int_{\mathbb{R}^n} e^{ap(r^2+\frac{|x|^2}{2})}\left(\int_{\mathbb{R}^n}|g(\sqrt{r^2+|x-y|^2},y)|^p dy\right)\\
&\quad\times\left({}^t\mathscr{R}\left(e^{-aq|(.,.)|^2}\right)(r,x)\right)^{\frac{p}{q}} dm_{n+1}(r,x)\\
&= \frac{1}{2^{n(p-\frac{1}{2})}\pi^{\frac{n}{2}}(aq)^{\frac{np}{2q}}}\int_0^{+\infty}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|g(\sqrt{r^2+|x-y|^2},y)|^p dy\right)dm_{n+1}(r,x)
\end{aligned}$$

and therefore

$$\begin{aligned}
\int_0^{+\infty}\int_{\mathbb{R}^n} e^{ap(r^2+\frac{|x|^2}{2})}\left|{}^t\mathscr{R}(e^{-a|(.,.)|^2}g)(r,x)\right|^p dm_{n+1}(r,x) &= \frac{\|{}^t\mathscr{R}(|g|^p)\|_{1,m_{n+1}}}{2^{n(p-1)}(aq)^{\frac{np}{2q}}}\\
&\leqslant \frac{\|g\|^p_{p,\nu_{n+1}}}{2^{n(p-1)}(aq)^{\frac{np}{2q}}}.
\end{aligned}$$

Hence, the function $e^{a(.^2+\frac{|.|^2}{2})}\,{}^t\mathscr{R}\left(e^{-a|(.,.)|^2}g\right)$ belongs to the space $L^p(dm_{n+1})$, and we have

$$\left\| e^{a(.^2+\frac{|.|^2}{2})}\,{}^t\mathscr{R}\left(e^{-a|(.,.)|^2}g\right)\right\|_{p,m_{n+1}} \leqslant \frac{1}{(4aq)^{\frac{n}{2q}}}\|g\|_{p,\nu_{n+1}}.$$

• If $p = 1$, then by the same way, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}^n} e^{a(r^2+\frac{|x|^2}{2})}\left|{}^t\mathscr{R}(e^{-a|(.,.)|^2}g)(r,x)\right| dm_{n+1}(r,x)\\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} e^{-a|y\sqrt{2}-\frac{x}{\sqrt{2}}|^2}g(\sqrt{r^2+|x-y|^2},y)dy\right| dm_{n+1}(r,x)\\
&\leqslant \frac{1}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|g(\sqrt{r^2+|x-y|^2},y)\right| dy\right)dm_{n+1}(r,x)\\
&= \left\|{}^t\mathscr{R}(|g|)(r,x)\right\|_{1,m_{n+1}} \leqslant \|g\|_{1,\nu_{n+1}}.
\end{aligned}$$

Hence,

$$\left\|e^{a(.^2+\frac{|.|^2}{2})\,t}\mathscr{R}\left(e^{-a|(.,.)|^2}g\right)\right\|_{1,m_{n+1}} \leqslant \|g\|_{1,\nu_{n+1}}.$$

- If $p=+\infty$, then for every $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, we have

$$\begin{aligned}
&\left|e^{a(r^2+\frac{|x|^2}{2})\,t}\mathscr{R}\left(e^{-a|(.,.)|^2}g\right)(r,x)\right|\\
&=\frac{e^{a(r^2+\frac{|x|^2}{2})}}{(2\pi)^{\frac{n}{2}}}\left|\int_{\mathbb{R}^n}e^{-a(r^2+|x-y|^2+|y|^2)}g(\sqrt{r^2+|x-y|^2},y)dy\right|\\
&\leqslant \frac{\|g\|_{\infty,\nu_{n+1}}}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-a|y\sqrt{2}-\frac{x}{\sqrt{2}}|^2}dy=\frac{\|g\|_{\infty,\nu_{n+1}}}{(2\sqrt{a})^n}.
\end{aligned}$$

Hence,

$$\left\|e^{a(.^2+\frac{|.|^2}{2})\,t}\mathscr{R}\left(e^{-a|(.,.)|^2}g\right)\right\|_{\infty,m_{n+1}} \leqslant \frac{\|g\|_{\infty,\nu_{n+1}}}{(2\sqrt{a})^n}. \qquad \square$$

LEMMA 4.4. *Let $p,q\in[1,+\infty]$, and let a be a positive real number. If f is a measurable function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable satisfying*

$$e^{a|(.,.)|^2}f\in L^p(d\nu_{n+1})+L^q(d\nu_{n+1}),$$

then $\mathscr{F}(f)$ is well defined and entire on $\mathbb{C}\times\mathbb{C}^n$. Moreover, there is a positive constant C such that, for every $(s,y)\in\mathbb{C}\times\mathbb{C}^n$, we have

$$|\mathscr{F}(f)(s,y)|\leqslant Ce^{\frac{|Ims|^2+2|Imy|^2}{4a}}. \tag{4.3}$$

Proof. Let $f_1\in L^p(d\nu_{n+1})$ and $f_2\in L^q(d\nu_{n+1})$ such that

$$f=e^{-a|(.,.)|^2}f_1+e^{-a|(.,.)|^2}f_2,$$

and let p' and q' be respectively the conjugate components of p and q, then by Hölder's inequality, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}^n}|f(r,x)|d\nu_{n+1}(r,x)\\
&\leqslant \int_0^{+\infty}\int_{\mathbb{R}^n}e^{-a|(r,x)|^2}(|f_1(r,x)|+|f_2(r,x)|)d\nu_{n+1}(r,x)\\
&\leqslant \left(\int_0^{+\infty}\int_{\mathbb{R}^n}e^{-ap'|(r,x)|^2}d\nu_{n+1}(r,x)\right)^{\frac{1}{p'}}\|f_1\|_{p,\nu_{n+1}}\\
&\quad+\left(\int_0^{+\infty}\int_{\mathbb{R}^n}e^{-aq'|(r,x)|^2}d\nu_{n+1}(r,x)\right)^{\frac{1}{q'}}\|f_2\|_{q,\nu_{n+1}}\\
&<+\infty.
\end{aligned}$$

Then, the function f belongs to the space $L^1(d\nu_{n+1})$ and $\mathscr{F}(f)$ is well defined.

According to relation (2.1), we know that for every $(r,x) \in \mathbb{R}\times\mathbb{R}^n$, the function $(s,y) \longmapsto j_{\frac{n-1}{2}}(rs)e^{-i\langle y|x\rangle}$ is entire on $\mathbb{C}\times\mathbb{C}^n$. Let K be a compact set of $\mathbb{C}\times\mathbb{C}^n$ and let b be a positive real number such that for every $(s,y)\in K$, we have $|(s,y)|\leqslant b$. Then for every $(s,y)\in K$ and by using relation (2.2), we have

$$\begin{aligned}\left|f(r,x)j_{\frac{n-1}{2}}(rs)e^{-i\langle x|y\rangle}\right| &\leqslant e^{|rIms|+|xImy|}|f(r,x)|\\ &\leqslant e^{-\frac{a}{2}(r^2+|x|^2)}e^{\frac{1}{2a}(|Ims|^2+|Imy|^2)}\left(|f_1(r,x)|+|f_2(r,x)|\right)\\ &\leqslant Ce^{-\frac{a}{2}(r^2+|x|^2)}\left(|f_1(r,x)|+|f_2(r,x)|\right).\end{aligned}$$

Since, the function $(r,x)\longmapsto e^{-\frac{a}{2}(r^2+|x|^2)}\left(|f_1(r,x)|+|f_2(r,x)|\right)$ is integrable over $\mathbb{R}\times\mathbb{R}^n$ with respect to the measure $d\nu_{n+1}$, this shows that $\widetilde{\mathscr{F}}(f)$ is analytic on $\mathbb{C}\times\mathbb{C}^n$, moreover since $\widetilde{\mathscr{F}}(f)$ is even with respect to the first variable then $\mathscr{F}(f)$ is also analytic on $\mathbb{C}\times\mathbb{C}^n$.

Let $(s,y)\in\mathbb{C}\times\mathbb{C}^n$, then by relations (3.1), (3.8) and (3.9), we have

$$\begin{aligned}|\mathscr{F}(f)(s,y)| &= \left|\int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\mathscr{R}\left(\cos(s.)e^{-i\langle y\,|\,.\rangle}\right)(r,x)d\nu_{n+1}(r,x)\right|\\ &= \left|\int_0^{+\infty}\int_{\mathbb{R}^n} {}^t\mathscr{R}(f)(r,x)\cos(sr)e^{-i\langle y\,|\,x\rangle}dm_{n+1}(r,x)\right|\\ &\leqslant \int_0^{+\infty}\int_{\mathbb{R}^n}\left|{}^t\mathscr{R}(f)(r,x)\right|e^{r|Ims|}e^{|x||Imy|}dm_{n+1}(r,x)\end{aligned}$$

Hence by a basic calculus, we get

$$\begin{aligned}&|\mathscr{F}(f)(s,y)|\\ &\leqslant e^{\frac{|Ims|^2+2|Imy|^2}{4a}}\int_0^{+\infty}\int_{\mathbb{R}^n} e^{a(r^2+\frac{|x|^2}{2})}\left|{}^t\mathscr{R}(f)(r,x)\right|e^{-(\sqrt{a}r-\frac{|Ims|}{2\sqrt{a}})^2}e^{-(\frac{\sqrt{a}|x|}{\sqrt{2}}-\frac{|Imy|}{\sqrt{2a}})^2}dm_{n+1}(r,x)\\ &\leqslant \sum_{i=1}^{2}e^{\frac{|Ims|^2+2|Imy|^2}{4a}}\int_0^{+\infty}\int_{\mathbb{R}^n} e^{a(r^2+\frac{|x|^2}{2})}\left|{}^t\mathscr{R}\left(e^{-a|(.,.)|^2}f_i\right)(r,x)\right|\\ &\quad\times e^{-(\sqrt{a}r-\frac{|Ims|}{2\sqrt{a}})^2}e^{-(\frac{\sqrt{a}|x|}{\sqrt{2}}-\frac{|Imy|}{\sqrt{2a}})^2}dm_{n+1}(r,x)\end{aligned}$$

And finally by relation (4.2), we deduce that for every $(s,y)\in\mathbb{C}\times\mathbb{C}^n$,

$$\begin{aligned}|\mathscr{F}(f)(s,y)| &\leqslant Ce^{\frac{|Ims|^2+2|Imy|^2}{4a}}\left(\|f_1\|_{p,\nu_{n+1}}+\|f_2\|_{q,\nu_{n+1}}\right)\\ &\leqslant C'e^{\frac{|Ims|^2+2|Imy|^2}{4a}}. \quad\square\end{aligned}$$

THEOREM 4.5. (Miyachi for $\mathscr{F}$) *Let a,b and δ be positive real numbers and let $p,q\in[1,+\infty]$. Let f be a measurable function on $\mathbb{R}\times\mathbb{R}^n$ even with respect to the first variable satisfying*

$$e^{a|(.,.)|^2}f\in L^p(d\nu_{n+1})+L^q(d\nu_{n+1}), \tag{4.4}$$

and

$$\int_{\mathbb{R}\times\mathbb{R}^n} \log^+ \frac{|\mathscr{F}(f)(r,x)|e^{b(r^2+2|x|^2)}}{\delta} drdx < +\infty. \tag{4.5}$$

Then

i) If $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.

ii) If $ab = \frac{1}{4}$, then $f = Cg_b$ with $|C| \leqslant \delta$, where g_b is the Gauss kernel given in definition 3.6.

iii) If $ab < \frac{1}{4}$, then for every $\sigma \in]b, \dfrac{1}{4a}[$, and for every polynomial function R on $\mathbb{R}\times\mathbb{R}^n$ even with respect to the first variable, the function $f = Rg_\sigma$ satisfies relations (4.4) and (4.5).

Proof. $i)$ Suppose that $ab > \frac{1}{4}$ and let h be the function defined on $\mathbb{C}\times\mathbb{C}^n$, by

$$h(z_0,z) = e^{\frac{z_0^2}{4a}} \left(\prod_{i=1}^{n} e^{\frac{z_i^2}{2a}}\right) \mathscr{F}(f)(z_0,z),$$

then by Lemma 4.4, it follows that h is analytic on $\mathbb{C}\times\mathbb{C}^n$, and by relation (4.3), we deduce that there is a positive constant C such that

$$\forall (z_0,z) \in \mathbb{C}\times\mathbb{C}^n,\ |h(z_0,z)| \leqslant Ce^{\frac{(Re\,z_0)^2+2|Re\,z|^2}{4a}}.$$

Now, by using the fact that for every positive real numbers x,y, we have $\log^+(xy) < \log^+(x) + y$, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}^n} \log^+ |h(r,x)|\, drdx \\
&= \int_{\mathbb{R}\times\mathbb{R}^n} \log^+ |\mathscr{F}(f)(r,x)| e^{\frac{r^2+2|x|^2}{4a}} drdx \\
&= \int_{\mathbb{R}\times\mathbb{R}^n} \log^+ \left(\frac{|\mathscr{F}(f)(r,x)|e^{b(r^2+2|x|^2)}}{\delta} e^{(\frac{1}{4a}-b)(r^2+2|x|^2)} \delta \right) drdx \\
&\leqslant \int_{\mathbb{R}\times\mathbb{R}^n} \log^+ \frac{|\mathscr{F}(f)(r,x)|e^{b(r^2+2|x|^2)}}{\delta} drdx + \delta \int_{\mathbb{R}\times\mathbb{R}^n} e^{(\frac{1}{4a}-b)(r^2+2|x|^2)} drdx
\end{aligned}$$

Since $ab > \frac{1}{4}$, then $\displaystyle\int_{\mathbb{R}\times\mathbb{R}^n} \log^+ |h(r,x)|\, drdx < +\infty$, and therefore h satisfies the assumptions of Lemma 4.2. Hence, h is constant on $\mathbb{C}\times\mathbb{C}^n$, which means that there is a constant C such that

$$\forall (r,x) \in \mathbb{R}\times\mathbb{R}^n,\ \mathscr{F}(f)(r,x) = Ce^{-\frac{r^2+2|x|^2}{4a}},$$

but in this case, relation (4.5) holds only whenever $C = 0$, and the result follows then by Theorem 3.3.

ii) Suppose that $ab = \frac{1}{4}$, then as showed in the previous case, there is a constant C such that

$$\forall (r,x) \in \mathbb{R} \times \mathbb{R}^n, \ \mathscr{F}(f)(r,x) = Ce^{-b(r^2+2|x|^2)},$$

and similarly, relation (4.5) holds only whenever $|C| \leqslant \delta$, and by combining relation (3.12) with Theorem 3.3 we get the desired result.

iii) Finally suppose that $ab < \frac{1}{4}$. Let $\sigma \in]b, \frac{1}{4a}[$ and suppose that $f = Rg_\sigma$ for some polynomial function $R \in \mathbb{C}_e[X^{n+1}]$, then by Proposition 3.7, there is a polynomial function $Q \in \mathbb{C}_e[X^{n+1}]$, such that

$$\mathscr{F}(Rg_\sigma) = Qe^{-\sigma(r^2+2|x|^2)},$$

and it is easy to verify, that in this case, f and $\mathscr{F}(f)$ satisfy relations (4.4) and (4.5). $\square$

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