Journal of
Mathematical
Inequalities
Volume 8, Number 3 (2014), 489–495 doi:10.7153/jmi-08-36

SOME REMARKS ON (s,m)–CONVEXITY IN THE SECOND SENSE

NOHA EFTEKHARI

(*Communicated by G. Toader*)

Abstract. The aim of this work is to establish several inequalities for functions whose first derivative in absolute value are (s,m)-convex in the second sense. Some estimates to the left hand side of the Hermite-Hadamard type inequality for (s,m)-convex functions in the second sense are given.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a convex function, the following double inequality is well known as the Hermite-Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leqslant\frac{f(a)+f(b)}{2}.$$

In 1984 [5], G. Toader defined the class of m-convex functions. This class is defined in the following way: a function $f:[0,b]\to\mathbb{R}$ is said to be m-convex, where $m\in[0,1]$, if for every $x,y\in[0,b]$ and $t\in[0,1]$ we have

$$f(tx+m(1-t)y)\leqslant tf(x)+m(1-t)f(y).$$

In 1993 [4], V. Mihesan introduced the class of (α,m)-convex functions as the following: a function $f:[0,b]\to\mathbb{R}$ is said to be (α,m)-convex, where $(\alpha,m)\in[0,1]^2$, if for every $x,y\in[0,b]$ and $t\in[0,1]$ we have

$$f(tx+m(1-t)y)\leqslant t^\alpha f(x)+m(1-t^\alpha)f(y).$$

For recent results and generalizations and new inequalities concerning (α,m)-convex functions, see [2, 6].

In 1978 [3], W. W. Breckner introduced the class of s-convex functions in the second sense, in the following way: a function $f:[0,\infty)\to\mathbb{R}$ is said to be s-convex in the second sense if for all $x,y\in[0,\infty)$, $t\in[0,1]$ and for some fixed $s\in(0,1]$, we have

$$f(tx+(1-t)y)\leqslant t^s f(x)+(1-t)^s f(y).$$

Now, we combine two definitions of m-convexity and s-convexity in the second sense and obtain the class of (s,m)-convex functions in the second sense as the following.

Mathematics subject classification (2010): 26D10, 39B62.
Keywords and phrases: Hermite-Hadamard inequality, (s,m)-convex function in the second sense, convexity, Hölder's inequality.

DEFINITION 1.1. A function $f:[0,b]\to\mathbb{R}$ is said to be (s,m)-convex in the second sense, where $(s,m)\in(0,1]^2$, if for every $x,y\in[0,b]$ and $t\in[0,1]$ we have

$$f(tx+m(1-t)y)\leqslant t^s f(x)+m(1-t)^s f(y).$$

The aim of this work is to establish some upper bounds to the left hand side of the Hermite-Hadamard type inequality for (s,m)-convex functions in the second sense.

2. Main results

In order to prove new theorems, we need Lemma 1 in [1] by setting $x=\frac{a+b}{2}$, which implies the following lemma.

LEMMA 2.1. *Let $f:I\to\mathbb{R}$, $I\subset\mathbb{R}$ be a differentiable function on I°. If $f'\in L^1[a,b]$, where $a,b\in I$ with $a<b$, then the following equality holds:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\\ =&\frac{b-a}{4}\int_0^1 t\left[f'\left(t\frac{a+b}{2}+(1-t)a\right)-f'\left(t\frac{a+b}{2}+(1-t)b\right)\right]\mathrm{d}t.\end{aligned}$$

The following results may be stated:

THEOREM 2.1. *Let $f:[a,b]\to\mathbb{R}$, $[a,b]\subset[0,\infty)$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$. If $|f'|$ is (s,m)-convex in the second sense on $[a,b]$ for $(s,m)\in(0,1]^2$, then the following inequality holds:*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\ \leqslant&\frac{b-a}{4(s+2)}\left[2\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{m}{s+1}\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right].\end{aligned}$$

Proof. By Lemma 2.1, it follows that

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\ \leqslant&\frac{b-a}{4}\int_0^1 t\left[\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|+\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right]\mathrm{d}t.\end{aligned}\tag{1}$$

Since $|f'|$ is (s,m)-convex in the second sense on $[a,b]$, for any $t\in[0,1]$, we obtain

$$\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|\leqslant t^s\left|f'\left(\frac{a+b}{2}\right)\right|+m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|,\tag{2}$$

$$\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\leqslant t^s\left|f'\left(\frac{a+b}{2}\right)\right|+m(1-t)^s\left|f'\left(\frac{b}{m}\right)\right|.\tag{3}$$

Now (1), (2) and (3) imply

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
\leqslant&\frac{b-a}{4}\int_0^1\left[2t^{s+1}\left|f'\left(\frac{a+b}{2}\right)\right|+mt(1-t)^s\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right]\mathrm{d}t \\
=&\left(\frac{b-a}{4}\right)\left[\frac{2}{s+2}\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{m}{(s+1)(s+2)}\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right] \\
=&\left(\frac{b-a}{4(s+2)}\right)\left[2\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{m}{s+1}\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right]. \quad \square
\end{aligned}$$

THEOREM 2.2. *Suppose that all the assumptions of Theorem 2.1 are satisfied. Then the following inequality holds:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \\
\leqslant&\frac{b-a}{2^{s+2}(s+2)}\left[|f'(a)|+|f'(b)|+\frac{m(2^{s+2}-s+3)}{s+1}\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right].
\end{aligned}$$

Proof. Since $t(\frac{a+b}{2})+(1-t)a=\frac{t}{2}b+(1-\frac{t}{2})a$ and $|f'|$ is (s,m)-convex in the second sense on $[a,b]$, then for any $t\in[0,1]$, one can get the following inequalities:

$$\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|\leqslant\left(\frac{t}{2}\right)^s|f'(b)|+m\left(1-\frac{t}{2}\right)^s\left|f'\left(\frac{a}{m}\right)\right|, \tag{4}$$

$$\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\leqslant\left(\frac{t}{2}\right)^s|f'(a)|+m\left(1-\frac{t}{2}\right)^s\left|f'\left(\frac{b}{m}\right)\right|, \tag{5}$$

so (1), (4) and (5) imply

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \\
\leqslant&\frac{b-a}{4}\int_0^1\left[\frac{t^{s+1}}{2^s}(|f'(a)|+|f'(b)|)+mt\left(1-\frac{t}{2}\right)^s\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right]\mathrm{d}t \\
=&\frac{b-a}{2^{s+2}(s+2)}\left[|f'(a)|+|f'(b)|+\frac{m(2^{s+2}-s+3)}{s+1}\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right]. \quad \square
\end{aligned}$$

THEOREM 2.3. *Let $f:[a,b]\to\mathbb{R}$, $[a,b]\subset[0,\infty)$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$. If $|f'|^q$ is (s,m)-convex in the second sense on $[a,b]$ for*

$(s,m) \in (0,1]^2$ *and* $q > 1$, *then the following inequalities hold:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right|$$
$$\leqslant \left(\frac{b-a}{4}\right)\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[2\left|f'\left(\frac{a+b}{2}\right)\right| + m^{\frac{1}{q}}\left(\left|f'\left(\frac{a}{m}\right)\right| + \left|f'\left(\frac{b}{m}\right)\right|\right)\right], \tag{6}$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right|$$
$$\leqslant \left(\frac{b-a}{8}\right)\left(\frac{2}{s+2}\right)^{\frac{1}{q}}\left[2\left|f'\left(\frac{a+b}{2}\right)\right| + \left(\frac{m}{s+1}\right)^{\frac{1}{q}}\left(\left|f'\left(\frac{a}{m}\right)\right| + \left|f'\left(\frac{b}{m}\right)\right|\right)\right]. \tag{7}$$

Proof. By applying (2), (3) for $|f'|^q$ and by using the Hölder's inequality for $q > 1$ and $p = \frac{q}{q-1}$, it follows that

$$\int_0^1 t\left|f'\left(t\left(\frac{a+b}{2}\right) + (1-t)a\right)\right|\mathrm{d}t$$
$$\leqslant \left(\int_0^1 t^p \mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(t\left(\frac{a+b}{2}\right) + (1-t)a\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}$$
$$\leqslant \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\int_0^1 \left(t^s\left|f'\left(\frac{a+b}{2}\right)\right|^q + m(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}$$
$$= \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + m\left|f'\left(\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}, \tag{8}$$

and similarly

$$\int_0^1 t\left|f'\left(t\left(\frac{a+b}{2}\right) + (1-t)b\right)\right|\mathrm{d}t$$
$$\leqslant \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}. \tag{9}$$

Therefore, the inequalities (1), (8), (9) and the following relation imply (6),

$$\sum_{i=1}^n (a_i + b_i)^r \leqslant \sum_{i=1}^n a_i^r + \sum_{i=1}^n b_i^r, \tag{10}$$

for $0 < r < 1$, $a_1, a_2, \ldots, a_n \geqslant 0$ and $b_1, b_2, \ldots, b_n \geqslant 0$.

Now, if we use the Hölder's inequality in the following way and by applying the inequalities (1), (10), then it follows (7),

$$\int_0^1 t\left|f'\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|\mathrm{d}t=\int_0^1 t^{1-\frac{1}{q}}\,t^{\frac{1}{q}}\,\left|f'\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|\mathrm{d}t$$

$$\leqslant\left(\int_0^1 t\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leqslant\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\int_0^1\left(t^{s+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+mt(1-t)^s\left|f'\left(\frac{a}{m}\right)\right|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}$$

$$=\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left(\frac{1}{s+2}\right)^{\frac{1}{q}}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{m}{s+1}\left|f'\left(\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}$$

$$\leqslant\left(\frac{1}{2}\right)\left(\frac{2}{s+2}\right)^{\frac{1}{q}}\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\left(\frac{m}{s+1}\right)^{\frac{1}{q}}\left|f'\left(\frac{a}{m}\right)\right|\right].\quad\square$$

THEOREM 2.4. *Suppose that all the assumptions of Theorem 2.3 are satisfied. Then the following inequalities hold:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right|\leqslant\left(\frac{b-a}{4}\right)\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{q}}$$
$$\times\left[|f'(a)|+|f'(b)|+(m(2^{s+1}-1))^{\frac{1}{q}}\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right],\tag{11}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right|\leqslant\left(\frac{b-a}{8}\right)\left(\frac{2}{s+2}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{q}}$$
$$\times\left[|f'(a)|+|f'(b)|+\left(\frac{m(2^{s+2}-s+3)}{s+1}\right)^{\frac{1}{q}}\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right].\tag{12}$$

Proof. By applying (4), (5) for $|f'|^q$ and by using the Hölder's inequality for $q>1$ and $p=\frac{q}{q-1}$, we obtain

$$\int_0^1 t\left|f'\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|\mathrm{d}t$$

$$\leqslant\left(\int_0^1 t^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leqslant\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\int_0^1\left(\left(\frac{t}{2}\right)^s|f'(b)|^q+m\left(1-\frac{t}{2}\right)^s\left|f'\left(\frac{a}{m}\right)\right|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}$$

$$=\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{q}}\left[\frac{|f'(b)|^q}{s+1}+m\frac{2^{s+1}-1}{s+1}\left|f'\left(\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}, \tag{13}$$

and similarly

$$\int_0^1 t\left|f'\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\right|\mathrm{d}t$$

$$\leqslant\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{q}}\left[\frac{|f'(a)|^q}{s+1}+m\frac{2^{s+1}-1}{s+1}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}. \tag{14}$$

Therefore, the inequalities (1), (10),(13), (14) imply (11).

Now, the inequalities (1), (10) and (4), (5) for $|f'|^q$ and by using the Hölder's inequality in the following way, imply (12)

$$\int_0^1 t\left|f'\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|\mathrm{d}t=\int_0^1 t^{1-\frac{1}{q}}\,t^{\frac{1}{q}}\left|f'\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|\mathrm{d}t$$

$$\leqslant\left(\int_0^1 t\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leqslant\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\int_0^1\left(t\left(\frac{t}{2}\right)^s|f'(b)|^q+mt\left(1-\frac{t}{2}\right)^s\left|f'\left(\frac{a}{m}\right)\right|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}$$

$$=\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{q}}\left[\frac{|f'(b)|^q}{s+2}+\frac{m(2^{s+2}-s+3)}{(s+1)(s+2)}\left|f'\left(\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}$$

$$\leqslant\left(\frac{1}{2}\right)^{\frac{s+q-1}{q}}\left(\frac{1}{s+2}\right)^{\frac{1}{q}}\left[|f'(b)|+\left(\frac{m(2^{s+2}-s+3)}{s+1}\right)^{\frac{1}{q}}\left|f'\left(\frac{a}{m}\right)\right|\right].\quad\square$$

3. Applications to special means

We consider the means for arbitrary positive numbers $\alpha,\beta\ (\alpha\neq\beta)$ as follows:

(1) The arithmetic mean: $A(\alpha,\beta)=\dfrac{\alpha+\beta}{2}$,

(2) The generalized log-mean:

$$L_p(\alpha,\beta)=\left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}},\qquad p\in\mathbb{R}\setminus\{-1,0\}.$$

Now by using the results of Section 2, we give some applications to special means of real numbers.

PROPOSITION 3.1. *Let $a,b \in (0,\infty)$, $a<b$ and $s \in (0,1)$. Then it follows that*

$$|A^s(a,b) - L_s^s(a,b)| \leqslant \frac{(b-a)s}{2(s+2)}\left[\frac{1}{s+1}A(a^{s-1},b^{s-1}) + A^{s-1}(a,b)\right],$$

and

$$|A^s(a,b) - L_s^s(a,b)| \leqslant \frac{(b-a)(2^s+1)s}{2^{s-1}(s+1)(s+2)}A(a^{s-1},b^{s-1}).$$

Proof. The assertions follow from Theorem 2.1 and Theorem 2.2, for $m=1$ applied to the s-convex function $f(x)=x^s$.

PROPOSITION 3.2. *Let $a,b \in (0,\infty)$, $a<b$ and $s \in (0,1)$. Then for all $q>1$, it follows that*

$$|A^s(a,b) - L_s^s(a,b)| \leqslant s\left(\frac{b-a}{4}\right)\left(\frac{2}{s+2}\right)^{\frac{1}{q}}\left[A^{s-1}(a,b) + \left(\frac{1}{s+1}\right)^{\frac{1}{q}} A(a^{s-1},b^{s-1})\right].$$

Proof. The assertion follow from Theorem 2.3, for $m=1$ applied to the s-convex function $f(x)=x^s$.

Acknowledgements. The author thanks the editor and the referees for their comments.

REFERENCES

[1] M. ALOMARI, M. DARUS, S. S. DRAGOMIR AND P. CERONE, *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense*, Appl. Math. Lett., **23**, (2010), 1071–1076.

[2] M. K. BAKULA, M. E. ÖZDEMIR AND J. PEČARIĆ, *Hadamard type inequalities for m-convex and (α,m)-convex functions*, J. Inequal. Pure Appl. Math., **9**, (2008), Article 96.

[3] W. W. BRECKNER, *Stetigkeitsaussagen für eine klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen*, Publ. Inst. Math. (Beograd), **23**, (1978), 13–20.

[4] V. MIHESAN, *A generalization of the convexity*, Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca, Romania, 1993.

[5] G. TOADER, *Some generalisations of the convexity*, Proc. Colloq. Approx. Optim., (1984), 329–338.

[6] G. TOADER, *The hierarchy of convexity and some classic inequalities*, J. Math. Ineq., **3**, 3 (2009), 305–313.

(Received November 22, 2012)

Noha Eftekhari
Shahrekord University
Faculty of Mathematical sciences
P. O. Box 115, Shahrekord
88186-34141, Iran
e-mail: eftekharinoha@yahoo.com