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MORE ON SOME HARDY TYPE INTEGRAL INEQUALITIES

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Abstract. In 2012, W. T. Sulaiman presented new kinds of Hardy's integral inequalities. In this paper, we derive some new extensions of the famous Hardy's integral inequality. The results present direct generalization of the original Hardy inequality. In addition, the corresponding reverse relation is also obtained.

1. Introduction

In 1920, Hardy [1] presented the following inequality.

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx,$$

where $f \geqslant 0$, $p > 1$, and

$$F(x) = \int_0^x f(t)dt.$$

The constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. This inequality is important in mathematical analysis and its applications.

In 1964, Levinson [2] presented the following inequality.

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_a^b f^p(x)dx,$$

where $0 < a < b < \infty$, $f \geqslant 0$, $p > 1$, and

$$F(x) = \int_0^x f(t)dt.$$

In 2012, W. T. Sulaiman [3] extended the Hardy's integral inequality as follows. If $f \geqslant 0$, $g > 0$, $\frac{x}{g(x)}$ is non-increasing, $p > 1$, $0 < a < 1$ and

$$F(x) = \int_0^x f(t)dt,$$

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then

$$\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx \leqslant \frac{1}{a(p-1)(1-a)^{p-1}} \int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^p dx. \tag{1}$$

Moreover, he [3] presented the reverse inequality as follows. If $f \geqslant 0,\ g > 0,\ \frac{x}{g(x)}$ is non-decreasing, $0 < p < 1$, $a > 0$ and

$$F(x) = \int_0^x f(t)dt,$$

then

$$\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx \geqslant \frac{1}{a(1-p)(1+a)^{p-1}} \int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^p dx. \tag{2}$$

In this paper, we present direct generalization of the original Hardy inequality. In addition, the corresponding reverse relation is also obtained.

2. Main results

In this section, we generalize the integral inequality (1) and the reverse integral inequality (2) by given the value q.

THEOREM 2.1. *Let $f \geqslant 0,\ g > 0,\ 0 < a < 1,\ p > 1,\ q > p - a(p-1)$ and*

$$F(x) = \int_0^x f(t)dt.$$

Assume that $\frac{x}{g(x)}$ is a non-increasing function. Then

$$\int_0^\infty \frac{F^p(x)}{g^q(x)} dx \leqslant \frac{1}{((a-1)(p-1)+q-1)(1-a)^{p-1}} \int_0^\infty \frac{(xf(x))^p}{g^q(x)} dx. \tag{3}$$

Proof. By the Hölder inequality and the assumption of the function $\frac{x}{g(x)}$, we obtain that

$$\begin{aligned}
\int_0^\infty \frac{F^p(x)}{g^q(x)} dx &= \int_0^\infty g^{-q}(x) \left(\int_0^x t^{\frac{-a(p-1)}{p}} t^{\frac{a(p-1)}{p}} f(t)dt\right)^p dx \\
&\leqslant \int_0^\infty g^{-q}(x) \left(\left(\int_0^x t^{-a}dt\right)^{\frac{p-1}{p}} \left(\int_0^x t^{a(p-1)} f^p(t)dt\right)^{\frac{1}{p}}\right)^p dx \\
&= \int_0^\infty g^{-q}(x) \left(\int_0^x t^{-a}dt\right)^{p-1} \int_0^x t^{a(p-1)} f^p(t)dtdx \\
&= \frac{1}{(1-a)^{p-1}} \int_0^\infty x^{(1-a)(p-1)} g^{-q}(x) \int_0^x t^{a(p-1)} f^p(t)dtdx \\
&= \frac{1}{(1-a)^{p-1}} \int_0^\infty t^{a(p-1)} f^p(t) \int_t^\infty x^{(1-a)(p-1)} g^{-q}(x)dxdt
\end{aligned}$$

$$\leqslant \frac{1}{(1-a)^{p-1}}\int_0^\infty t^{a(p-1)} f^p(t)\left(\frac{t}{g(t)}\right)^q \int_t^\infty x^{(1-a)(p-1)-q}dxdt$$
$$= \frac{1}{((a-1)(p-1)+q-1)(1-a)^{p-1}}\int_0^\infty \frac{(tf(t))^p}{g^q(t)}dt. \quad \square$$

THEOREM 2.2. *Let $f \geqslant 0,\ g>0,\ 0<p<1,\ a>0,\ q>p+a(p-1)$ and*

$$F(x)=\int_0^x f(t)dt.$$

Assume that $\frac{x}{g(x)}$ is a non-decreasing function. Then

$$\int_0^\infty \frac{F^p(x)}{g^q(x)}dx \geqslant \frac{1}{((1+a)(1-p)+q-1)(1+a)^{p-1}}\int_0^\infty \frac{(xf(x))^p}{g^q(x)}dx. \tag{4}$$

Proof. By the reverse Hölder inequality and the assumption of the function $\frac{x}{g(x)}$, we obtain that

$$\int_0^\infty \frac{F^p(x)}{g^q(x)}dx = \int_0^\infty g^{-q}(x)\left(\int_0^x t^{\frac{a(p-1)}{p}}t^{\frac{-a(p-1)}{p}}f(t)dt\right)^p dx$$
$$\geqslant \int_0^\infty g^{-q}(x)\left(\left(\int_0^x t^a dt\right)^{\frac{p-1}{p}}\left(\int_0^x t^{-a(p-1)}f^p(t)dt\right)^{\frac{1}{p}}\right)^p dx$$
$$= \int_0^\infty g^{-q}(x)\left(\int_0^x t^a dt\right)^{p-1}\int_0^x t^{-a(p-1)}f^p(t)dtdx$$
$$= \frac{1}{(1+a)^{p-1}}\int_0^\infty x^{(1+a)(p-1)}g^{-q}(x)\int_0^x t^{-a(p-1)}f^p(t)dtdx$$
$$= \frac{1}{(1+a)^{p-1}}\int_0^\infty t^{-a(p-1)}f^p(t)\int_t^\infty x^{(1+a)(p-1)}g^{-q}(x)dxdt$$
$$\geqslant \frac{1}{(1+a)^{p-1}}\int_0^\infty t^{-a(p-1)}f^p(t)\left(\frac{t}{g(t)}\right)^q\int_t^\infty x^{(1+a)(p-1)-q}dxdt$$
$$= \frac{1}{((1+a)(1-p)+q-1)(1+a)^{p-1}}\int_0^\infty \frac{(tf(t))^p}{g^q(t)}dt. \quad \square$$

3. Applications

COROLLARY 3.1. [3] *Let $f \geqslant 0,\ g>0,\ p>1,\ 0<a<1$ and*

$$F(x)=\int_0^x f(t)dt.$$

Assume that $\frac{x}{g(x)}$ is a non-increasing function. Then

$$\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx \leqslant \frac{1}{a(p-1)(1-a)^{p-1}}\int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^p dx.$$

Proof. This follows from Theorem 2.1 where $q = p$. $\square$

COROLLARY 3.2. [3] *Let* $f \geqslant 0$, $g > 0$, $a > 0$, $0 < p < 1$ *and*

$$F(x) = \int_0^x f(t)dt.$$

Assume that $\frac{x}{g(x)}$ *is a non-decreasing function. Then*

$$\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx \geqslant \frac{1}{a(1-p)(1+a)^{p-1}} \int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^p dx.$$

Proof. This follows from Theorem 2.2 where $q = p$. $\square$

COROLLARY 3.3. *Let* $f \geqslant 0$, $p > 1$, $0 < a < 1$, $q > p - a(p-1)$ *and*

$$F(x) = \int_0^x f(t)dt.$$

Then

$$\int_0^\infty \frac{F^p(x)}{x^q} dx \leqslant \frac{1}{((a-1)(p-1)+q-1)(1-a)^{p-1}} \int_0^\infty \frac{f^p(x)}{x^{q-p}} dx.$$

Proof. This follows from Theorem 2.1 where $g(x) = x$. $\square$

COROLLARY 3.4. *Let* $f \geqslant 0$, $a > 0$, $0 < p < 1$, $q > p + a(p-1)$ *and*

$$F(x) = \int_0^x f(t)dt.$$

Then

$$\int_0^\infty \frac{F^p(x)}{x^q} dx \geqslant \frac{1}{((1+a)(1-p)+q-1)(1+a)^{p-1}} \int_0^\infty \frac{f^p(x)}{x^{q-p}} dx.$$

Proof. This follows from Theorem 2.2 where $g(x) = x$. $\square$

4. Open problem

In this section, we pose a question that how to generalize the integral inequality (3) and the reverse integral inequality (4) if we replace

$$\int_0^\infty \frac{F^p(x)}{g^q(x)} dx$$

by

$$\int_0^\infty \frac{F^p(x)}{G^q(x)} dx,$$

where

$$G(x) = \int_0^x g(t)dt.$$

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