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SHARPENING OF THE INEQUALITIES OF SCHUR, EBERLEIN, KRESS AND HUANG, AND NEW LOCATION OF EIGENVALUES

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Abstract. Let $A=(a_{ij})$ be an $n\times n$ complex matrix with eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$. We determine the new upper bounds of $\sum_{j=1}^{n}|\lambda_j|^2$, which will sharpen Schur, Eberlein, Kress and Huang's inequalities. We also exhibit new methods to locate the eigenvalues of a given complex matrix, which are more exact than those existing in previous literature. Numerical examples are provided to show the effectiveness of our results.

1. Introduction

Let A be an $n\times n$ complex matrix with eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$, $C^{n\times n}$ stands for the set of all $n\times n$ complex matrices. For any $A\in C^{n\times n}$, we denote the conjugate transpose of A by A^*, the Euclidean norm of A by $\|A\|$, the trace of A by $\mathrm{tr}A$. And we write $[A,B]=AB-BA$.

The estimation of $\sum_{j=1}^{n}|\lambda_j|^2$ plays an important role in location of eigenvalues. In 1909, Schur [1] first put forward the following well-known inequality:

$$\sum_{j=1}^{n}|\lambda_j|^2\leqslant\psi_1=\|A\|^2. \tag{1.1}$$

In [3], Eberlein sharpened Schur's above inequality, especially for non-normal matrix, where he gave the following bound:

$$\sum_{j=1}^{n}|\lambda_j|^2\leqslant\psi_2=\|A\|^2-\frac{\|[A,A^*]\|^2}{6\|A\|^2}. \tag{1.2}$$

Kress et al. showed another different bound (see [4]):

$$\sum_{j=1}^{n}|\lambda_j|^2\leqslant\psi_3=(\|A\|^4-\frac{1}{2}\|[A,A^*]\|^2)^{\frac{1}{2}}. \tag{1.3}$$

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In [5], Huang and Wang obtained a tighter upper bound for $\sum_{j=1}^{n}|\lambda_j|^2$, that is

$$\sum_{j=1}^{n}|\lambda_j|^2 \leqslant \psi_4 = \left(\left(\|A\|^2 - \frac{|\mathrm{tr}A|^2}{n}\right)^2 - \frac{1}{2}\|[A,A^*]\|^2\right)^{\frac{1}{2}} + \frac{|\mathrm{tr}A|^2}{n}. \tag{1.4}$$

In this paper, we will continue with the topic of exploring upper bounds for $\sum_{j=1}^{n}|\lambda_j|^2$ and the localization of eigenvalues of a given matrix, but the difference is the process of dealing with the problems. The paper is structured as follows. In Section 2, a new upper bound for $\sum_{j=1}^{n}|\lambda_j|^2$ is provided, which is more precise than ψ_4. We also derive five new determinant inequalities. Section 3 is aimed at exploring new methods to locate eigenvalues of a given matrix. We prove that all eigenvalues of a given complex matrix can be located in only one closed disk, which are more precise than those existing in [1, 5–7]. Furthermore, we use rectangle regions to locate all eigenvalues of a given complex matrix. The paper ends with several numerical examples, and they will show the validity of our results in Section 4.

2. New upper bounds for $\sum_{j=1}^{n}|\lambda_j|^2$ and determinant inequalities

THEOREM 2.1. *Let $M \in \mathrm{C}^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$,*

$$M = \begin{bmatrix} A_{k\times k} & B_{k\times(n-k)} \\ C_{(n-k)\times k} & D_{(n-k)\times(n-k)} \end{bmatrix} \quad \textit{and} \quad M(x) = \begin{bmatrix} A_{k\times k} & xB_{k\times(n-k)} \\ x^{-1}C_{(n-k)\times k} & D_{(n-k)\times(n-k)} \end{bmatrix},$$

where $A_{k\times k}$ is $k\times k$ principal submatrix of M $(1 \leqslant k \leqslant n-1)$ and x is one of non-zero real numbers. Then

$$\sum_{j=1}^{n}|\lambda_j|^2 \leqslant \min_{x\neq 0}\min_{1\leqslant k\leqslant n-1} f_M(k,x), \tag{2.1}$$

where

$$f_M(k,x) = \left((\Delta_M(k,x))^2 - \frac{1}{2}\|[M(x),M(x)^*]\|^2\right)^{\frac{1}{2}} + \frac{|\mathrm{tr}M|^2}{n}, \tag{2.2}$$

$$\Delta_M(k,x) = \|M\|^2 - \left((1-x^2)\|B_{k\times(n-k)}\|^2 + (1-x^{-2})\|C_{(n-k)\times k}\|^2\right) - \frac{|\mathrm{tr}M|^2}{n}. \tag{2.3}$$

Proof. Since

$$\begin{aligned} M(x) &= \begin{bmatrix} A_{k\times k} & xB_{k\times(n-k)} \\ x^{-1}C_{(n-k)\times k} & D_{(n-k)\times(n-k)} \end{bmatrix} \\ &= \begin{bmatrix} xI_k & \mathbf{0} \\ \mathbf{0} & I_{n-k} \end{bmatrix} \begin{bmatrix} A_{k\times k} & B_{k\times(n-k)} \\ C_{(n-k)\times k} & D_{(n-k)\times(n-k)} \end{bmatrix} \begin{bmatrix} x^{-1}I_k & \mathbf{0} \\ \mathbf{0} & I_{n-k} \end{bmatrix}, \end{aligned}$$

where I_k is a $k\times k$ unit matrix, $M(x)$ is similar to M, and then $\lambda_1,\ \lambda_2,\cdots,\lambda_n$ are eigenvalues of $M(x)$. We let $N=M(x)-\frac{\mathrm{tr}M}{n}I$, where I is an $n\times n$ unit matrix, then $\lambda_j-\frac{\mathrm{tr}M}{n}\ \ (j=1,2,\cdots,n)$ are eigenvalues of N.

By (1.3), we educe that

$$\sum_{j=1}^{n}\left|\lambda_j-\frac{\mathrm{tr}M}{n}\right|^2\leqslant\left(\|N\|^4-\frac{1}{2}\|[N,N^*]\|^2\right)^{\frac{1}{2}}. \tag{2.4}$$

We note that

$$\sum_{j=1}^{n}\left|\lambda_j-\frac{\mathrm{tr}M}{n}\right|^2=\sum_{j=1}^{n}|\lambda_j|^2-\frac{|\mathrm{tr}M|^2}{n}, \tag{2.5}$$

$$\begin{aligned}\|N\|^4&=\left(\mathrm{tr}\left(\left(M(x)-\frac{\mathrm{tr}M}{n}I\right)\left(M(x)-\frac{\mathrm{tr}M}{n}I\right)^*\right)\right)^2\\&=\left(\|M\|^2-\left((1-x^2)\|B_{k\times(n-k)}\|^2+(1-x^{-2})\|C_{(n-k)\times k}\|^2\right)-\frac{|\mathrm{tr}M|^2}{n}\right)^2\\&=(\Delta_M(k,x))^2,\end{aligned} \tag{2.6}$$

and

$$[N,N^*]=\left[M(x)-\frac{\mathrm{tr}M}{n}I,M(x)^*-\frac{\overline{\mathrm{tr}M}}{n}I\right]=[M(x),M(x)^*]. \tag{2.7}$$

Combining (2.4)–(2.7), we can conclude the inequality (2.1). □

Now we consider some special cases of the above the theorem. If choosing $x=1$ in (2.1), we can get (1.4), i.e., $f_M(k,1)=\psi_4$, and then $f_M(k,x)\leqslant\psi_4$. Therefore Theorem 2.1 is superior to (1.4). It sharpens up Schur, Eberlein, Kress and Huang's inequalities. Further, if we choosing choose $x=\sqrt{\frac{\|C_{(n-k)\times k}\|}{\|B_{k\times(n-k)}\|}}=\delta\neq 0$ in (2.1), then we will get a new estimate for $\sum_{j=1}^{n}|\lambda_j|^2$ and five determinant inequalities.

COROLLARY 2.1. *Let $M\in \mathrm{C}^{n\times n}$ with eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$,*

$$M=\begin{bmatrix}A_{k\times k} & B_{k\times(n-k)}\\ C_{(n-k)\times k} & D_{(n-k)\times(n-k)}\end{bmatrix}\quad \textit{and}\quad M_k=\begin{bmatrix}A_{k\times k} & \delta B_{k\times(n-k)}\\ \delta^{-1}C_{(n-k)\times k} & D_{(n-k)\times(n-k)}\end{bmatrix},$$

where $A_{k\times k}$ is $k\times k$ principal submatrix of M $(1\leqslant k\leqslant n-1)$. Then

$$\sum_{j=1}^{n}|\lambda_j|^2\leqslant\psi_5=\min_{1\leqslant k\leqslant n-1}\left((\Delta_M(k))^2-\frac{1}{2}\|[M_k,M_k^*]\|^2\right)^{\frac{1}{2}}+\frac{|\mathrm{tr}M|^2}{n}, \tag{2.8}$$

where

$$\Delta_M(k)=\|M\|^2-\left(\|B_{k\times(n-k)}\|-\|C_{(n-k)\times k}\|\right)^2-\frac{|\mathrm{tr}M|^2}{n}. \tag{2.9}$$

THEOREM 2.2. *Let $M \in \mathrm{C}^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, $M(x)$ be defined as Theorem 2.1. If M is non-singular, then*

$$|\det M| \leqslant \left(\frac{\Psi_j}{n}\right)^{\frac{n}{2}}, \ \ (j = 1, 2, \cdots, 5). \tag{2.10}$$

Proof. We note that $|\det M| = |\prod_{j=1}^{n} \lambda_j| = \left(\left(\prod_{j=1}^{n} |\lambda_j|^2\right)^{\frac{1}{n}}\right)^{\frac{n}{2}} \leqslant \left(\frac{\sum_{j=1}^{n} |\lambda_j|^2}{n}\right)^{\frac{n}{2}}$, combining (1.1)–(1.4) and Corollary 2.1, we can conclude (2.10). □

3. The new localization of eigenvalues of $n \times n$ complex matrices

In 1931, Geršgorin gave a well-known theorem, which is called Geršgorin disk theorem. It specifically states that for a given complex matrix $A = (a_{ij}) \in \mathrm{C}^{n\times n}$, all its eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are included in set $\Gamma(A) = \bigcup_{i=1}^{n} \Gamma_i(A)$, where $\Gamma_i(A) = \{z \in C : |z - a_{ii}| \leqslant r_i(A)\}$ and $r_i(A) = \sum_{j=1, j\neq i}^{n} |a_{ij}|$. That is, all eigenvalues of a given $n \times n$ complex matrix must be located in the union of the following n disks:

$$|\lambda - a_{ii}| \leqslant \sum_{j=1, j\neq i}^{n} |a_{ij}|, \ \ i = 1, 2, \cdots, n. \tag{3.1}$$

However, from Geršgorin's theorem we can know that all eigenvalues of a given matrix are located in the union (called the Geršgorin set) of many subsets, it means that there is a problem with Geršgorin's theorem, that is, it still needs people to determine the position (small disk) of the eigenvalues of a given matrix further. In addition, it will also encounter a problem that two or more similar matrices have the same eigenvalues. According to Geršgorin's theorem, there will be much more Geršgorin sets containing these eigenvalues and it will be a difficult matter to find out which is the smallest region and an explicit and calculable numerical formula to express such set.

We note that Y. X. Gu proposed a new method which uses only one closed disk to locate all eigenvalues of a given $n \times n$ complex matrix (see [6]). He proved that all eigenvalues of a given $n \times n$ complex matrix A can be included in the following disk:

$$\left|\lambda - \frac{\mathrm{tr}A}{n}\right| \leqslant \left(\frac{n-1}{n}\left(\|A\|^2 - \frac{|\mathrm{tr}A|^2}{n}\right)\right)^{\frac{1}{2}}. \tag{3.2}$$

It is obvious that Gu's method can avoid the troubles that present in Geršgorin's.

Furthermore, we also note that L. M. Zou and Y. Y. Jiang gave the following more precise disk to locate all eigenvalues of a given $n \times n$ complex matrix M (see [7]):

$$\left|\lambda - \frac{\mathrm{tr}M}{n}\right| \leqslant \min_{1\leqslant k\leqslant n-1} \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \left(\|M\|^2 - \frac{|\mathrm{tr}M|^2}{n} - (\|B_{k\times(n-k)}\| - \|C_{(n-k)\times k}\|)^2\right)^{\frac{1}{2}}. \tag{3.3}$$

In this section, we should sharpen Y. X. Gu, L. M. Zou and Y. Y. Jiang's results. It means that we should put forward much smaller disks to contain all eigenvalues of a given $n\times n$ complex matrix. In addition, we also use rectangle regions to contain all of the eigenvalues of a given $n\times n$ complex matrix.

In [8], authors gave the following

LEMMA 3.1. *If $z_1,z_2,\cdots,z_n$ are complex numbers, then*

$$\left|z_i-\frac{1}{n}\sum_{j=1}^{n} z_j\right|^2 \leqslant \frac{n-1}{n}\left(\sum_{j=1}^{n}|z_j|^2-\frac{1}{n}\left|\sum_{j=1}^{n} z_j\right|^2\right) \quad (i=1,2,\cdots,n).$$

Lemma 3.1 shows that for any n complex numbers, they can be included in disk

$$\left|z-\frac{1}{n}\sum_{j=1}^{n} z_j\right| \leqslant \left(\frac{n-1}{n}\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}|z_j|^2-\frac{1}{n}\left|\sum_{j=1}^{n} z_j\right|^2\right)^{\frac{1}{2}}.$$

THEOREM 3.1. *Let $M\in \mathrm{C}^{n\times n}$ with eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$, $f_M(k,x)$ be defined as Theorem 2.1. Then all eigenvalues of M are included in the following disk:*

$$\left|\lambda-\frac{\mathrm{tr}M}{n}\right| \leqslant \min_{x\neq 0}\min_{1\leqslant k\leqslant n-1}\left(\frac{n-1}{n}\right)^{\frac{1}{2}}\left(f_M(k,x)-\frac{|\mathrm{tr}M|^2}{n}\right)^{\frac{1}{2}} \tag{3.4}$$

Proof. If we let $z_j=\lambda_j (j=1,2,\cdots,n)$ in Lemma 3.1 and combining Theorem 2.1, we have

$$\begin{aligned}
\left|\lambda-\frac{1}{n}\sum_{j=1}^{n}\lambda_j\right|^2 &\leqslant \frac{n-1}{n}\left(\sum_{j=1}^{n}|\lambda_j|^2-\frac{1}{n}\left|\sum_{j=1}^{n}\lambda_j\right|^2\right)\\
&\leqslant \frac{n-1}{n}\left(\min_{x\neq 0}\min_{1\leqslant k\leqslant n-1} f_M(k,x)-\frac{1}{n}\left|\sum_{j=1}^{n}\lambda_j\right|^2\right)\\
&\leqslant \frac{n-1}{n}\left(\min_{x\neq 0}\min_{1\leqslant k\leqslant n-1} f_M(k,x)-\frac{|\mathrm{tr}M|^2}{n}\right)\\
&\leqslant \min_{x\neq 0}\min_{1\leqslant k\leqslant n-1}\left(\frac{n-1}{n}\right)\left(f_M(k,x)-\frac{|\mathrm{tr}M|^2}{n}\right).
\end{aligned}$$

So, we can deduce (3.4). □

COROLLARY 3.1. *Let $M\in \mathrm{C}^{n\times n}$ with eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$, M_k be defined as Corollary 2.1. Then all eigenvalues of M are included in the following two disks respectively:*

$$\left|\lambda-\frac{\mathrm{tr}M}{n}\right| \leqslant \left(\frac{n-1}{n}\right)^{\frac{1}{2}}\left(\left(\|M\|^2-\frac{|\mathrm{tr}M|^2}{n}\right)^2-\frac{1}{2}\|[M,M^*]\|^2\right)^{\frac{1}{4}}, \tag{3.5}$$

$$\left|\lambda - \frac{\mathrm{tr}M}{n}\right| \leqslant \min_{1\leqslant k\leqslant n-1}\left(\frac{n-1}{n}\right)^{\frac{1}{2}}\left((\Delta_M(k))^2 - \frac{1}{2}\|[M_k, M_k^*]\|^2\right)^{\frac{1}{4}}. \tag{3.6}$$

where $\Delta_M(k)$ *be defined as (2.9).*

Obviously, the radiuses of disks (3.5) and (3.6) are smaller than that in (3.2) and (3.3) respectively. So if we use (3.5) and (3.6) to estimate the eigenvalues of a given $n\times n$ complex matrix, then (3.5) and (3.6) is superior to (3.2) and (3.3), respectively.

THEOREM 3.2. *Let* $M \in \mathrm{C}^{n\times n}$ *with eigenvalues* $\lambda_1, \lambda_2, \cdots, \lambda_n$, $f_M(k,x)$ *be defined as Theorem 2.1. Then all eigenvalues of* M *are included in the following disk:*

$$\left|\lambda - \frac{\mathrm{tr}M}{n}\right| \leqslant \min_{x\neq 0}\min_{1\leqslant k\leqslant n-1}\left(\frac{n-1}{2n}\right)^{\frac{1}{2}}\left(f_M(k,x) - \frac{|\mathrm{tr}M|^2}{n} + \left|\mathrm{tr}M^2 - \frac{\mathrm{tr}^2M}{n}\right|\right)^{\frac{1}{2}}. \tag{3.7}$$

Proof. Let $z_j = \mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}\lambda_j)$, where $\theta = \arg\left(\overline{\lambda_p} - \frac{\overline{\mathrm{tr}M}}{n}\right)$, $j = 1,2,\cdots,n$. By Lemma 3.1, for any $1\leqslant p\leqslant n$, we educe that

$$\left|z_p - \frac{1}{n}\sum_{j=1}^{n}\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}\lambda_j)\right| \leqslant \left(\frac{n-1}{n}\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}\left(\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}\lambda_j)\right)^2 - \frac{1}{n}\left(\sum_{j=1}^{n}\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}\lambda_j)\right)^2\right)^{\frac{1}{2}}. \tag{3.8}$$

We note that the following equalities are hold:

$$\begin{aligned}\left|z_p - \frac{1}{n}\sum_{j=1}^{n}\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}\lambda_j)\right| &= \left|\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}\lambda_p) - \mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}\frac{\mathrm{tr}M}{n})\right| \\ &= \left|\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta}\left(\lambda_p - \frac{\mathrm{tr}M}{n}\right)\right)\right| = \left|\lambda_p - \frac{\mathrm{tr}M}{n}\right|,\end{aligned} \tag{3.9}$$

$$\begin{aligned}\sum_{j=1}^{n}\left(\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}\lambda_j)\right)^2 &= \frac{1}{2}\sum_{j=1}^{n}\left(\sqrt{2}\cos\theta\mathrm{Re}\lambda_j - \sqrt{2}\sin\theta\mathrm{Im}\lambda_j\right)^2 \\ &= \frac{1}{2}\sum_{j=1}^{n}\left(|\lambda_j|^2 + \mathrm{Re}(\mathrm{e}^{2\mathrm{i}\theta}\lambda_j^2)\right) = \frac{1}{2}\sum_{j=1}^{n}|\lambda_j|^2 + \frac{1}{2}\mathrm{Re}(\mathrm{e}^{2\mathrm{i}\theta}\mathrm{tr}M^2),\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\frac{1}{n}\left(\sum_{j=1}^{n}\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}\lambda_j)\right)^2 &= \frac{1}{2n}\left(\sqrt{2}\cos\theta\mathrm{Re}(\mathrm{tr}M) - \sqrt{2}\sin\theta\mathrm{Im}(\mathrm{tr}M)\right)^2 \\ &= \frac{1}{2n}\left(|\mathrm{tr}M|^2 + \cos 2\theta\mathrm{Re}(\mathrm{tr}^2M) - \sin 2\theta\mathrm{Im}(\mathrm{tr}^2M)\right) \\ &= \frac{1}{2n}|\mathrm{tr}M|^2 + \frac{1}{2n}\mathrm{Re}(\mathrm{e}^{2\mathrm{i}\theta}\mathrm{tr}^2M).\end{aligned} \tag{3.11}$$

Combining (3.8)–(3.11), we deduce that

$$\begin{aligned}\left|\lambda_p-\frac{\operatorname{tr}M}{n}\right| &\leqslant \left(\frac{n-1}{2n}\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}|\lambda_j|^2+\operatorname{Re}(\mathrm{e}^{2\mathrm{i}\theta}\operatorname{tr}M^2)-\frac{1}{n}\left(|\operatorname{tr}M|^2+\operatorname{Re}(\mathrm{e}^{2\mathrm{i}\theta}\operatorname{tr}^2M)\right)\right)^{\frac{1}{2}}\\ &\leqslant \left(\frac{n-1}{2n}\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}|\lambda_j|^2-\frac{|\operatorname{tr}M|^2}{n}+\left|\operatorname{tr}M^2-\frac{\operatorname{tr}^2M}{n}\right|\right)^{\frac{1}{2}}.\end{aligned}$$

Furthermore, applying Theorem 2.1, we have

$$\left|\lambda_p-\frac{\operatorname{tr}M}{n}\right| \leqslant \left(\frac{n-1}{2n}\right)^{\frac{1}{2}}\left(f_M(k,x)-\frac{|\operatorname{tr}M|^2}{n}+\left|\operatorname{tr}M^2-\frac{\operatorname{tr}^2M}{n}\right|\right)^{\frac{1}{2}}.$$

And then we know that (3.7) is hold as $1\leqslant k\leqslant n-1$ and $x\neq 0$. The proof process is completed. □

COROLLARY 3.2. *Let $M\in \mathrm{C}^{n\times n}$ with eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$, M_k be defined as Corollary 2.1. Then all eigenvalues of M are included in the following two disks respectively:*

$$\left|\lambda-\frac{\operatorname{tr}M}{n}\right| \leqslant \left(\frac{n-1}{2n}\right)^{\frac{1}{2}}\left(\left(\left(\|M\|^2-\frac{|\operatorname{tr}M|^2}{n}\right)^2-\frac{1}{2}\|[M,M^*]\|^2\right)^{\frac{1}{2}}+\left|\operatorname{tr}M^2-\frac{\operatorname{tr}^2M}{n}\right|\right)^{\frac{1}{2}}, \tag{3.12}$$

$$\left|\lambda-\frac{\operatorname{tr}M}{n}\right| \leqslant \min_{1\leqslant k\leqslant n-1}\left(\frac{n-1}{2n}\right)^{\frac{1}{2}}\left(\left((\Delta_M(k))^2-\frac{1}{2}\|[M_k,M_k^*]\|^2\right)^{\frac{1}{2}}+\left|\operatorname{tr}M^2-\frac{\operatorname{tr}^2M}{n}\right|\right)^{\frac{1}{2}}. \tag{3.13}$$

We note that the following

$$\begin{aligned}\left|\operatorname{tr}M^2-\frac{\operatorname{tr}^2M}{n}\right| &= \left|\sum_{j=1}^{n}\lambda_j^2-\frac{\operatorname{tr}^2M}{n}\right| = \left|\sum_{j=1}^{n}\left(\lambda_j-\frac{\operatorname{tr}M}{n}\right)^2\right|\\ &\leqslant \sum_{j=1}^{n}\left|\lambda_j-\frac{\operatorname{tr}M}{n}\right|^2=\sum_{j=1}^{n}|\lambda_j|^2-\frac{|\operatorname{tr}M|^2}{n}.\end{aligned}$$

Combing Theorem 2.1, we deduce (3.12) and (3.13) is superior to (3.5) and (3.6), respectively.

COROLLARY 3.3. *Let $M\in \mathrm{C}^{n\times n}$ with eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$, $f_M(k,x)$ be defined as Theorem 2.1. Then the spectral radius $\rho(M)$ meet the following inequality:*

$$\rho(M) \leqslant \min_{x\neq 0}\min_{1\leqslant k\leqslant n-1}\left(\frac{n-1}{2n}\right)^{\frac{1}{2}}\left(f_M(k,x)-\frac{|\operatorname{tr}M|^2}{n}+\left|\operatorname{tr}M^2-\frac{\operatorname{tr}^2M}{n}\right|\right)^{\frac{1}{2}}+\frac{|\operatorname{tr}M|}{n}. \tag{3.14}$$

THEOREM 3.3. *Let* $M \in \mathbb{C}^{n\times n}$ *with eigenvalues* $\lambda_1, \lambda_2, \cdots, \lambda_n$, $f_M(k,x)$ *be defined as Theorem 2.1. Let* m *be an integer satisfying* $\operatorname{rank}(M) \leqslant m \leqslant n$. *Then all its eigenvalues are included in the following disk:*

$$\left|\lambda - \frac{\operatorname{tr}M}{m}\right| \leqslant \min_{x\neq 0} \min_{1\leqslant k\leqslant n-1} \left(\frac{m-1}{2m}\right)^{\frac{1}{2}} \left(f_M(k,x) - \frac{|\operatorname{tr}M|^2}{m} + \left|\operatorname{tr}M^2 - \frac{\operatorname{tr}^2 M}{m}\right|\right)^{\frac{1}{2}}. \quad (3.15)$$

The proof is similar to that of Theorem 3.2 and hence we omit it here. We note that $f_M(k,x) \leqslant \psi_4$, and therefore our result (3.15) is superior to [5, Theorem 2.1].

THEOREM 3.4. *Let* $M \in \mathbb{C}^{n\times n}$ *with eigenvalues* $\lambda_1, \lambda_2, \cdots, \lambda_n$, $f_M(k,x)$ *be defined as Theorem 2.1. Then all the eigenvalues of* M *are located in the following rectangle region:*

$$\left[\frac{\operatorname{Re}(\operatorname{tr}M)}{n} - \alpha, \frac{\operatorname{Re}(\operatorname{tr}M)}{n} + \alpha\right] \times \left[\frac{\operatorname{Im}(\operatorname{tr}M)}{n} - \beta, \frac{\operatorname{Im}(\operatorname{tr}M)}{n} + \beta\right],$$

where

$$\alpha = \min_{x\neq 0} \min_{1\leqslant k\leqslant n-1} \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \left(\frac{1}{2}\left(f_M(k,x) + \operatorname{Re}(\operatorname{tr}M^2)\right) - \frac{(\operatorname{Re}(\operatorname{tr}M))^2}{n}\right)^{\frac{1}{2}},$$

$$\beta = \min_{x\neq 0} \min_{1\leqslant k\leqslant n-1} \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \left(\frac{1}{2}\left(f_M(k,x) - \operatorname{Re}(\operatorname{tr}M^2)\right) - \frac{(\operatorname{Im}(\operatorname{tr}M))^2}{n}\right)^{\frac{1}{2}}.$$

Proof. Let $z_j = \operatorname{Re}\lambda_j$ and $z_j = \operatorname{Im}\lambda_j$ in Lemma 3.1 respectively, we have

$$\left|\operatorname{Re}\lambda_j - \frac{\operatorname{Re}(\operatorname{tr}M)}{n}\right| \leqslant \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} (\operatorname{Re}\lambda_j)^2 - \frac{(\operatorname{Re}(\operatorname{tr}M))^2}{n}\right)^{\frac{1}{2}},$$

$$\left|\operatorname{Im}\lambda_j - \frac{\operatorname{Im}(\operatorname{tr}M)}{n}\right| \leqslant \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} (\operatorname{Im}\lambda_j)^2 - \frac{(\operatorname{Im}(\operatorname{tr}M))^2}{n}\right)^{\frac{1}{2}}.$$

We note that

$$\sum_{j=1}^{n} (\operatorname{Re}\lambda_j)^2 = \frac{1}{2}\left(\sum_{j=1}^{n} |\lambda_j|^2 + \operatorname{Re}(\operatorname{tr}M^2)\right),$$

$$\sum_{j=1}^{n} (\operatorname{Im}\lambda_j)^2 = \frac{1}{2}\left(\sum_{j=1}^{n} |\lambda_j|^2 - \operatorname{Re}(\operatorname{tr}M^2)\right).$$

We deduce that

$$\left|\operatorname{Re}\lambda_j - \frac{\operatorname{Re}(\operatorname{tr}M)}{n}\right| \leqslant \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \left(\frac{1}{2}\left(\sum_{j=1}^{n} |\lambda_j|^2 + \operatorname{Re}(\operatorname{tr}M^2)\right) - \frac{(\operatorname{Re}(\operatorname{tr}M))^2}{n}\right)^{\frac{1}{2}},$$

$$\left|\operatorname{Im}\lambda_j - \frac{\operatorname{Im}(\operatorname{tr}M)}{n}\right| \leqslant \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \left(\frac{1}{2}\left(\sum_{j=1}^{n} |\lambda_j|^2 - \operatorname{Re}(\operatorname{tr}M^2)\right) - \frac{(\operatorname{Im}(\operatorname{tr}M))^2}{n}\right)^{\frac{1}{2}}.$$

Combining Theorem 2.1, we can conclude Theorem 3.4. □

COROLLARY 3.4. *Let $M \in C^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, M_k be defined as Corollary 2.1. Then all eigenvalues of are included in the following rectangle regions:*

$$\left[\frac{\mathrm{Re}(\mathrm{tr}M)}{n} - \alpha_j, \frac{\mathrm{Re}(\mathrm{tr}M)}{n} + \alpha_j\right] \times \left[\frac{\mathrm{Im}(\mathrm{tr}M)}{n} - \beta_j, \frac{\mathrm{Im}(\mathrm{tr}M)}{n} + \beta_j\right], \ (j = 1,2) \tag{3.16}$$

where

$$\alpha_1 = \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \left(\frac{1}{2}\left(\eta + \mathrm{Re}(\mathrm{tr}M^2)\right) - \frac{(\mathrm{Re}(\mathrm{tr}M))^2}{n}\right)^{\frac{1}{2}},$$

$$\beta_1 = \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \left(\frac{1}{2}\left(\eta - \mathrm{Re}(\mathrm{tr}M^2)\right) - \frac{(\mathrm{Im}(\mathrm{tr}M))^2}{n}\right)^{\frac{1}{2}},$$

$$\alpha_2 = \min_{1\leqslant k\leqslant n-1} \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \left(\frac{1}{2}\left(\varphi + \mathrm{Re}(\mathrm{tr}M^2)\right) - \frac{(\mathrm{Re}(\mathrm{tr}M))^2}{n}\right)^{\frac{1}{2}},$$

$$\beta_2 = \min_{1\leqslant k\leqslant n-1} \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \left(\frac{1}{2}\left(\varphi - \mathrm{Re}(\mathrm{tr}M^2)\right) - \frac{(\mathrm{Im}(\mathrm{tr}M))^2}{n}\right)^{\frac{1}{2}}.$$

where

$$\eta = \left(\left(\|M\|^2 - \frac{|\mathrm{tr}M|^2}{n}\right)^2 - \frac{1}{2}\|[M,M^*]\|^2\right)^{\frac{1}{2}} + \frac{|\mathrm{tr}M|^2}{n},$$

$$\varphi = \min_{1\leqslant k\leqslant n-1} \left((\Delta_M(k))^2 - \frac{1}{2}\|[M_k,M_k^*]\|^2\right)^{\frac{1}{2}} + \frac{|\mathrm{tr}M|^2}{n}.$$

where $\Delta_M(k)$ be defined as (2.9).

4. Numerical examples

In this section, we give some numerical examples to show the effectiveness of our results.

EXAMPLE 4.1. Let

$$M = \begin{bmatrix} 26 & 5 & 15 \\ 3 & 80 & 17 \\ 1 & 7 & 10 \end{bmatrix}$$

By (1.1), we have $\psi_1 = 7.7740\mathrm{e}+003$.
By (1.2), we get $\psi_2 = 7.7336\mathrm{e}+003$.

By (1.3), we get $\psi_3 = 7.7132\mathrm{e}+003$.
By (1.4), we have $\psi_4 = 7.6275\mathrm{e}+003$.
By Corollary 2.1, we get $\psi_5 = 7.5137\mathrm{e}+003$.
As a result it can be easily seen that our result $\psi_5 \leqslant \psi_j (j = 1,2,3,4)$.

EXAMPLE 4.2. Let

$$M = \begin{bmatrix} 2 & 5+\mathrm{i} & 1 \\ 7 & -\mathrm{i} & 9 \\ 12 & 3 & 2-\mathrm{i} \end{bmatrix}$$

If using Gerschgorin's disk theorem to estimate the eigenvalues of M, then we know that all eigenvalues of M are located in the following set:

$$\{\lambda : |\lambda - 2| \leqslant 6.0990\} \cup \{\lambda : |\lambda - (-\mathrm{i})| \leqslant 16\} \cup \{\lambda : |\lambda - (2-\mathrm{i})| \leqslant 15\}.$$

If we use (3.2) to estimate the eigenvalues of M, then we know that all eigenvalues of M are located in disk $\left|\lambda - \frac{4-2\mathrm{i}}{3}\right| \leqslant 14.4530$.

If we use (3.3) to estimate the eigenvalues of M, then we know that all eigenvalues of M are located in disk $\left|\lambda - \frac{4-2\mathrm{i}}{3}\right| \leqslant 12.5886$.

If we use a series of our results to estimate the eigenvalues of M, we have the following:
By (3.5), we have $\left|\lambda - \frac{4-2\mathrm{i}}{3}\right| \leqslant 13.0795$.
By (3.6), we have $\left|\lambda - \frac{4-2\mathrm{i}}{3}\right| \leqslant 12.4702$.
By (3.12), we have $\left|\lambda - \frac{4-2\mathrm{i}}{3}\right| \leqslant 11.6532$.
By (3.13), we have $\left|\lambda - \frac{4-2\mathrm{i}}{3}\right| \leqslant 10.1039$.
As a result it can be easily seen that our result (3.13) is superior to others.

By (3.16), we know that all eigenvalues of are located in the following rectangles $[-10.3087, 12.9753] \times [-6.6280, 5.2946]$, $[-9.7711, 12.4377] \times [-5.4945, 4.1611]$.

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