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ON THE MAXIMAL INEQUALITIES FOR CONDITIONAL DEMIMARTINGALES

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Abstract. In this paper, based on a Fubini formula of conditional expectation and a maximal inequality for conditional demimartingales, we extend some inequalities of demimartingales to the case of conditional demimartingales. Meanwhile, we obtain some maximal ϕ-inequalities for conditional demimartingales and some maximal inequalities of concave Young functions for conditional demimartingales.

1. Introduction

Let $\{S_n : n \geqslant 1\}$ or $\{X_n : n \geqslant 1\}$ denote a sequence of random variables defined on a fixed probability space $(\Omega, \mathscr{A}, P)$.

First, we recall the definitions of demimartingales.

DEFINITION 1. A sequence of random variables $\{S_n : n \geqslant 1\}$ satisfying $E|S_n| < \infty$ for all $n \in \mathbb{N}$ is called a demimartingale if for all $i \in \mathbb{N}$,

$$E\{(S_{i+1} - S_i)f(S_1, S_2, \cdots, S_i)\} \geqslant 0$$

for every function $f : \mathbb{R}^i \to \mathbb{R}$ that is nondecreasing componentwise and such that the expectation is defined. If in addition f is assumed to be nonnegative, the sequence $\{S_n : n \geqslant 1\}$ is called a demisubmartingale.

Definition 1 is due to Newman and Wright [12]. Many authors have studied this concept providing interesting results and applications. Newman and Wright [12] extended Doob type maximal inequality and upcrossing inequality to the case of demimartingales and pointed out that the partial sum of a sequence of mean zero associated random variables is a demimartingale. Christofides [4] showed that the Chow type maximal inequality for (sub)martingales can be extended to the case of demi(sub)martingales. Christofides [5] constructed some U-statistics based on associated random

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variables and proved it to be a demimartingale. Wang [18] obtained Doob's type inequality for more general demimartingales. Prakasa Rao [13] established some maximal inequalities for demisubmartingales. Wang and Hu [20] obtained some maximal inequalities for demimartingales and gave an equivalent condition of uniform integrability for demisubmartingales. Wang et al. [21] obtained some maximal inequalities for functions of demimartingales including Doob's type inequality, strong laws of large numbers and growth rate for demimartingales. Gong [9] obtained some maximal ϕ-inequalities for demimartingales. Prakasa Rao [15] gave an alternate approach for deriving maximal inequalities for nonnegative demisubmartingales. Hu et al. [11] investigated the Marshall type inequalities for demimartingales. Christofides and Hadjikyriakou [6] provided some maximal and moment inequalities for demimartingales. Wang et al. [22] established some maximal inequalities for demimartingales based on concave Young functions and so forth. For more details about demimartingales, one can refers to Prakasa Rao [16].

Let X and Y be random variables defined on a probability space $(\Omega,\mathscr{A},P)$ with $EX^2<\infty$ and $EY^2<\infty$. Let $\mathscr{F}$ be a sub-σ algebra of $\mathscr{A}$. Prakasa Rao [14] defined the notion of the conditional covariance of X and Y given $\mathscr{F}$ ($\mathscr{F}$-covariance, for short) as

$$Cov^{\mathscr{F}}(X,Y)=E^{\mathscr{F}}\Big((X-E^{\mathscr{F}}X)(Y-E^{\mathscr{F}}Y)\Big),$$

where $E^{\mathscr{F}}Z$ denotes the conditional expectation of a random variable Z given $\mathscr{F}$. In contrast to the ordinary concept of variance, conditional variance of X given $\mathscr{F}$ is defined as $Var^{\mathscr{F}}X=Cov^{\mathscr{F}}(X,X)$.

Christofides and Hadjikyriakou introduced the following definition (see [7, 10]).

DEFINITION 2. A sequence of random variables $\{S_n : n\geqslant 1\}$ satisfying $E|S_n|<\infty$ for all $n\in\mathbb{N}$ is called an $\mathscr{F}$-demimartingale if for all $i<j$, $i,j\in\mathbb{N}$,

$$E^{\mathscr{F}}\{(S_j-S_i)f(S_1,S_2,\cdots,S_i)\}\geqslant 0 \ a.s. \tag{1}$$

for every function $f:\mathbb{R}^i\to\mathbb{R}$ that is nondecreasing componentwise and such that the conditional expectation is defined. If in addition f is assumed to be nonnegative, the sequence $\{S_n : n\geqslant 1\}$ is called an $\mathscr{F}$-demisubmartingale.

It is easy to check that for all $i\in\mathbb{N}$, (1) is equivalent to

$$E^{\mathscr{F}}\{(S_{i+1}-S_i)f(S_1,S_2,\cdots,S_i)\}\geqslant 0 \ a.s.$$

Christofides and Hadjikyriakou [7] established some maximal inequalities and asymptotic results for conditional demimartingales. Wang and Wang [19] obtained some maximal inequalities for conditional demi(sub)martingales and minimal inequalities for nonnegative conditional demimartingales.

Next, recall the notion of conditional association introduced by Prakasa Rao [14].

DEFINITION 3. A finite collection of random variables $\{X_i : 1\leqslant i\leqslant n\}$ is said to be $\mathscr{F}$-associated if for any $f:\mathbb{R}^n\to\mathbb{R}$ and $g:\mathbb{R}^n\to\mathbb{R}$, which are two componentwise nondecreasing functions such that

$$Cov^{\mathscr{F}}(f(X_1,X_2,\cdots,X_n),g(X_1,X_2,\cdots,X_n)) \geqslant 0 \ \ a.s.,$$

whenever the conditional covariance exists. An infinite collection $\{X_n : n \geqslant 1\}$ is said to be $\mathscr{F}$-associated if every finite subcollection is $\mathscr{F}$-associated.

REMARK 1. Proposition 2 of Newman and Wright [12] shows that the partial sum of a sequence of mean zero associated random variables (c.f. [8]) is a demimartingale. It is easy to verify that the partial sum of a sequence of $\mathscr{F}$-associated random variables with conditional mean zero is an $\mathscr{F}$-demimartingale by the definition of $\mathscr{F}$-association. The details on $\mathscr{F}$-association are refereed to Prakasa Rao [14], Roussas [17], Yuan and Yang [23] and others. Yuan and Yang [23] took examples to point out that conditional association of random variables does not imply association and the opposite implication is also not true.

From the property of conditional expectations that $E(E(Z|\mathscr{F})) = E(Z)$ for any random variable Z with $E|Z| < \infty$, it follows that $\mathscr{F}$-demi(sub)martingales defined on a probability space $(\Omega,\mathscr{A},P)$ are demi(sub)martingales on the probability space $(\Omega,\mathscr{A},P)$, but the converse is not true. Hadjikyriakou [10] gave an example which is a demimartingale but not an $\mathscr{F}$-demisubmartingale.

In this paper, inspired by the above authors, based on a Fubini formula of conditional expectation and a maximal inequality for conditional demimartingales, we establish some maximal ϕ-inequalities and some maximal inequalities of concave Young functions for conditional demimartingales. In particular, Doob's type inequality for conditional demimartingales is presented. These results extend some corresponding ones of [9, 21, 22] to the cases of functions of conditional demimartingales.

The paper is organized as follows: auxiliary lemmas are obtained in Section 2, maximal ϕ-inequalities including Doob's type inequality for $\mathscr{F}$-demimartingales are presented in Section 3 and some maximal inequalities for $\mathscr{F}$-demimartingales based on concave Young functions are presented in Section 4.

Throughout this paper, let $I(A)$ denote the indicator function of the set A and let $P^{\mathscr{F}}(A) = E^{\mathscr{F}}I(A)$, $S_0 \doteq 0$, $a \vee b = \max(a,b)$, $x^+ = 0 \vee x$, $\log x = \log_e x = \ln x$ and $\log^+ x = \ln(x \vee 1)$.

2. Lemmas

In this section, some very useful lemmas are given to prove the main results of the paper.

LEMMA 1. *Let $X(\cdot,\cdot) : \Omega \times \mathbb{R} \to \mathbb{R}$ be $\mathscr{A} \times \mathscr{B}$-measurable. Suppose further that $X(\cdot,\cdot)$ is either nonnegative or integrable with respect to $P \times \mu$, where μ is the Lebesgue measure. Then*

$$E^{\mathscr{F}} \int_{\mathbb{R}} X(\cdot,t)dt = \int_{\mathbb{R}} E^{\mathscr{F}} X(\cdot,t)dt \ \ a.s. \tag{2}$$

Proof. Let $Y = \int_{\mathbb{R}} X(\cdot,t)dt$, then Y is $\mathscr{A}$-measurable. As following from the proof of Theorem 4.1 of [17], by the properties of conditional expectations (c.f. [3], p. 204) and Fubini theorem, for any $B \in \mathscr{F}$, $E^{\mathscr{F}}Y$ is $\mathscr{F}$-measurable and

$$\begin{aligned}\int_B E^{\mathscr{F}} Y dP = \int_B Y dP &= \int_B \Big(\int_{\mathbb{R}} X(\cdot,t)dt\Big)dP \\ &= \int_{\mathbb{R}} \Big(\int_B X(\cdot,t)dP\Big)dt = \int_{\mathbb{R}} \Big(\int_B E^{\mathscr{F}} X(\cdot,t)dP\Big)dt \\ &= \int_B \Big(\int_{\mathbb{R}} E^{\mathscr{F}} X(\cdot,t)dt\Big)dP.\end{aligned}$$

Furthermore, $\int_{\mathbb{R}} E^{\mathscr{F}} X(\cdot,t)dt$ is $\mathscr{F}$-measurable. It follows that

$$E^{\mathscr{F}} Y = \int_{\mathbb{R}} E^{\mathscr{F}} X(\cdot,t)dt \ \ a.s. \quad \square$$

COROLLARY 1. *Let X be a nonnegative random variable, then*

$$E^{\mathscr{F}} X = \int_0^\infty P^{\mathscr{F}}(X \geqslant t)dt \ \ a.s. \tag{3}$$

Proof. Taking $X(\cdot,t) = I(X \geqslant t), t \geqslant 0$ in Lemma 1, we get (3) immediately. $\square$

REMARK 2. The inequality (3) in Corollary 1 is obtained by the properties of conditional expectations and Fubini theorem. One can also refer to Lemma 20 of [7].

A maximal inequality for conditional demimartingales is obtained as follows.

LEMMA 2. *Let $\{S_n : n \geqslant 1\}$ be an $\mathscr{F}$-demimartingale. Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative convex function satisfying $g(0) = 0$ and $Eg(S_i) < \infty$ for every $i \in \mathbb{N}$. Let $m : \mathbb{R} \to \mathbb{R}$ be a nonnegative nondecreasing function. Suppose that $\{c_n : n \geqslant 1\}$ is a positive nonincreasing sequence of $\mathscr{F}$-measurable random variables and define $S_n^* = \max_{1 \leqslant i \leqslant n} c_i g(S_i)$. Then*

$$E^{\mathscr{F}}\Big(\int_0^{S_n^*} u dm(u)\Big) \leqslant \sum_{i=1}^n E^{\mathscr{F}}\Big(c_i(g(S_i) - g(S_{i-1}))m(S_n^*)\Big) \ \ a.s. \tag{4}$$

In particular, for any $\mathscr{F}$-measurable random variable $\varepsilon > 0$ a.s.,

$$\varepsilon P^{\mathscr{F}}(S_n^* \geqslant \varepsilon) \leqslant \sum_{i=1}^n E^{\mathscr{F}}\Big(c_i(g(S_i) - g(S_{i-1}))I_{(S_n^* \geqslant \varepsilon)}\Big) \ \ a.s. \tag{5}$$

Proof. Let

$$u(x) = \begin{cases} g(x), & x \geqslant 0, \\ 0, & x < 0, \end{cases} \qquad v(x) = \begin{cases} 0, & x \geqslant 0, \\ g(x), & x < 0. \end{cases}$$

It is easy to check that $u(x)$ is a nonnegative nondecreasing convex function, $v(x)$ is a nonnegative nonincreasing convex function and

$$g(x) = u(x) + v(x) = u(x) \vee v(x), x \in \mathbb{R},$$

Define $S_n' = \max_{1 \leqslant i \leqslant n} c_i u(S_i)$ and $S_n'' = \max_{1 \leqslant i \leqslant n} c_i v(S_i)$. Then

$$E^{\mathscr{F}}\Big(\int_0^{S_n^*} udm(u)\Big) \leqslant E^{\mathscr{F}}\Big(\int_0^{S_n'} udm(u)\Big) + E^{\mathscr{F}}\Big(\int_0^{S_n''} udm(u)\Big) \ \ a.s. \tag{6}$$

Firstly, we prove that

$$E^{\mathscr{F}}\Big(\int_0^{S_n'} udm(u)\Big) \leqslant \sum_{i=1}^{n} c_i E^{\mathscr{F}}\Big((u(S_i) - u(S_{i-1}))m(S_n^*)\Big) \ \ a.s. \tag{7}$$

By the definitions of S_n' and m, it follows that for $S_i' \geqslant S_{i-1}'$, either $S_i' = c_i u(S_i)$ or $m(S_i') = m(S_{i-1}')$. Then we have

$$\begin{aligned} E^{\mathscr{F}}\Big(\int_0^{S_n'} udm(u)\Big) &\leqslant \sum_{i=1}^{n} c_i E^{\mathscr{F}}\Big(u(S_i)(m(S_i') - m(S_{i-1}'))\Big) \\ &\leqslant \sum_{i=1}^{n} c_i E^{\mathscr{F}}\Big((u(S_i) - u(S_{i-1}))m(S_n')\Big) \\ &\quad -\Big\{\sum_{i=1}^{n-1} E^{\mathscr{F}}\Big((c_{i+1}u(S_{i+1}) - c_i u(S_i))m(S_i')\Big) \\ &\quad + \sum_{i=1}^{n-1} (c_i - c_{i+1}) E^{\mathscr{F}}(u(S_i)m(S_n'))\Big\} \ \ a.s. \end{aligned}$$

Let

$$A = \sum_{i=1}^{n-1} E^{\mathscr{F}}\Big((c_{i+1}u(S_{i+1}) - c_i u(S_i))m(S_i')\Big) + \sum_{i=1}^{n-1} (c_i - c_{i+1}) E^{\mathscr{F}}(u(S_i)m(S_n'))$$

and $h(x) = \lim\limits_{\Delta x \to 0^-} \frac{u(x+\Delta x) - u(x)}{\Delta x}$. It follows that $h(x)$ is a nondecreasing function from the convexity of $u(x)$. Hence by the monotonicity of $\{c_k : k \geqslant 1\}$ and the convexity of $u(x)$,

$$\begin{aligned} A &\geqslant \sum_{i=1}^{n-1} E^{\mathscr{F}}\Big((c_{i+1}u(S_{i+1}) - c_i u(S_i))m(S_i')\Big) + \sum_{i=1}^{n-1} (c_i - c_{i+1}) E^{\mathscr{F}}(u(S_i)m(S_i')) \\ &= \sum_{i=1}^{n-1} c_{i+1} E^{\mathscr{F}}\Big((u(S_{i+1}) - u(S_i))m(S_i')\Big) \\ &\geqslant \sum_{i=1}^{n-1} c_{i+1} E^{\mathscr{F}}\Big((S_{i+1} - S_i)h(S_i)m(S_i')\Big) \geqslant 0 \ \ a.s., \end{aligned} \tag{8}$$

where the last inequality follows from the definition of $\mathscr{F}$-demimartingales and the fact that $h(S_i)m(S_i')$ is a nondecreasing function of $S_1, S_2, \cdots, S_i$. Hence

$$\begin{aligned}
E^{\mathscr{F}}\left(\int_0^{S_n'} udm(u)\right) &\leqslant \sum_{i=1}^{n} c_i E^{\mathscr{F}}\Big((u(S_i)-u(S_{i-1}))m(S_n')\Big) \\
&= \sum_{i=1}^{n-1}(c_i-c_{i+1})E^{\mathscr{F}}(u(S_i)m(S_n'))+c_nE^{\mathscr{F}}(u(S_n)m(S_n')) \\
&\leqslant \sum_{i=1}^{n-1}(c_i-c_{i+1})E^{\mathscr{F}}(u(S_i)m(S_n^*))+c_nE^{\mathscr{F}}(u(S_n)m(S_n^*)) \\
&= \sum_{i=1}^{n} c_i E^{\mathscr{F}}\Big((u(S_i)-u(S_{i-1}))m(S_n^*)\Big) \ \ a.s.
\end{aligned}$$

Similarly, we have

$$E^{\mathscr{F}}\left(\int_0^{S_n'} vdm(v)\right) \leqslant \sum_{i=1}^{n} c_i E^{\mathscr{F}}\Big((v(S_i)-v(S_{i-1}))m(S_n^*)\Big) \ \ a.s. \tag{9}$$

Hence (6), (7) and (9) yield (4).

Taking $m(t)=I(t\geqslant\varepsilon)$ in (4), we get (5) immediately. □

REMARK 3. Under the same conditions as Lemma 2, Christofides and Hadjikyriakou [7] obtained the inequalities

$$\begin{aligned}
\varepsilon P^{\mathscr{F}}(S_n^*\geqslant\varepsilon) &\leqslant \sum_{i=1}^{n} c_i E^{\mathscr{F}}\Big(g(S_i)-g(S_{i-1})\Big) - c_nE^{\mathscr{F}}(g(S_n)I(S_n^*<\varepsilon)) \\
&\leqslant \sum_{i=1}^{n} E^{\mathscr{F}}\Big(c_i(g(S_i)-g(S_{i-1}))\Big) \ \ a.s.
\end{aligned}$$

3. Maximal ϕ-inequalities for conditional nonnegative demimartingales

Let $\mathscr{C}$ denote the class of *Orlicz functions*, that is, unbounded, nondecreasing convex functions $\phi:[0,\infty)\to[0,\infty)$ with $\phi(0)=0$. If the right derivative ϕ' is unbounded, then the function ϕ is called a Young function and we denote the subclass of such functions by $\mathscr{C}'$. It is easy to show that

$$\phi(x)=\int_0^x \phi'(s)ds$$

from Alsmeyer and Rösler [2].

Let

$$\mathscr{C}_0=\{\phi\in\mathscr{C}:\phi'(0)=0,\phi'(x)/x \text{ is integrable on } (0,\varepsilon) \text{ for some } \varepsilon>0\}.$$

Given $\phi\in\mathscr{C}$ and $a\geqslant 0$, define

$$\Phi_a(x)=\int_a^x\int_a^s \frac{\phi'(r)}{r}drds, \ \ x>0.$$

Set $\Phi(x) = \Phi_0(x)$, $x > 0$.

In this section, we will use the important inequality (4) to obtain some maximal ϕ-inequalities. In particular, Doob's type inequality for $\mathscr{F}$-demimartingales is presented. We will state our main results and give their proofs in the following.

THEOREM 1. *Let* $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. *Assume that* $\{S_n : n \geqslant 1\}$ *is a nonnegative* $\mathscr{F}$*-demimartingale and* $g:\mathbb{R}\to\mathbb{R}$ *is a nonnegative convex function satisfying* $g(0)=0$ *and* $E(g(S_k))^p<\infty$ *for each* $k\in\mathbb{N}$. *Suppose that* $\{c_n : n\geqslant 1\}$ *is a positive nonincreasing sequence of* $\mathscr{F}$*-measurable random variables and define* $S_n^* = \max_{1\leqslant i\leqslant n} c_i g(S_i)$. *Then for* $\phi\in\mathscr{C}_0$,

$$E^{\mathscr{F}}\phi(S_n^*) \leqslant \Big(E^{\mathscr{F}}\big(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\big)^p\Big)^{\frac{1}{p}}\Big(E^{\mathscr{F}}(\Phi'(S_n^*))^q\Big)^{\frac{1}{q}} \quad a.s. \tag{10}$$

In particular,

$$E^{\mathscr{F}}\phi\Big(\max_{1\leqslant k\leqslant n} g(S_k)\Big) \leqslant \Big(E^{\mathscr{F}}\big(g(S_n)\big)^p\Big)^{\frac{1}{p}}\Big(E^{\mathscr{F}}(\Phi'(\max_{1\leqslant k\leqslant n} g(S_k)))^q\Big)^{\frac{1}{q}} \quad a.s. \tag{11}$$

Proof. Using Lemmas 1 and 2 and the conditional Hölder inequality (c.f. [3], p. 219), we have that

$$\begin{aligned}
E^{\mathscr{F}}\phi(S_n^*) &= E^{\mathscr{F}}\Big(\int_0^{S_n^*}\phi'(t)dt\Big) = E^{\mathscr{F}}\Big(\int_0^{\infty}\phi'(t)I(S_n^*\geqslant t)dt\Big)\\
&= \int_0^{\infty} E^{\mathscr{F}}\Big(\phi'(t)I(S_n^*\geqslant t)\Big)dt \quad \text{(by (2))}\\
&= \int_0^{\infty}\phi'(t)P^{\mathscr{F}}(S_n^*\geqslant t)dt\\
&\leqslant \int_0^{\infty}\frac{\phi'(t)}{t}E^{\mathscr{F}}\Big(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))I(S_n^*\geqslant t)\Big)dt\\
&= E^{\mathscr{F}}\Big(\int_0^{S_n^*}\frac{\phi'(t)}{t}\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\Big)dt\\
&= E^{\mathscr{F}}\Big(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\Phi'(S_n^*)\Big)\\
&\leqslant \Big(E^{\mathscr{F}}\big(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\big)^p\Big)^{\frac{1}{p}}\Big(E^{\mathscr{F}}(\Phi'(S_n^*))^q\Big)^{\frac{1}{q}} \quad a.s.
\end{aligned} \tag{12}$$

□

COROLLARY 2. *Let* $\{S_n : n\geqslant 1\}$ *be a nonnegative* $\mathscr{F}$*-demimartingale. Assume that* $g:\mathbb{R}\to\mathbb{R}$ *is a nonnegative convex function satisfying* $g(0)=0$ *and* $E(g(S_k))^p<\infty$ *for each* $k\in\mathbb{N}$ *and some* $p>1$. *Suppose that* $\{c_n : n\geqslant 1\}$ *is a positive nonincreasing sequence of* $\mathscr{F}$*-measurable random variables. Then for each* $n\in\mathbb{N}$,

$$E^{\mathscr{F}}(S_n^*)^p \leqslant \Big(\frac{p}{p-1}\Big)^p E^{\mathscr{F}}\Big(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\Big)^p \quad a.s., \tag{13}$$

$$E^{\mathscr{F}} S_n^* \leqslant \frac{e}{e-1}\Big\{1+E^{\mathscr{F}}\Big(\Big(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))\Big) \times \log^+\Big(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))\Big)\Big)\Big\} \ \ a.s. \tag{14}$$

In particular,

$$E^{\mathscr{F}}(\max_{1\leqslant k\leqslant n} g(S_k))^p \leqslant \Big(\frac{p}{p-1}\Big)^p E^{\mathscr{F}}(g(S_n))^p \ \ a.s., \tag{15}$$

$$E^{\mathscr{F}}(\max_{1\leqslant k\leqslant n} g(S_k)) \leqslant \frac{e}{e-1}\{1+E^{\mathscr{F}}(g(S_n)\log^+ g(S_n))\} \ \ a.s. \tag{16}$$

Proof. By taking $\phi(x)=x^p,\ p>1$ in Theorem 1, we get $\Phi'(x)=\frac{p}{p-1}x^{p-1}$. Then (13) follows from (10).

Taking $\phi(x)=(x-1)^+$ in Theorem 1, it follows that $\Phi'(x)=\int_0^x \frac{\phi'(r)}{r}dr$. Hence by the proof of (12),

$$\begin{aligned} E^{\mathscr{F}}(S_n^*-1) &\leqslant E^{\mathscr{F}}(S_n^*-1)^+ \\ &\leqslant E^{\mathscr{F}}\Big(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))\int_0^{S_n^*}\frac{I(r\geqslant 1)}{r}dr\Big) \\ &= E^{\mathscr{F}}\Big(\Big(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))\Big)\log^+ S_n^*\Big) \ \ a.s. \end{aligned} \tag{17}$$

Using the inequality $a\log^+ b \leqslant a\log^+ a + be^{-1},\ a>0,\ b>0$, it follows that

$$E^{\mathscr{F}}(S_n^*-1) \leqslant E^{\mathscr{F}}\Big(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))\log^+\Big(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))\Big)\Big) + e^{-1}E^{\mathscr{F}}S_n^* \ \ a.s.,$$

which implies that (14) holds. □

REMARK 4. Letting $g(x)=|x|$ in (14), we have

$$E^{\mathscr{F}}(\max_{1\leqslant k\leqslant n} c_k|S_k|) \leqslant \frac{e}{e-1}\Big\{1+E^{\mathscr{F}}\Big(\Big(\sum_{j=1}^{n} c_j(|S_j|-|S_{j-1}|)\Big) \times \log^+\Big(\sum_{j=1}^{n} c_j(|S_j|-|S_{j-1}|)\Big)\Big)\Big\} \ \ a.s.$$

Taking $g(x)=|x|$ in Corollary 2, we can get the following corollary immediately, i.e., Doob's type inequality for conditional demimartingales is obtained.

COROLLARY 3. *Let* $p > 1$ *and let* $\{S_n : n \geqslant 1\}$ *be an* $\mathscr{F}$*-demimartingale. Suppose that* $E|S_k|^p < \infty$ *for each* $k \in \mathbb{N}$*, then for each* $n \in \mathbb{N}$*,*

$$E^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n}|S_k|\Big)^p \leqslant \Big(\frac{p}{p-1}\Big)^p E^{\mathscr{F}}|S_n|^p \;\; a.s., \tag{18}$$

$$E^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n}|S_k|\Big) \leqslant \frac{e}{e-1}\Big(1+E^{\mathscr{F}}(|S_n|\log^+|S_n|)\Big) \;\; a.s. \tag{19}$$

THEOREM 2. *Let* $\{S_n : n \geqslant 1\}$ *be an* $\mathscr{F}$*-demimartingale and let* $g : \mathbb{R} \to \mathbb{R}$ *be a nonnegative function satisfying* $g(0) = 0$*. Assume that* $\{c_n : n \geqslant 1\}$ *is a positive nonincreasing sequence of* $\mathscr{F}$*-measurable random variables. Denote* $T_n = \sum_{j=1}^n c_j(g(S_j) - g(S_{j-1}))$*. Then for all* $n \in \mathbb{N}$*,* $t > 0$ *and* $0 < \lambda < 1$*,*

$$\begin{aligned} P^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} c_k g(S_k) \geqslant t\Big) &\leqslant \frac{\lambda}{(1-\lambda)t}\int_t^\infty P^{\mathscr{F}}\Big(T_n \geqslant \lambda s\Big)ds \\ &= \frac{\lambda}{(1-\lambda)t} E^{\mathscr{F}}\Big(\frac{T_n}{\lambda} - t\Big)^+ \;\; a.s. \end{aligned} \tag{20}$$

Furthermore, for $\phi \in \mathscr{C}_0$*,* $n \in \mathbb{N}$*,* $a > 0$*,* $b > 0$ *and* $0 < \lambda < 1$*,*

$$\begin{aligned} E^{\mathscr{F}}\Big(\phi(\max_{1\leqslant k\leqslant n} c_k g(S_k))\Big) &\leqslant \phi(b) + \frac{\lambda}{1-\lambda}E^{\mathscr{F}}\Big\{\Phi_b\Big(\frac{T_n}{\lambda}\Big)I(T_n \geqslant \lambda b)\Big\} \\ &= \phi(b) + \frac{\lambda}{1-\lambda}E^{\mathscr{F}}\Big\{\Big(\Phi_a\Big(\frac{T_n}{\lambda}\Big) - \Phi_a(b) \\ &\quad -\Phi_a'(b)\Big(\frac{T_n}{\lambda} - b\Big)\Big)I(T_n \geqslant \lambda b)\Big\} \;\; a.s. \end{aligned} \tag{21}$$

Proof. Using Lemma 2 and Corollary 1, it follows that for $t > 0$ and $0 < \lambda < 1$,

$$\begin{aligned} P^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} c_k g(S_k) \geqslant t\Big) &\leqslant \frac{1}{t}E^{\mathscr{F}}\Big(T_n I(\max_{1\leqslant k\leqslant n} c_k g(S_k) \geqslant t)\Big) \\ &= \frac{1}{t}\int_0^\infty P^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} c_k g(S_k) \geqslant t, T_n \geqslant s\Big)ds \\ &\leqslant \frac{1}{t}\int_0^{\lambda t} P^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} c_k g(S_k) \geqslant t\Big)ds + \frac{1}{t}\int_{\lambda t}^\infty P^{\mathscr{F}}\Big(T_n \geqslant s\Big)ds \\ &= \lambda P^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} c_k g(S_k) \geqslant t\Big) + \frac{\lambda}{t}\int_t^\infty P^{\mathscr{F}}\Big(T_n \geqslant \lambda s\Big)ds \;\; a.s., \end{aligned}$$

which implies that

$$P^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} c_k g(S_k) \geqslant t\Big) \leqslant \frac{\lambda}{(1-\lambda)t}\int_t^\infty P^{\mathscr{F}}\Big(T_n \geqslant \lambda s\Big)ds \;\; a.s.$$

It is easy to check that

$$\int_t^\infty I\Big(\frac{T_n}{\lambda} \geqslant s\Big)ds = \Big(\frac{T_n}{\lambda} - t\Big)^+, \;\; t \geqslant 0.$$

Hence it follows that from (2),

$$\int_t^\infty P^{\mathscr{F}}\Big(T_n \geqslant \lambda s\Big)ds = E^{\mathscr{F}}\Big(\frac{T_n}{\lambda}-t\Big)^+ \ \ a.s., \tag{22}$$

which implies that (20) holds.

By (20), for $n\in\mathbb{N}$, $a>0$, $b>0$ and $0<\lambda<1$, we have that

$$\begin{aligned}
E^{\mathscr{F}}\Big(\phi(\max_{1\leqslant k\leqslant n} c_k g(S_k))\Big) &= \int_0^\infty \phi'(t)P^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} c_k g(S_k)\geqslant t\Big)dt \\
&= \int_0^b \phi'(t)P^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} c_k g(S_k)\geqslant t\Big)dt \\
&\quad + \int_b^\infty \phi'(t)P^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} c_k g(S_k)\geqslant t\Big)dt \\
&\leqslant \phi(b) + \int_b^\infty \phi'(t)P^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} c_k g(S_k)\geqslant t\Big)dt \\
&\leqslant \phi(b) + \frac{\lambda}{1-\lambda}\int_b^\infty \frac{\phi'(t)}{t}\Big(\int_t^\infty P^{\mathscr{F}}(T_n\geqslant \lambda s)ds\Big)dt \\
&= \phi(b) + \frac{\lambda}{1-\lambda}\int_b^\infty \frac{\phi'(t)}{t}E^{\mathscr{F}}\Big(\frac{T_n}{\lambda}-t\Big)^+dt \\
&= \phi(b) + E^{\mathscr{F}}\Big(\int_b^\infty \frac{\phi'(t)}{t}\Big(\frac{T_n}{\lambda}-t\Big)^+dt\Big) \quad \text{(by (2))} \\
&= \phi(b) + E^{\mathscr{F}}\Big(\int_b^\infty \frac{\phi'(t)}{t}\Big(\int_t^\infty I(T_n\geqslant \lambda s)ds\Big)dt\Big) \\
&= \phi(b) + E^{\mathscr{F}}\Big(\int_b^{T_n/\lambda}\Big(\int_b^s \frac{\phi'(t)}{t}dt\Big)ds\cdot I(T_n\geqslant \lambda b)\Big) \\
&= \phi(b) + E^{\mathscr{F}}\Big(\Phi_b\Big(\frac{T_n}{\lambda}\Big)I(T_n\geqslant \lambda b)\Big).
\end{aligned} \tag{23}$$

Since

$$\begin{aligned}
\Phi_a\Big(\frac{T_n}{\lambda}\Big)-\Phi_a(b)-\Phi_b\Big(\frac{T_n}{\lambda}\Big) &= \int_b^{T_n/\lambda}\int_a^s \frac{\phi'(r)}{r}drds - \int_b^{T_n/\lambda}\int_b^s \frac{\phi'(r)}{r}drds \\
&= \int_b^{T_n/\lambda}\int_a^b \frac{\phi'(r)}{r}drds = \int_a^b \frac{\phi'(r)}{r}dr\Big(\frac{T_n}{\lambda}-b\Big) \\
&= \Phi_a'(b)\Big(\frac{T_n}{\lambda}-b\Big),
\end{aligned} \tag{24}$$

(21) follows from (23). □

REMARK 5. Taking $c_k\equiv 1$ for each $k\in\mathbb{N}$ in Theorem 2, then for $n\in\mathbb{N}$, $t>0$ and $0<\lambda<1$,

$$\begin{aligned}
P^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} g(S_k)\geqslant t\Big) &\leqslant \frac{\lambda}{(1-\lambda)t}\int_t^\infty P^{\mathscr{F}}(g(S_n)>\lambda s)ds \\
&= \frac{\lambda}{(1-\lambda)t}E^{\mathscr{F}}\Big(\frac{g(S_n)}{\lambda}-t\Big)^+ \ \ a.s.
\end{aligned}$$

Furthermore, for $\phi \in \mathscr{C}_0$, $n \in \mathbb{N}$, $a > 0$, $b > 0$, $t > 0$ and $0 < \lambda < 1$,

$$\begin{aligned} E^{\mathscr{F}}\Big(\phi(\max_{1\leqslant k\leqslant n} g(S_k))\Big) &\leqslant \phi(b) + \frac{\lambda}{1-\lambda} E^{\mathscr{F}}\Big\{\Phi_b\Big(\frac{g(S_n)}{\lambda}\Big) I(T_n \geqslant \lambda b)\Big\} \\ &= \phi(b) + \frac{\lambda}{1-\lambda} E^{\mathscr{F}}\Big\{\Big(\Phi_a\Big(\frac{g(S_n)}{\lambda}\Big) - \Phi_a(b) \\ &\qquad -\Phi_a'(b)\Big(\frac{g(S_n)}{\lambda} - b\Big)\Big) I(T_n \geqslant \lambda b)\Big\} \ \ a.s. \end{aligned}$$

REMARK 6. It is easy to have that for $\phi \in \mathscr{C}_0$, $n \in \mathbb{N}$, $a > 0$ and $0 < \lambda < 1$,

$$E^{\mathscr{F}}\Big(\phi(\max_{1\leqslant k\leqslant n} g(S_k))\Big) \leqslant \phi(a) + \frac{\lambda}{1-\lambda} E^{\mathscr{F}}\Big(\Phi_a\Big(\frac{g(S_n)}{\lambda}\Big) I(g(S_n) \geqslant \lambda a)\Big) \ \ a.s. \quad (25)$$

Let $\lambda = \frac{1}{2}$ in (25), then

$$E^{\mathscr{F}}\Big(\phi(\max_{1\leqslant k\leqslant n} g(S_k))\Big) \leqslant \phi(a) + E^{\mathscr{F}}\Big(\Phi_a(2g(S_n)) I(g(S_n) \geqslant \frac{a}{2})\Big) \ \ a.s.$$

COROLLARY 4. *Let $\{S_n : n \geqslant 1\}$ be an $\mathscr{F}$-demimartingale and let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative function satisfying $g(0) = 0$ and $Eg(S_k) < \infty$ for each $k \in \mathbb{N}$. Assume that $\{c_n : n \geqslant 1\}$ is a positive nonincreasing sequence of $\mathscr{F}$-measurable random variables. Denote $T_n = \sum_{j=1}^n c_j(g(S_j) - g(S_{j-1}))$. Then for all $n \in \mathbb{N}$ and $b > 1$,*

$$E^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} c_k g(S_k)\Big) \leqslant b + \frac{b}{b-1}\Big(E^{\mathscr{F}}(T_n \log^+ T_n) - E^{\mathscr{F}}(T_n - 1)^+\Big) \ \ a.s. \quad (26)$$

Proof. Let $\phi(x) = x$ in Theorem 2, then $\Phi_1(x) = x\log x - x + 1$ and $\Phi_1'(x) = \log x$. (21) with $a = 1$ reduces that

$$\begin{aligned} E^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} c_k g(S_k)\Big) &\leqslant b + \frac{\lambda}{1-\lambda} E^{\mathscr{F}}\Big\{\Big(\frac{T_n}{\lambda}\log\frac{T_n}{\lambda} - \frac{T_n}{\lambda} + b - (\log b)\frac{T_n}{\lambda}\Big) I(T_n > \lambda b)\Big\} \\ &= b + \frac{1}{1-\lambda} E^{\mathscr{F}}\Big\{\Big(T_n \log T_n - T_n(\log\lambda + \log b + 1) \\ &\qquad + \lambda b\Big) I(T_n > \lambda b)\Big\} \ \ a.s. \end{aligned}$$

for all $b > 0$ and $0 < \lambda < 1$. Let $b > 1$ and $\lambda = \frac{1}{b}$, then

$$E^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} c_k g(S_k)\Big) \leqslant b + \frac{b}{b-1} E^{\mathscr{F}}\Big(\int_1^{T_n \vee 1} \log t\, dt\Big) \ \ a.s. \quad (27)$$

Since

$$\int_1^x \log y\, dy = x\log^+ x - (x-1), \ \ x \geqslant 1,$$

the inequality (27) can be written in the form

$$E^{\mathscr{F}}\Big(\max_{1\leqslant k\leqslant n} c_k g(S_k)\Big) \leqslant b + \frac{b}{b-1}\Big(E^{\mathscr{F}}(T_n \log^+ T_n) - E^{\mathscr{F}}(T_n - 1)^+\Big) \ \ a.s. \quad \square$$

REMARK 7. As a special case of Corollary 4, choosing $b=e$ in (26), then we get the maximal inequality

$$E^{\mathscr{F}}\left(\max_{1\leqslant k\leqslant n} c_k g(S_k)\right) \leqslant e+\frac{e}{e-1}\left(E^{\mathscr{F}}(T_n\log^+ T_n)-E^{\mathscr{F}}(T_n-1)^+\right) \ a.s.,\ n\in\mathbb{N}.$$

4. Maximal inequalities for conditional demimartingales based on concave Young Functions

Let ψ be a right continuous decreasing function on $(0,\infty)$ which satisfies the condition

$$\psi(\infty):=\lim_{t\to\infty}\psi(t)=0.$$

Assume further that ψ is also integrable with respect to the Lebesgue measure on any finite interval $(0,x)$. Let

$$\Psi(x)=\int_0^x \psi(t)dt,\ \ x\geqslant 0.$$

Then the function $\Psi(x)$ is a nonnegative increasing concave function such that $\Psi(0)=0$. If $\Psi(\infty)=\infty$, then Ψ is called a concave Young function.

For more details and properties of concave Young functions, one can refer to Agbeko [1]. An example of such a function is $\Psi(x)=x^p$, $0<p<1$.

THEOREM 3. *Let $\{S_n : n\geqslant 1\}$ be an $\mathscr{F}$-demimartingale. Let $g:\mathbb{R}\to\mathbb{R}$ be a nonnegative convex function satisfying $g(0)=0$ and $Eg(S_i)<\infty$ for every $i\in\mathbb{N}$. Suppose that $\{c_n : n\geqslant 1\}$ is a positive nonincreasing sequence of $\mathscr{F}$-measurable random variables and define $S_n^*=\max_{1\leqslant i\leqslant n} c_i g(S_i)$. Further assume that $\Psi(x)$ is a concave Young function and*

$$\int_1^\infty \frac{\psi(t)}{t}dt=C_\psi<\infty, \tag{28}$$

where C_ψ is a positive constant depending only on ψ, then

$$E^{\mathscr{F}}\Psi(S_n^*)\leqslant \Psi(1)+C_\psi\sum_{j=1}^n c_j E^{\mathscr{F}}(g(S_j)-g(S_{j-1})) \ \ a.s. \tag{29}$$

Proof. By Lemma 2, integrating the inequality (5) on $[1,\infty)$, with respect to the measure generated by the nondecreasing function $\int_0^x \frac{\psi(t)}{t}dt$, $x\geqslant 1$, it follows that from

Lemma 1 and (5) that

$$\begin{aligned}\int_1^\infty P^{\mathscr{F}}(S_n^* \geqslant t)\psi(t)dt &\leqslant \sum_{j=1}^n c_j \int_1^\infty E^{\mathscr{F}}\Big((g(S_j)-g(S_{j-1}))I(S_n^* \geqslant t)\Big)\frac{\psi(t)}{t}dt \\ &= \sum_{j=1}^n c_j E^{\mathscr{F}}\left((g(S_j)-g(S_{j-1}))\int_1^{S_n^*\vee 1}\frac{\psi(x)}{x}dx\right) \\ &= \sum_{j=1}^{n-1}(c_j-c_{j+1})E^{\mathscr{F}}\left(g(S_j)\int_1^{S_n^*\vee 1}\frac{\psi(x)}{x}dx\right) \\ &\quad + c_n E^{\mathscr{F}}\left(g(S_n)\int_1^{S_n^*\vee 1}\frac{\psi(x)}{x}dx\right) \\ &\leqslant C_\psi \sum_{j=1}^{n-1}(c_j-c_{j+1})E^{\mathscr{F}}g(S_j) + C_\psi c_n E^{\mathscr{F}}g(S_n) \\ &= C_\psi \sum_{j=1}^n c_j E^{\mathscr{F}}(g(S_j)-g(S_{j-1})) \ \ a.s. \end{aligned} \tag{30}$$

On the other hand, by Lemma 1, it follows that

$$\begin{aligned}\int_1^\infty P^{\mathscr{F}}(S_n^* \geqslant t)\psi(t)dt &= E^{\mathscr{F}}\left(\int_1^{S_n^*\vee 1}\psi(t)dt\right) \\ &= E^{\mathscr{F}}\Psi(S_n^*\vee 1) - \Psi(1) \\ &\geqslant E^{\mathscr{F}}\Psi(S_n^*) - \Psi(1) \ \ a.s. \end{aligned} \tag{31}$$

We can get (29) from (30) and (31) immediately. □

COROLLARY 5. *Let $\{S_n : n \geqslant 1\}$ be an $\mathscr{F}$-demimartingale. Let $g:\mathbb{R}\to\mathbb{R}$ be a nonnegative convex function satisfying $g(0)=0$ and $Eg(S_i)<\infty$ for every $i\in\mathbb{N}$. Suppose that $\{c_n : n\geqslant 1\}$ is a positive nonincreasing sequence of $\mathscr{F}$-measurable random variables and define $S_n^* = \max_{1\leqslant i\leqslant n} c_i g(S_i)$. Let $0<p<1$. Then for every $n\in\mathbb{N}$,*

$$E^{\mathscr{F}}(S_n^*)^p \leqslant 1 + \frac{p}{1-p}\sum_{j=1}^n c_j E^{\mathscr{F}}(g(S_j)-g(S_{j-1})) \ \ a.s. \tag{32}$$

In particular,

$$E^{\mathscr{F}}\left(\max_{1\leqslant k\leqslant n} g(S_k)\right)^p \leqslant 1 + \frac{p}{1-p}E^{\mathscr{F}}g(S_n) \ \ a.s. \tag{33}$$

Proof. Taking $\Psi(x)=x^p$, $0<p<1$ in Theorem 3, it follows that $\psi(x)=px^{p-1}$ and $C_\psi = \int_1^\infty \frac{\psi(t)}{t}dt = \frac{p}{1-p}$. Thus, (29) implies (32). If $c_k\equiv 1$ for each $k\in\mathbb{N}$ in (32), we get (33) immediately. □

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