Journal of Mathematical Inequalities
Volume 8, Number 3 (2014), 559–579 doi:10.7153/jmi-08-42

L_p–MIXED INTERSECTION BODIES

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(*Communicated by G. Sinnamon*)

Abstract. This paper presents the concept of L_p-mixed intersection body, and study the monotonicity of the operator $I_{p,i}$ for L_p-mixed intersection body. Meanwhile, we establish Busemann-type inequalities and the dual Brunn-Minkowski type inequalities for L_p-mixed intersection body.

1. Introduction

Projection body was originally invented by Minkowski [1] in 1934. Minkowski mapping has a special role in the Banach spaces based on the fact that affine equivalent convex body has the properties of affine equivalent projection body. The research on the projection body has attracted much attention. Because the projection body has a variety of new applications such as the combinatorics, the stereology, the stochastic geometry, the random determinants, etc.

In [6], Lutwak presented the concept of intersection body and given the duality relation between the projection body and intersection body.

We recall the definition of the L_p-projection body of a convex body in $\mathbb{R}^n$ (see [7]): For every convex body K in $\mathbb{R}^n$ and real $p \geqslant 1$, the L_p-projection body, $\Pi_p K$, of K is an origin-symmetric convex body whose support function is given by

$$h(\Pi_p K, u) = \left(\frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v) \right)^{\frac{1}{p}}, \text{ for all } u \in S^{n-1}, \tag{1}$$

where

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}.$$

The unusual normalization of the definition (1) is chosen so that for the unit ball B in $\mathbb{R}^n$, we have $\Pi_p B = B$.

Here we use the L_p-mixed volume to rewrite the definition of L_p-projection body that its intrinsic geometric nature is even more clear:

$$h(\Pi_p K, u) = \left(\frac{V_p(K, [-u, u])}{V_p(B, [-u, u])} \right)^{\frac{1}{p}}, \text{ for all } u \in S^{n-1}, \tag{2}$$

Mathematics subject classification (2010): 52A40, 52A20.
Keywords and phrases: Convex body, star body, intersection body, L_p-mixed intersection body.
This work was supported by the National Natural Science Foundation of China (Grant No. 11161019, 11371224).

where $[-u,u]$ denotes the line segment joining $-u$ and u.

Similar to the definition of L_p-projection body, Wang and Leng in [14] gave the definition of L_p-mixed projection body as follow: Let K be a convex body origin in its interior, real $p \geqslant 1$ and $i = 0,1,\cdots,n-1$, the L_p-mixed projection body, $\Pi_p K$, of K be origin-symmetric convex body whose support function is given by

$$h(\Pi_{p,i}K,u) = \left(\frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |u\cdot v|^p dS_{p,i}(K,v)\right)^{\frac{1}{p}}, \ \text{for all } u \in S^{n-1}. \tag{3}$$

Here the positive Borel measure $S_{p,i}(K,\cdot)$ on S^{n-1} is called the L_p-mixed surface area measure of K which is introduced by Lutwak (see [8]).

Using the concept of mixed quermassintegrals, we can rewrite the L_p-mixed projection body as follows:

$$h(\Pi_{p,i}K,u) = \left(\frac{W_{p,i}(K,[-u,u])}{W_{p,i}(B,[-u,u])}\right)^{\frac{1}{p}}, \ \text{for all } u \in S^{n-1}. \tag{4}$$

There is a duality relation between the projection body and intersection body, from this point, we believe that there is a certain duality relations between the L_p-mixed projection body and the mixed intersection body. So that we can use the dual L_p-mixed quermassintegrals to definite the L_p-mixed intersection bodies are as follows:

DEFINITION 1.1. Let K be a star body in $\mathbb{R}^n$, $p \geqslant 1$ and $i \in \mathbb{R}$, the L_p-mixed intersection body, $I_{p,i}K$, of K is defined by

$$\rho(I_{p,i}K,u) = \left(\frac{\widetilde{W}_{p,i}(K,B\cap u^{\perp})}{\widetilde{W}_{p,i}(B,B\cap u^{\perp})}\right)^{\frac{1}{p}} = \left(\frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1}\cap u^{\perp}} \rho_K^{n-p-i}(v)dS_{n-2}(v)\right)^{\frac{1}{p}} \tag{5}$$

for all $u \in S^{n-1}$. Where $\widetilde{W}_{p,i}(K,L)$ denotes the dual L_p-mixed quermassintegrals of star bodies K and L in $\mathbb{R}^n$. The definition of $\widetilde{W}_{p,i}(K,L)$ will be introduced in Section 2.

REMARK. The intersection body was first defined by Lutwak in [6], after the unusual normalization we can give the definition of the classic intersection body as follows:

$$\rho(IK,u) = \frac{v(K\cap u^{\perp})}{v(B\cap u^{\perp})} = \frac{\widetilde{V}_1(K,B\cap u^{\perp})}{\widetilde{V}_1(B,B\cap u^{\perp})}, \ \text{for all } u \in S^{n-1}, \tag{6}$$

where $v(\cdot)$ denote $(n-1)$-dimensional volume of convex body.

1994, Zhang in [17] further proposed the concept of ith intersection body, and ith intersection body of the unusual normalization are defined as follows:

$$\rho(I_iK,u) = \frac{\widetilde{w}_i(K\cap u^{\perp})}{\widetilde{w}_i(B\cap u^{\perp})} = \frac{\widetilde{W}_{1,i}(K,B\cap u^{\perp})}{\widetilde{W}_{1,i}(B,B\cap u^{\perp})}, \ \text{for all } u \in S^{n-1}, \tag{7}$$

where $\widetilde{w}_i(\cdot)$ denote $(n-1)$-dimensional ith dual quermassintegrals of star body. From the definition of L_p-mixed intersection body, we have

$$I_{1,i}K = I_iK, \ \ I_{1,0} = IK.$$

From a geometric point of view, the L_p-mixed intersection body which we defined is a generalization of the classical intersection body.

The main purpose of this paper is to study the monotonicity of the operator $I_{p,i}$ base on our definition of L_p-mixed intersection body $I_{p,i}K$. Also the Busemann-type inequalities for L_p-mixed intersection bodies and the dual Brunn-Minkowski type inequalities for L_p-mixed intersection bodies are established.

The structural arrangements of this article is: In Section 2 we introduces some notations in convex geometry and preparation knowledge in order to discuss in this article; In Section 3 we discusses the property of operator $I_{p,i}$ of L_p-mixed intersection bodied; In Section 4 we will establish the Busemann-type inequalities for L_p-mixed intersection bodies; In Section 5 we further propose and prove that dual Brunn-Minkowski type inequalities of L_p-intersection bodies.

2. Notation and ready knowledge

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geqslant 2)$. Let $\mathscr{K}^n$ denotes the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies, we write as $\mathscr{K}_o^n$ and $\mathscr{K}_c^n$, respectively. Let $\mathscr{S}_o^n$ denotes the set of all star bodies (about the origin) in $\mathbb{R}^n$, $\mathscr{S}_c^n$ denotes set of all origin-symmetric star bodies in $\mathscr{S}_o^n$. Let B denote the unit ball in $\mathbb{R}^n$ with the unit sphere S^{n-1}. We use $V(K)$ for n-dimensional volume of geometry body K and denote $\omega_n = V(B)$.

If $K \in \mathscr{K}^n$, then its support function, $h_K = h(K,\cdot) : \mathbb{R}^n \longrightarrow (0,\infty)$, is defined by

$$h(K,x) = \max\{(x,y)|y \in K\}, \text{ for all } x \in \mathbb{R}^n,$$

where (x,y) denotes the standard inner product of x and y in $\mathbb{R}^n$.

The proof of Proposition 3.1 in Section 3 we will also be used a property of the inner product (see [3]): If $\phi \in GL_n, x, y \in \mathbb{R}^n$, then

$$(x,\phi y) = (\phi^t x, y), \tag{8}$$

where GL_n denotes the non-singular affine (or linear) transformation group, ϕ^t denote transpose of the transformation ϕ.

If $L \in \mathscr{S}_o^n$, its radial function about the origin, $\rho_L(x) = \rho(L,\cdot) : \mathbb{R}^n \setminus \{0\} \longrightarrow [0,\infty)$, is defined by

$$\rho(L,x) = \max\{\lambda \geqslant 0|\lambda x \in L\}, \text{ for all } x \in \mathbb{R}^n \setminus \{0\}.$$

Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

From the definition of radial function, we know that: If $K, L \in \mathscr{S}_o^n$ and any $x \in \mathbb{R}^n$, then (see [3,13])

(i) For the radial function ρ_K, if $c > 0$, then

$$\rho_K(cx) = c^{-1}\rho_K(x);$$

(ii) For the radial function ρ_K, if $\phi \in GL_n$, then

$$\rho_{\phi K}(x) = \rho_K(\phi^{-1}x),$$

where ϕ^{-1} denotes inverse transform of ϕ;

(iii) For $\lambda > 0, \rho_K(u) \leqslant \lambda \rho_L(u)$ for any $u \in S^{n-1}$ if and only if $K \subseteq \lambda L$.

If $K \in \mathscr{K}_o^n$, the polar body, K^*, of K is defined by

$$K^* = \{x \in \mathbb{R}^n | (x,y) \leqslant 1, y \in K\}.$$

From the definitions of the support and radial functions and the definition of the polar body, it follows that for $K \in \mathscr{K}_o^n$ and any $u \in S^{n-1}$,

$$h(K^*,u) = \frac{1}{\rho(K,u)}, \text{ and } \rho(K^*,u) = \frac{1}{h(K,u)}.$$

Let f be a Borel function on S^{n-1}, the spherical Radon transform of the function f is defined by (see [4])

$$(Rf)(u) = \int_{S^{n-1}\cap u^{\perp}} f(v)dS_{n-2}(v). \tag{9}$$

Using the spherical Radon transform, we can rewrite the definition of L_p-mixed intersection body $I_{p,i}K$ as follows

$$\rho(I_{p,i}K,u)^p = \frac{1}{(n-1)\omega_{n-1}} R\big(\rho_K^{n-p-i}\big)(u), \text{ for any } u \in S^{n-1}. \tag{10}$$

Spherical Radon transform has the following two important properties:

(i) The spherical Radon transform is a continuous bijection from $C_e^{\infty}(S^{n-1})$ to itself;

(ii) The spherical Radon transform is self-adjoint, i.e., if f and g are defined on S^{n-1} bounded Borel functional, then

$$\int_{S^{n-1}} f(u)Rg(u)dS(u) = \int_{S^{n-1}} Rf(u)g(u)dS(u). \tag{11}$$

Radial sum of the vector $x_1, x_2, \cdots, x_r$ in $\mathbb{R}^n$ is defined as follows: If $x_1, x_2, \cdots, x_r$ are coplanar lines, then $x_1 \tilde{+} \cdots \tilde{+} x_r$ is the usual vector addition, otherwise it is zero vector.

If $K_1, K_2, \cdots, K_r \in \mathscr{S}_o^n$ and $\lambda_1, \lambda_2, \cdots, \lambda_r \in \mathbb{R}$, then the radial Minkowski linear combination is defined as (see [9]):

$$\lambda_1 K_1 \tilde{+} \cdots + \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r | x_i \in K_i, i = 1, 2, \cdots, r\}.$$

It is easy to show that for any $K, L \in \mathscr{S}_o^n$ and $\alpha, \beta \geqslant 0$,

$$\rho(\alpha K \tilde{+} \beta L, u) = \alpha\rho(K,u) + \beta\rho(L,u). \tag{12}$$

Using the polar coordinate formula of the volume, then the volume of the radial Minkowski linear combination $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$ is defined as

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum_{1 \leqslant i_1, \cdots, i_n \leqslant r} \widetilde{V}(K_{i_1}, \cdots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}, \tag{13}$$

where the sum is taken over all n-tuples $(i_1, \cdots, i_n)$ of positive integers not exceeding r. The coefficient $\widetilde{V}(K_{i_1}, \cdots, K_{i_n})$ depends only on the bodies $K_{i_1}, \cdots, K_{i_n}$, and is uniquely determined by (13), it is called the dual mixed volume of star bodies $K_{i_1}, \cdots, K_{i_n}$.

If $K_1, \cdots, K_n \in \mathscr{S}_o^n$, then the integral representation of the dual mixed volume $\widetilde{V}(K_1, \cdots, K_n)$ of $K_1, \cdots, K_n$ can be expressed as follows:

$$\widetilde{V}(K_1, \cdots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{K_1}(u) \cdots \rho_{K_n}(u) dS(u). \tag{14}$$

In particular, $\widetilde{V}(K, \cdots, K) = V(K)$.

If $K, L \in \mathscr{S}_o^n$, then

$$\widetilde{V}_i(K,L) = \widetilde{V}(\underbrace{K, \cdots, K}_{n-i}, \underbrace{L, \cdots, L}_{i}) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-i} \rho_L(u)^i dS(u).$$

Apparently, for $K, L, M \in \mathscr{S}_o^n$ and $\alpha, \beta \geqslant 0$, we have

$$\widetilde{V}_{n-1}(\alpha K \tilde{+} \beta L, M) = \alpha \widetilde{V}_{n-1}(K, M) + \beta \widetilde{V}_{n-1}(L, M).$$

In particular, $\widetilde{V}_i(K, B)$ is said to be an ith dual quermassintegrals of star body K and is written as $\widetilde{W}_i(K)$, namely

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-i} dS(u). \tag{15}$$

DEFINITION 2.1. For $K, L \in \mathscr{S}_o^n$, real $p > 0$ and α, $\beta \geqslant 0$ (not both zero), the L_p-radial linear combination, $\alpha \cdot K \tilde{+}_p \beta \cdot L$, of K and L is the star body whose radial function is defined by

$$\rho(\alpha \cdot K \tilde{+}_p \beta \cdot L, \cdot)^p = \alpha \rho(K, \cdot)^p + \beta \rho(L, \cdot)^p.$$

Corresponds to the L_p-mixed quermassintegrals and the L_p-radial linear combination, we will introduce the concept of dual L_p-mixed quermassintegrals as follows:

DEFINITION 2.2. If $K, L \in \mathscr{S}_o^n$, for real $p \geqslant 1$ and real $i \neq n$, the dual L_p-mixed quermassintegrals, $\widetilde{W}_{p,i}(K, L)$, of K and L is defined by

$$\frac{n-i}{p} \widetilde{W}_{p,i}(K, L) = \lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K \tilde{+}_p \varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon}.$$

Apparently, for $p = 1$, $\widetilde{W}_{p,i}(K, L) = \widetilde{W}_i(K, L)$; For any $p \geqslant 1$,

$$\widetilde{W}_{p,i}(K, K) = \widetilde{W}_i(K). \tag{16}$$

According to the above definition 2.2 and the polar coordinate formula for volume, we easily get the following integral representation of the dual L_p-mixed quermassintegrals $\widetilde{W}_{p,i}(K, L)$ of $K, L \in \mathscr{S}_o^n$.

LEMMA 2.1. *If $K,L \in \mathscr{S}_o^n$, for real $p \geqslant 1$ and real $i \neq n$, $i \neq n-p$, the integral representation of dual L_p-mixed quermassintegrals, $\widetilde{W}_{p,i}(K,L)$, of K and L can be expressed as follows:*

$$\widetilde{W}_{p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i-p}\rho(L,u)^p dS(u). \tag{17}$$

Proof. According to the definition of $\widetilde{W}_{p,i}(K,L)$ and $\widetilde{W}_i(K)$, together with the dual L_p-radial linear property, we have

$$\begin{aligned}
\frac{n-i}{p}\widetilde{W}_{p,i}(K,L) &= \lim_{\varepsilon\to 0^+} \frac{\frac{1}{n}\int_{S^{n-1}}\left(\rho(K\tilde{+}_p\varepsilon\cdot L,u)^{n-i} - \rho(K,u)^{n-i}\right)dS(u)}{\varepsilon} \\
&= \frac{1}{n}\int_{S^{n-1}} \lim_{\varepsilon\to 0^+} \frac{\left(\rho(K,u)^p + \varepsilon\rho(L,u)^p\right)^{\frac{n-i}{p}} - \rho(K,u)^{n-i}}{\varepsilon} dS(u) \\
&= \frac{n-i}{np}\int_{S^{n-1}} \rho(K,u)^{n-i-p}\rho(L,u)^p dS(u).
\end{aligned}$$

This complete the proof of Lemma 2.1. □

In the following we will give the dual L_p-mixed Minkowski inequality.

LEMMA 2.2. *If $K,L \in \mathscr{S}_o^n$, and $p \geqslant 1$, while real $i \neq n, n-p$. Then for $i < n-p$,*

$$\widetilde{W}_{p,i}(K,L)^{n-i} \leqslant \widetilde{W}_i(K)^{n-i-p}\widetilde{W}_i(L)^p, \tag{18}$$

with equality if and only if K and L are dilates. For $n-p < i < n$ or $i > n$, inequality (18) is reversed.

Proof. First we need the following extended Hölder's inequality (see [5]): Suppose f and g are two almost everywhere nonnegative bounded Borel functions on a set X, and real $p,q \in \mathbb{R}^+$ satisfies $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\int_X f(x)g(x)d\mu(x) \leqslant \left(\int_X f(x)^p d\mu(x)\right)^{\frac{1}{p}} \left(\int_X g(x)^p d\mu(x)\right)^{\frac{1}{q}}. \tag{19}$$

If one is negative of both p and q, the above inequality is reversed. For both cases the equality holds if and only if one of the two functions for measure μ is almost everywhere zero, or there exists insufficiency zero non-negative constants a,b, such that $af^p = bg^q$ for the measure μ almost everywhere in the establishment.

Therefore, for $i < n-p$, we use integral representation (17) of $\widetilde{W}_{p,i}(K,L)$ and Hölder's inequality (19) can be obtained

$$\begin{aligned}
\widetilde{W}_{p,i}(K,L) &= \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i-p}\rho(L,u)^p dS(u) \\
&\leqslant \left(\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i}dS(u)\right)^{\frac{n-i-p}{n-i}} \left(\frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{n-i}dS(u)\right)^{\frac{p}{n-i}} \\
&= \widetilde{W}_i(K)^{\frac{n-i-p}{n-i}}\widetilde{W}_i(L)^{\frac{p}{n-i}}.
\end{aligned}$$

Thus, we obtain inequality (18). For $n-p<i<n$ or $i>n$ we similarly can prove the reverse inequality.

Note that the equality holds in Hölder's inequality if and only if $\rho(K,u)^{n-i}=c\rho(L,u)^{n-i}$ (c is a constant), i.e., $\rho(K,u)=\lambda\rho(L,u)$ (λ is a constant). Thus the equality holds in inequality (18) if and only if K and L are dilates. □

LEMMA 2.3. *Suppose $K,L\in\mathscr{S}_o^n$, and $i\in\mathbb{R}, n-i\neq p\geqslant 1$. If*

$$\widetilde{W}_{p,i}(K,Q)=\widetilde{W}_{p,i}(L,Q) \text{ or } \widetilde{W}_{p,i}(Q,K)=\widetilde{W}_{p,i}(Q,L), \text{ for all } Q\in\mathscr{S}_o^n,$$

then $K=L$. And vice versa.

Proof. We only show the first conclusion. Taking $Q=K$, using the known conditions $\widetilde{W}_{p,i}(K,Q)=\widetilde{W}_{p,i}(L,Q)$, we have

$$\widetilde{W}_i(K)=\widetilde{W}_{p,i}(K,K)=\widetilde{W}_{p,i}(L,K),$$

for $i<n-p$, using Lemma 2.2, it is easy to know

$$\widetilde{W}_i(K)^{n-i}\leqslant\widetilde{W}_i(L)^{n-i-p}\widetilde{W}_i(K)^p.$$

then $\widetilde{W}_i(K)\leqslant\widetilde{W}_i(L)$, with equality if and only if L and K are dilates.

For $i<n-p$, taking $Q=L$, we have $\widetilde{W}_i(L)\leqslant\widetilde{W}_i(K)$ with equality if and only if K and L are dilates.

Thus, $\widetilde{W}_i(K)=\widetilde{W}_i(L)$, and K and L are dilates each other. Therefore $K=L$.

Similarly, we can prove that for $n-p<i<n$ or $i>n$, $\widetilde{W}_{p,i}(K,Q)=\widetilde{W}_{p,i}(L,Q)$ is also implication $K=L$. We have completed the proof of Lemma 2.3. □

We need to use the famous Jenesen inequality in the proof of Proposition 3.3 (see Reference [5], Chapter 6). Suppose $p\neq 0,\mu$ is a finite Borel measure in set X, and f and ω are almost everywhere non-negative μ-integrable functions on X. Then pth weighted mean, $M_{p,\omega}f$, of f is defined as

$$M_{p,\omega}f=\left(\frac{1}{\omega(X)}\int_X f(x)^p\omega(x)d\mu(x)\right)^{\frac{1}{p}},$$

$$\lim_{p\to\infty}M_{p,\omega}f=ess\sup_{x\in X}f(x),$$

there $\omega(X)=\int_X\omega(x)d\mu(x)$. And we are easy to know

$$\lim_{p\to 0}M_{p,\omega}f=\exp\left(\frac{1}{\omega(X)}\int_X\omega(x)\log f(x)d\mu(x)\right).$$

Jensen's inequality may be stated that: If $p,q\neq 0$, $p\leqslant q$, and both $M_{p,\omega}f$ and $M_{q,\omega}f$ are existence, then

$$M_{p,\omega}f\leqslant M_{q,\omega}f, \tag{20}$$

with equality for $p\neq q$ if and only if f is a constant or if and only if $p=q$.

The following the concept of L_p-mixed harmonic Blaschke plus $K\dot{+}_pL$ will be used later.

DEFINITION 2.3. [12] Suppose $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$, and any real $i\neq n$, $i\neq n+p$. We denote $\xi>0$, i.e.,

$$\xi^{\frac{p}{n+p-i}}=\frac{1}{n}\int_{S^{n-1}}\left[\widetilde{W}_i(K)^{-1}\rho(K,u)^{n+p-i}+\widetilde{W}_i(L)^{-1}\rho(L,u)^{n+p-i}\right]^{\frac{n-i}{n+p-i}}dS(u). \quad (21)$$

Then the L_p-mixed harmonic Blaschke plus, $K\dot{+}_pL\in\mathscr{S}_o^n$, of K and L is a star body contains the origin in its internal whose radial function is defined by

$$\xi^{-1}\rho(K\dot{+}_pL,\cdot)^{n+p-i}=\widetilde{W}_i(K)^{-1}\rho(K,u)^{n+p-i}+\widetilde{W}_i(L)^{-1}\rho(L,u)^{n+p-i}. \quad (22)$$

For $i=0$, Definition 2.3 is introduced by Yuan in [15] whose it is called L_p-harmonic Blaschke plus and write $K\bar{+}_pL$. For $i=0$, $p=1$, Definition 2.3 is introduced by Lutwak in [10] whose it is called harmonic Blaschke plus and write $K\hat{+}L$.

3. The property of the operator $I_{p,i}$

PROPOSITION 3.1. *Suppose $K\in\mathscr{S}_o^n$, and real $p\geqslant 1$. If $\phi\in GL_n$, then*

$$I_{p,i}\phi K=|\det\phi|^{-\frac{1}{p}}\phi^{-t}(I_{p,i}K).$$

Proof. Let $u\in S^{n-1}$. Using three properties (i), (ii), (iii) of the radial functions, and the property (8) of the inner product, we have

$$\begin{aligned}
\rho(I_{p,i}\phi K,u)^p &= \frac{1}{(n-1)\omega_{n-1}}\int_{S^{n-1}\cap u^{\perp}}\rho_{\phi K}^{n-p-i}(v)dS_{n-2}(v)\\
&= \frac{1}{(n-1)\omega_{n-1}}\int_{\{v:(v,u)=0\}}\rho_K^{n-p-i}(\phi^{-1}v)dS_{n-2}(v)\ \ (\text{Let } y=\phi^{-1}v)\\
&= \frac{1}{(n-1)\omega_{n-1}}\int_{\{\phi y:(\phi y,u)=0\}}\rho_K^{n-p-i}(y)|\det\phi|dS_{n-2}(y)\\
&= |\det\phi|\frac{1}{(n-1)\omega_{n-1}}\int_{\{y:(y,\phi^t u)=0\}}\rho_K^{n-p-i}(y)dS_{n-2}(y)\\
&= |\det\phi|\rho(I_{p,i}K,\phi^t u)^p\\
&= |\det\phi|\rho(\phi^{-t}I_{p,i}K,u)^p.
\end{aligned}$$

From this, we obtain Proposition 3.1. The proof of Proposition 3.1 is completed. □

PROPOSITION 3.2. *If $K,L\in\mathscr{S}_o^n, p\geqslant 1$, and $i,j\in\mathbb{R}$, then*

$$\widetilde{W}_{p,j}(K,I_{p,i}L)=\widetilde{W}_{p,i}(L,I_{p,j}K). \quad (23)$$

Proof. Using Lemma 2.1, (10) and (11), we easily show that the results (23).
Now define a class

$$Z_{p,i}^p=\{I_{p,i}K : K\in\mathscr{S}_o^n\}.$$

We establish the monotonicity of the operator $I_{p,i}$ $(p\geqslant 1)$, the result is the following proposition. □

PROPOSITION 3.3. *Let* $K, L \in \mathscr{S}_o^n$, *and real* $p \geqslant 1$. *Then*
(a) If $p \leqslant q < \infty$, *then*

$$\left(\frac{\omega_{n-1}}{\widetilde{w}_i(K \cap u^{\perp})}\right)^{\frac{1}{p}} I_{p,i}K \subseteq \left(\frac{\omega_{n-1}}{\widetilde{w}_i(K \cap u^{\perp})}\right)^{\frac{1}{q}} I_{q,i}K,$$

with equality if and only if $p = q$ *or* K *is a ball centered at the origin.*
(b) Let $K \in \mathscr{S}_o^n$ *and any real* $i \in \mathbb{R}$. *If*

$$\widetilde{W}_{p,i}(K,Q) \leqslant \widetilde{W}_{p,i}(L,Q)$$

holds for all $Q \in \mathscr{S}_o^n$, *then*

$$\widetilde{W}_j(I_{p,i}K) \leqslant \widetilde{W}_j(I_{p,i}L) \text{ for } j < n-p,$$

and

$$\widetilde{W}_j(I_{p,i}K) \geqslant \widetilde{W}_j(I_{p,i}L) \text{ for } j > n-p,$$

with equality if and only if $K = L$.
(c) Let $K \in Z_{p,i}^n$ *and* $L \in \mathscr{S}_o^n$. *If for* $1 \leqslant p < n-i$ *satisfying* $I_{p,i}K \subseteq I_{p,i}L$, *then*

$$\widetilde{W}_i(K) \leqslant \widetilde{W}_i(L),$$

with equality if and only if $K = L$.
Let $L \in Z_{p,i}^n$ *and* $K \in \mathscr{S}_o^n$. *If for* $p > n-i$ *satisfying* $I_{p,i}K \subseteq I_{p,i}L$, *then*

$$\widetilde{W}_i(K) \geqslant \widetilde{W}_i(L),$$

with equality if and only if $K = L$.
(d) If $K \subseteq L$, *then*

$$I_{p,i}K \subseteq I_{p,i}L \text{ for } i < n-p,$$

and

$$I_{p,i}K \supseteq I_{p,i}L \text{ for } i > n-p,$$

with equality if and only if $K = L$.

Proof. (a) Note that for any $u \in S^{n-1}$, we have

$$\int_{S^{n-1} \cap u^{\perp}} \rho_K^{n-i}(v) dS_{n-2}(v) = (n-1)\widetilde{w}_i(K \cap u^{\perp}).$$

From the definition 1.1 of L_p-mixed intersection body, Jenesen's inequality (20) and the property (iii) of the radial functions, we have

$$\begin{aligned}
&\rho(I_{p,i}K,u)\\
&=\left(\frac{1}{(n-1)\omega_{n-1}}\int_{S^{n-1}\cap u^{\perp}}\rho_K^{n-p-i}(v)dS_{n-2}(v)\right)^{\frac{1}{p}}\\
&=\left(\frac{\widetilde{w}_i(K\cap u^{\perp})}{\omega_{n-1}}\right)^{\frac{1}{p}}\left(\frac{1}{(n-1)\widetilde{w}_i(K\cap u^{\perp})}\int_{S^{n-1}\cap u^{\perp}}\left(\frac{1}{\rho_K(v)}\right)^{p}\rho_K^{n-i}(v)dS_{n-2}(v)\right)^{\frac{1}{p}}\\
&\leqslant\left(\frac{\widetilde{w}_i(K\cap u^{\perp})}{\omega_{n-1}}\right)^{\frac{1}{p}}\left(\frac{1}{(n-1)\widetilde{w}_i(K\cap u^{\perp})}\int_{S^{n-1}\cap u^{\perp}}\left(\frac{1}{\rho_K(v)}\right)^{q}\rho_K^{n-i}(v)dS_{n-2}(v)\right)^{\frac{1}{q}}\\
&=\left(\frac{\widetilde{w}_i(K\cap u^{\perp})}{\omega_{n-1}}\right)^{\frac{1}{p}}\left(\frac{\omega_{n-1}}{\widetilde{w}_i(K\cap u^{\perp})}\right)^{\frac{1}{q}}\rho(I_{q,i}K,u).
\end{aligned}$$

Namely,

$$\left(\frac{\omega_{n-1}}{\widetilde{w}_i(K\cap u^{\perp})}\right)^{\frac{1}{p}}\rho(I_{p,i}K,u)\leqslant\left(\frac{\omega_{n-1}}{\widetilde{w}_i(K\cap u^{\perp})}\right)^{\frac{1}{q}}\rho(I_{q,i}K,u).$$

Therefore, we get

$$\left(\frac{\omega_{n-1}}{\widetilde{w}_i(K\cap u^{\perp})}\right)^{\frac{1}{p}}I_{p,i}K\subseteq\left(\frac{\omega_{n-1}}{\widetilde{w}_i(K\cap u^{\perp})}\right)^{\frac{1}{q}}I_{q,i}K.$$

We easily verify equation is true if and only if $p=q$. Or according to the conditions of equality hold in Jenesen's inequality, we know that the equality holds in (a) for $p\neq q$ if and only if $\rho_K(v)$ is a constant for $v\in S^{n-1}$. Namely, K is a ball contains at the origin.

(b) For any $Q\in\mathscr{S}_o^n$ and real $p\geqslant 1$, we have that $\widetilde{W}_{p,i}(K,Q)\leqslant\widetilde{W}_{p,i}(L,Q)$. Taking $Q=I_{p,i}M$ with $M\in\mathscr{S}_o^n$, we have

$$\widetilde{W}_{p,i}(K,I_{p,i}M)\leqslant\widetilde{W}_{p,i}(L,I_{p,i}M). \tag{24}$$

By Lemma 2.3 we know that the equality holds in (24) if and only if $K=L$. Using Proposition 3.2 in (24), we get

$$\widetilde{W}_{p,j}(M,I_{p,i}K)\leqslant\widetilde{W}_{p,j}(M,I_{p,i}L).$$

For $j<n-p$, taking $M=I_{p,i}K$, using (16) and L_p-mixed Minkowski inequality (18), the above inequality can be turned into

$$\widetilde{W}_j(I_{p,i}K)^{n-j}\leqslant\widetilde{W}_{p,j}(I_{p,i}K,I_{p,i}L)^{n-j}\leqslant\widetilde{W}_j(I_{p,i}K)^{n-p-j}\widetilde{W}_j(I_{p,i}L)^{p}.$$

Simplified it into

$$\widetilde{W}_j(I_{p,i}K)\leqslant\widetilde{W}_j(I_{p,i}L), \tag{25}$$

with equality if and only if $I_{p,i}K$ and $I_{p,i}L$ are dilates.

According to the condition of equality hold in the inequalities (24) and (25), we know that the equality hold in (25) if and only if $K=L$.

For $n-p<j<n$ or $i>n$, using the same argument as in the first part of the proof, we get

$$\widetilde{W}_j(I_{p,i}K) \geqslant \widetilde{W}_j(I_{p,i}L).$$

(c) If $K \in Z^n_{p,i}, L \in \mathscr{S}^n_o$ and $1 \leqslant p < n-i$. Let $M \in \mathscr{S}^n_o$, from Proposition 3.2 and Lemma 2.1, we can get

$$\widetilde{W}_{p,i}(K, I_{p,i}M) = \widetilde{W}_{p,i}(M, I_{p,i}K) = \frac{1}{n}\int_{S^{n-1}} \rho_M(u)^{n-i-p}\rho_{I_{p,i}K}(u)^p dS(u).$$

Similarly,

$$\widetilde{W}_{p,i}(L, I_{p,i}M) = \widetilde{W}_{p,i}(M, I_{p,i}L) = \frac{1}{n}\int_{S^{n-1}} \rho_M(u)^{n-i-p}\rho_{I_{p,i}L}(u)^p dS(u).$$

According to the condition $I_{p,i}K \subseteq I_{p,i}L$, we can get

$$\widetilde{W}_{p,i}(K, I_{p,i}M) \leqslant \widetilde{W}_{p,i}(L, I_{p,i}M) \text{ for any } M \in \mathscr{S}^n_o. \tag{26}$$

For each fixed $i < n-p$, taking $I_{p,i}M = K$ in (26), and using the inequality (18), we get

$$\widetilde{W}_i(K) = \widetilde{W}_{p,i}(K,K) \leqslant \widetilde{W}_{p,i}(L,K) \leqslant \widetilde{W}_i(L)^{\frac{n-i-p}{n-i}} \widetilde{W}_i(K)^{\frac{p}{n-i}}.$$

By this inequality, we immediately get

$$\widetilde{W}_i(K) \leqslant \widetilde{W}_i(L).$$

According to the condition of equality hold in the inequality (18), and known conditions $I_{p,i}K \subseteq I_{p,i}L$, it is easy to know the equality holds in $\widetilde{W}_i(K) \leqslant \widetilde{W}_i(L)$ if and only if $K = L$.

If$L \in Z^n_{p,i}, K \in \mathscr{S}^n_o$ and $p > n-i$. Taking $I_{p,i}M = L$ in (26), and using the inequality (18), we have

$$\widetilde{W}_i(L) = \widetilde{W}_{p,i}(L,L) \geqslant \widetilde{W}_{p,i}(K,L) \geqslant \widetilde{W}_i(K)^{\frac{n-i-p}{n-i}} \widetilde{W}_i(L)^{\frac{p}{n-i}}.$$

Therefore, $\widetilde{W}_i(K) \geqslant \widetilde{W}_i(L)$, with equality holds if and only if $K = L$.

(d) By the property (iii) of the radial functions, for any $u \in S^{n-1}$, $\rho_K(u) \leqslant \rho_L(u)$. Therefore, for any $u \in S^{n-1}$ and $i < n-p$, $\rho_K^{n-p-i}(u) \leqslant \rho_L^{n-p-i}(u)$. Together with the definition of L_p-mixed intersection body, we have $\rho^p_{I_{p,i}K}(u) \leqslant \rho^p_{I_{p,i}L}(u)$. Thus we get $I_{p,i}K \subseteq I_{p,i}L$, with equality if and only if $I_{p,i}K = I_{p,i}L$. Because for any $Q \in \mathscr{S}^n_o$, we have

$$\widetilde{W}_{p,i}(Q, I_{p,i}K) = \widetilde{W}_{p,i}(Q, I_{p,i}L).$$

Using Proposition 3.2, from the above equation, we can get

$$\widetilde{W}_{p,i}(K, I_{p,i}Q) = \widetilde{W}_{p,i}(L, I_{p,i}Q).$$

Further, by Lemma 2.3, we immediately have $K = L$.

For $i > n-p$, using the same argument as in the first part of the proof, we get $K \subseteq L$ implies $I_{p,i}K \supseteq I_{p,i}L$ with equality if and only if $K = L$. □

4. The Busemann-type inequalities for L_p-mixed intersection bodies

A classic affine inequalities in convex geometry and affine geometry is the Busemann intersecting inequalities (see [2]). It can be expressed as follows: Let K be a convex body in $\mathbb{R}^n$, then

$$V(IK)V(K)^{1-n} \leqslant \omega_n^{2-n}, \tag{27}$$

with equality if and only if K is an ellipsoid.

In this section we will establish the Busemann-type inequalities for L_p-mixed intersection bodies. To this end, we need the following some lemmas.

LEMMA 4.1. (see [11]) *If $K \in \mathscr{K}^n$ and $i = 0, 1, \cdots, n-1$, then*

$$W_i(K) \geqslant \omega_n^{\frac{i}{n}} V(K)^{\frac{n-i}{n}}, \tag{28}$$

with equality if and only if K is a n-ball.

LEMMA 4.2. (see [[9]]) *If $K \in \mathscr{K}^n$, $i \in \mathbb{R}$ and $0 < i < n$, then*

$$\widetilde{W}_i(K) \leqslant \omega_n^{\frac{i}{n}} V(K)^{\frac{n-i}{n}}, \tag{29}$$

with equality if and only if K is a n-ball centered at the origin. For $i = 0, n$, the equality hold in inequality (29) for any convex body K in $\mathbb{R}^n$.

Now we establish the following Busemann-type inequalities for L_p-mixed intersection bodies.

THEOREM 4.1. *Let $K \in \mathscr{K}^n$, $p > 1$, $i, j \in \mathbb{R}$ and $0 \leqslant i, j \leqslant n$. Then for $1 < p < n - i$,*

$$\widetilde{W}_j(I_{p,i}K) W_j(K)^{1-\frac{n-i}{p}} \leqslant \omega_n^{2-\frac{n-i}{p}}. \tag{30}$$

For $p > n - i$, the inequality (30) is reversed. When $p \neq n - i$, the equality holds if and only if K is a ball centered at the origin in the above two inequalities.

Proof. If $1 < p < n - i$, using (5), (6) and Hölder's inequality (19), we have

$$\begin{aligned}
\rho_{I_{p,i}K}(u)^{n-j} &= \left(\frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1}\cap u^\perp} \rho_K(u)^{n-p-i} dS_{n-2}(v) \right)^{\frac{n-j}{p}} \\
&\leqslant \left[\left(\frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1}\cap u^\perp} \rho_K(u)^{n-1} dS_{n-2}(v) \right)^{\frac{n-p-i}{n-1}} \right. \\
&\quad \left. \times \left(\frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1}\cap u^\perp} 1^{\frac{n-1}{p+i-1}} dS_{n-2}(v) \right)^{\frac{p+i-1}{n-1}} \right]^{\frac{n-j}{p}} \\
&= \rho_{IK}(u)^{\frac{(n-p-i)(n-j)}{(n-1)p}}.
\end{aligned} \tag{31}$$

By (15), (31), Hölder's inequality (19) and Lemma 4.2, we can get

$$\begin{aligned}
\widetilde{W}_j(I_{p,i}K) &= \frac{1}{n}\int_{S^{n-1}} \rho_{I_{p,i}K}(u)^{n-j}dS(u) \\
&\leqslant \frac{1}{n}\int_{S^{n-1}} \rho_{IK}(u)^{\frac{(n-p-i)(n-j)}{(n-1)p}}dS(u) \\
&\leqslant \left(\frac{1}{n}\int_{S^{n-1}} \rho_{IK}(u)^{n-j}dS(u)\right)^{\frac{n-p-i}{(n-1)p}} \left(\frac{1}{n}\int_{S^{n-1}} 1^{\frac{(n-1)p}{n(p-1)+i}}dS(u)\right)^{\frac{n(p-1)+i}{(n-1)p}} \\
&= \omega_n^{\frac{n(p-1)+i}{(n-1)p}} \widetilde{W}_j(IK)^{\frac{n-p-i}{(n-1)p}} \\
&\leqslant \omega_n^{\frac{n^2(p-1)+j(n-p-i)+in}{n(n-1)p}} V(IK)^{\frac{(n-j)(n-p-i)}{n(n-1)p}}. \qquad (32)
\end{aligned}$$

Using Lemma 4.1 and (27), and note that $1-\frac{n-i}{p}<0$, from (32) we have

$$\begin{aligned}
\widetilde{W}_j(I_{p,i}K)W_j(K)^{1-\frac{n-i}{p}} &\leqslant \omega_n^{\frac{n^2(p-1)+j(n-p-i)+in}{n(n-1)p}} \times \omega_n^{\frac{j(p-n+i)}{np}} \times \left(V(IK)V(K)^{1-n}\right)^{\frac{(n-j)(n-p-i)}{n(n-1)p}} \\
&\leqslant \omega_n^{\frac{n^2(p-1)+j(n-p-i)+in}{n(n-1)p}} \times \omega_n^{\frac{-j(n-p-i)}{np}} \times \omega_n^{\frac{(2-n)(n-j)(n-p-i)}{n(n-1)p}} \\
&= \omega_n^{2-\frac{n-i}{p}}. \qquad (33)
\end{aligned}$$

Namely

$$\widetilde{W}_j(I_{p,i}K)W_j(K)^{1-\frac{n-i}{p}} \leqslant \omega_n^{2-\frac{n-i}{p}}.$$

We have proved the first part of Theorem 4.1.

If $p>n-i$, then $0<\frac{n-1}{p+i-1}<1$ and $0<\frac{(n-1)p}{n(p-1)+i}<1$. Therefore, according to Hölder's inequality we see that the inequalities in (31) and (32) are reversed. And noted that $1-\frac{n-i}{p}>0$ and $\frac{(n-j)(n-p-i)}{n(n-1)p}<0$, the inequality in (33) is reversed. This proves the second part of the Theorem 4.1.

In the following we consider the conditions of equality holds in the inequality (30). According to the conditions of equality hold in Hölder's inequality (19), we know that the equality hold in (31) and (32) if and only if $\rho_K(\cdot)$ is a constant. Namely, K is a ball in $\mathbb{R}^n$. This, combined with the conditions of equality holds in (27) and (28), we see that for $p \neq n-i$ the equality hold in Theorem 4.1 if and only if K is a ball centered at the origin. This completes the proof. □

Taking $j=0$ in Theorem 4.1, we obtain that

COROLLARY 4.1. *Let $K\in\mathscr{K}^n$, $p>1$, $i\in\mathbb{R}^n$ and $0\leqslant i\leqslant n$, then for $1<p<n-i$,*

$$V(I_{p,i}K)V(K)^{1-\frac{n-i}{p}} \leqslant \omega_n^{2-\frac{n-i}{p}}. \qquad (34)$$

For $p>n-i$ the inequality (34) is reversed. The equality holds for $p\neq n-i$ if and only if K is a ball centered at the origin.

Taking $i=0$ in Corollary 4.1, we obtain that

COROLLARY 4.2. *Let $K \in \mathscr{K}^n$ and $p > 1$, then for $1 < p < n$,*

$$V(I_pK)V(K)^{1-\frac{n}{p}} \leqslant \omega_n^{2-\frac{n}{p}}. \tag{35}$$

For $p > n$ the inequality (35) is reversed. The equality hold for $p \neq n$ if and only if K is a ball centered at the origin.

Corollary 4.2 is just Theorem 2.1 in [16]. Taking $p = 1$ in Corollary 4.2, we get well-known Busemann intersecting inequalities in [2] (see (27) in our article).

Below, we will establish a relationship between $I_{p+q,i}K$, $I_{p,i}K$ and $I_{q,i}K$.

THEOREM 4.2. *Let $K \in \mathscr{S}_o^n$, $i, j \in \mathbb{R}$ and $p, q \geqslant 1$. Then for $p > q$,*

$$\left(\frac{\widetilde{W}_j(I_{p+q,i}K)}{\widetilde{W}_j(I_{p,i}K)}\right)^{p^2} \geqslant \left(\frac{\widetilde{W}_j(I_{p+q,i}K)}{\widetilde{W}_j(I_{q,i}K)}\right)^{q^2}, \tag{36}$$

with equality holds if and only if K is a ball.

Proof. According to the definition of the L_p-mixed intersection bodies and Hölder's inequality (19), we have

$$\begin{aligned}
\rho_{I_{p+q,i}K}(u)^{p+q} &= \frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1}\cap u^\perp} \rho_K(v)^{n-p-q-i} dS_{n-2}(v) \\
&= \frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1}\cap u^\perp} \left(\rho_K(v)^{n-p-i}\right)^{\frac{p}{p-q}} \left(\rho_K(v)^{n-q-i}\right)^{-\frac{q}{p-q}} dS_{n-2}(v) \\
&\geqslant \left[\frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1}\cap u^\perp} \rho_K(v)^{n-p-i} dS_{n-2}(v)\right]^{\frac{p}{p-q}} \\
&\quad \times \left[\frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1}\cap u^\perp} \rho_K(v)^{n-q-i} dS_{n-2}(v)\right]^{-\frac{q}{p-q}} \\
&= \rho_{I_{p,i}K}(u)^{\frac{p^2}{p-q}} \rho_{I_{p,i}K}(u)^{-\frac{q^2}{p-q}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\widetilde{W}_j(I_{p+q,i}K)^{p^2-q^2} &= \left[\frac{1}{n}\int_{S^{n-1}} \rho_{I_{p+q,i}K}(u)^{n-j} dS(u)\right]^{p^2-q^2} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left(\rho_{I_{p+q,i}K}(u)^{p+q}\right)^{\frac{n-j}{p+q}} dS(u)\right]^{p^2-q^2} \\
&\geqslant \left[\frac{1}{n}\int_{S^{n-1}} \left(\rho_{I_{p,i}}(u)^{\frac{p^2}{p-q}} \rho_{I_{q,i}K}(u)^{-\frac{q^2}{p-q}}\right)^{\frac{n-j}{p+q}} dS(u)\right]^{p^2-q^2} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left(\rho_{I_{p,i}K}(u)^{n-j}\right)^{\frac{p^2}{p^2-q^2}} \left(\rho_{I_{p,i}K}(u)^{n-j}\right)^{-\frac{q^2}{p^2-q^2}} dS(u)\right]^{p^2-q^2} \\
&\geqslant \left(\frac{1}{n}\int_{S^{n-1}} \rho_{I_{p,i}K}(u)^{n-j} dS(u)\right)^{p^2} \left(\frac{1}{n}\int_{S^{n-1}} \rho_{I_{q,i}K}(u)^{n-j} dS(u)\right)^{-q^2} \\
&= \widetilde{W}_j(I_{p,i}K)^{p^2} \widetilde{W}_j(I_{q,i}K)^{-q^2}.
\end{aligned}$$

According to the condition of equality holds in the Hölder's inequality (19), we know that the equality holds in (36) if and only if K is a ball. This completes the proof. □

From Definition 1.1 of the L_p-mixed intersection bodies and Theorem 4.2, we have

COROLLARY 4.3. *If $K \in \mathscr{S}_o^n$, $p > q \geqslant 1$, $i, j \in \mathbb{R}$ and $0 \leqslant i \leqslant n-1$. Then for $p+q+i=n$,*

$$\widetilde{W}_j(I_{p,i}K)^{p^2}\widetilde{W}_j(I_{q,i}K)^{-q^2} \leqslant \omega_n^{p^2-q^2}, \tag{37}$$

with equality if and only if K is a ball.

5. Dual Brunn-Minkowski-type inequalities for L_p-mixed intersection bodies

In this section we establish the dual Brunn-Minkowski-type inequalities for L_p-mixed intersection bodies. To do this, we first give following two lemmas.

LEMMA 5.1. (Minkowski integral inequality, see [5]) *Suppose f and g are two nonnegative bounded Borel functions in a set X. Then for $p \geqslant 1$,*

$$\left(\int_X (f(x)+g(x))^p d\mu(x)\right)^{\frac{1}{p}} \leqslant \left(\int_X f(x)^p d\mu(x)\right)^{\frac{1}{p}} + \left(\int_X g(x)^p d\mu(x)\right)^{\frac{1}{p}}; \tag{38}$$

For $0 < p < 1$, the inequality (38) is the reverse. In both cases, the equalities holds if and only if f and g are proportional.

LEMMA 5.2. *Let $K, L \in \mathscr{S}_o^n$, $p \geqslant 1$, $q > 0$, and λ, $\mu \geqslant 0$ (not both zero). Then for $i < n-p-q$,*

$$\rho_{I_{p,i}(\lambda K \widetilde{+}_q \mu L)}(u)^{\frac{pq}{n-p-i}} \leqslant \lambda \rho_{I_{p,i}K}(u)^{\frac{pq}{n-p-i}} + \mu \rho_{I_{p,i}L}(u)^{\frac{pq}{n-p-i}}. \tag{39}$$

For $n-p-q < i < n-p$, the inequality (39) is the reverse. In both cases, the equality hold if and only if K and L be dilates. When $q = n-p-i$, the equality also holds in inequality (39).

Proof. Since $p \geqslant 1$, $q > 0$, $i < n-p-q$, this yields $\frac{n-p-i}{q} > 1$. Then from (38), we have

$$\begin{aligned}
\rho_{I_{p,i}(\lambda K \widetilde{+}_q \mu L)}(u)^{\frac{pq}{n-p-i}} &= \left[\frac{1}{(n-1)\omega_{n-1}}\int_{S^{n-1}\cap u^\perp} \rho_{(\lambda K \widetilde{+}_q \mu L)}(v)^{n-p-i} dS_{n-2}(v)\right]^{\frac{q}{n-p-i}} \\
&= \left[\frac{1}{(n-1)\omega_{n-1}}\int_{S^{n-1}\cap u^\perp} \left(\lambda \rho_K(v)^q + \mu \rho_L(v)^q\right)^{\frac{n-p-i}{q}} dS_{n-2}(v)\right]^{\frac{q}{n-p-i}} \\
&\leqslant \left[\frac{1}{(n-1)\omega_{n-1}}\int_{S^{n-1}\cap u^\perp} \lambda^{\frac{n-p-i}{q}} \rho_K(v)^{n-p-i} dS_{n-2}(v)\right]^{\frac{q}{n-p-i}} \\
&\quad + \left[\frac{1}{(n-1)\omega_{n-1}}\int_{S^{n-1}\cap u^\perp} \mu^{\frac{n-p-i}{q}} \rho_L(v)^{n-p-i} dS_{n-2}(v)\right]^{\frac{q}{n-p-i}} \\
&= \lambda \rho_{I_{p,i}K}(u)^{\frac{pq}{n-p-i}} + \mu \rho_{I_{p,i}L}(u)^{\frac{pq}{n-p-i}}.
\end{aligned}$$

From this, we know that the inequality holds in (39).

On the other hand, if $p \geqslant 1$, $q > 0$, $n-p-q < i < n-p$, then $0 < \frac{n-p-i}{q} < 1$. Similar to the proof of the above process, we can prove the reverse inequality of (39).

According to the conditions of equality holds in Minkowski integral inequality (38), we know that the equalities holds in (39) and its the reverse inequality if and only if K and L are dilates. The proof of Lemma 5.2 is completed. □

Particularly, taking $q = n-p-i$, we obtain that

$$\rho_{I_{p,i}(\lambda K \widetilde{+}_{n-p-i} \mu L)}(u)^p = \lambda \rho_{I_{p,i}K}(u)^p + \mu \rho_{I_{p,i}L}(u)^p. \tag{40}$$

Here we establish dual Brunn-Minkowski-type inequalities for L_p-mixed intersection bodies.

THEOREM 5.1. *Let $K,L \in \mathscr{S}_o^n$, λ, $\mu \geqslant 0$ (not both zero), $p \geqslant 1$, $q > 0$ and i,j are real numbers. Then for $i \leqslant n-p-q$, $j \leqslant n-p$,*

$$\widetilde{W}_j(I_{p,i}(\lambda K \widetilde{+}_q \mu L))^{\frac{pq}{(n-p-i)(n-j)}} \leqslant \lambda \widetilde{W}_j(I_{p,i}K)^{\frac{pq}{(n-p-i)(n-j)}} + \mu \widetilde{W}_j(I_{p,i}L)^{\frac{pq}{(n-p-i)(n-j)}}. \tag{41}$$

For $n-p-q < i < n-p, n-p < j < n$, the inequality (41) is the reverse. In both cases, the equality holds if and only if K and L are dilates.

Proof. By the known conditions $i \leqslant n-p-q$, $j \leqslant n-p$, we know

$$\frac{n-p-i}{q} \geqslant 1, \qquad \frac{(n-p-i)(n-j)}{pq} \geqslant 1.$$

From Lemma 2.1, Lemma 5.2 and Minkowski integral inequality (38), we get

$$\widetilde{W}_j(I_{p,i}(\lambda K \widetilde{+}_q \mu L))^{\frac{pq}{(n-p-i)(n-j)}} = \left[\frac{1}{n}\int_{S^{n-1}} \rho_{I_{p,i}(\lambda K \widetilde{+}_q \mu L)}(u)^{n-j} dS(u)\right]^{\frac{pq}{(n-p-i)(n-j)}}$$
$$\leqslant \left[\frac{1}{n}\int_{S^{n-1}} \left(\lambda \rho_{I_{p,j}K}(u)^{\frac{pq}{n-p-i}} + \mu \rho_{I_{p,j}L}(u)^{\frac{pq}{n-p-i}}\right)^{\frac{(n-p-i)(n-j)}{pq}} dS(u)\right]^{\frac{pq}{(n-p-i)(n-j)}}$$
$$\leqslant \left(\frac{1}{n}\lambda \int_{S^{n-1}} \rho_{I_{p,j}K}(u)^{n-j} dS(u)\right)^{\frac{pq}{(n-p-i)(n-j)}} + \left(\frac{1}{n}\lambda \int_{S^{n-1}} \rho_{I_{p,j}L}(u)^{n-j} dS(u)\right)^{\frac{pq}{(n-p-i)(n-j)}}$$
$$= \lambda \widetilde{W}_j(I_{p,i}K)^{\frac{pq}{(n-p-i)(n-j)}} + \mu \widetilde{W}_j(I_{p,i}L)^{\frac{pq}{(n-p-i)(n-j)}}.$$

From this, we immediately gives the inequality (41).

According to the conditions of equality holds in the inequality (38), we know that equality holds in the inequality (41) if and only if $\rho_{I_{p,i}K}(\cdot)$ and $\rho_{I_{p,i}L}(\cdot)$ are proportional. Therefore, the equality holds in (41) if and only if K and L are dilates.

On the other hand, since $n-p-q < i < n-p$, $n-p < j < n$, this yields

$$0 < \frac{(n-p-i)(n-j)}{pq} < 1, \qquad 0 < \frac{n-p-i}{q} < 1.$$

Similar to the proof in the first case, from Minkowski integral inequality (38), we can obtain the reverse inequality of inequality (41), with equality in the reverse inequality if and only if K and L are dilates. □

Particularly, taking $q = n-p-i > 0$ in Theorem 5.1, we obtain that

COROLLARY 5.1. *Let $K, L \in \mathscr{S}_o^n$, λ, $\mu \geqslant 0$ (not both zero), $p \geqslant 1$ and i, j are real numbers. Then for $i < n-p$, $j \leqslant n-p$,*

$$\widetilde{W}_j(I_{p,i}(\lambda K \widetilde{+}_{n-p-i} \mu L))^{\frac{p}{n-j}} \leqslant \lambda \widetilde{W}_j(I_{p,i}K)^{\frac{p}{n-j}} + \mu \widetilde{W}_j(I_{p,i}L)^{\frac{p}{n-j}}. \tag{42}$$

For $i < n-p$, $n-p < j < n$, the inequality (42) is the reverse. In both cases, the equality holds if and only if K and L are dilates.

Together with Theorem 5.1 and Lemma 5.1, we can get the following an isolated form of inequality (42).

THEOREM 5.2. *Let $K, L \in \mathscr{S}_o^n$, $\alpha \in [0,1]$, $p \geqslant 1$ and i, j are real numbers. Then for $i < n-p$, $j \leqslant n-p$,*

$$\begin{aligned}
&\widetilde{W}_j(I_{p,i}(K \widetilde{+}_{n-p-i} L))^{\frac{p}{n-j}} \\
&\leqslant \widetilde{W}_j(I_{p,i}(\alpha \cdot K \widetilde{+}_{n-p-i} (1-\alpha) \cdot L))^{\frac{p}{n-j}} + \widetilde{W}_j(I_{p,i}((1-\alpha) \cdot K \widetilde{+}_{n-p-i} \alpha \cdot L))^{\frac{p}{n-j}} \\
&\leqslant \widetilde{W}_j(I_{p,i}K)^{\frac{p}{n-j}} + \widetilde{W}_j(I_{p,i}L)^{\frac{p}{n-j}},
\end{aligned} \tag{43}$$

with equality if and only if K and L are dilates. For $i < n-p$, $n-p < j < n$, the inequality (43) is the reverse.

Proof. For all $\alpha \in [0,1]$, let $M = \alpha \cdot K \widetilde{+}_{n-p-i} (1-\alpha) \cdot L$, $N = (1-\alpha) \cdot K \widetilde{+}_{n-p-i} \alpha \cdot L$. Since $K, L \in \mathscr{S}_o^n$, then $M, N \in \mathscr{S}_o^n$. By the formula (15), (40) and Definition 2.1, we have

$$\begin{aligned}
\widetilde{W}_j(I_{p,i}(K \widetilde{+}_{n-p-i} L)) &= \frac{1}{n} \int_{S^{n-1}} \rho(I_{p,i}(K \widetilde{+}_{n-p-i} L), u)^{n-j} dS(u) \\
&= \frac{1}{n} \int_{S^{n-1}} \left(\rho_{I_{p,i}K}(u)^p + \rho_{I_{p,i}L}(u)^p \right)^{\frac{n-j}{p}} dS(u) \\
&= \frac{1}{n} \int_{S^{n-1}} \Big[\left(\alpha \rho_{I_{p,i}K}(u)^p + (1-\alpha) \rho_{I_{p,i}L}(u)^p \right) \\
&\quad + \left((1-\alpha) \rho_{I_{p,i}K}(u)^p + \alpha \rho_{I_{p,i}L}(u)^p \right) \Big]^{\frac{n-j}{p}} dS(u) \\
&= \frac{1}{n} \int_{S^{n-1}} \Big[\rho^p(I_{p,i}(\alpha \cdot K \widetilde{+}_{n-p-i} (1-\alpha) \cdot L), u) \\
&\quad + \rho^p(I_{p,i}((1-\alpha) \cdot K \widetilde{+}_{n-p-i} \alpha \cdot L), u) \Big]^{\frac{n-j}{p}} dS(u) \\
&= \frac{1}{n} \int_{S^{n-1}} \rho(I_{p,i}(M \widetilde{+}_{n-p-i} N), u)^{n-j} dS(u) \\
&= \widetilde{W}_j(I_{p,i}(M \widetilde{+}_{n-p-i} N)).
\end{aligned}$$

According to Corollary 5.1, for $i<n-p,\ j\leqslant n-p$, we obtain that

$$\widetilde{W}_j(I_{p,i}(K\widetilde{+}_{n-p-i}L))=\widetilde{W}_j(I_{p,i}(M\widetilde{+}_{n-p-i}N))\leqslant \widetilde{W}_j(I_{p,i}M)^{\frac{p}{n-j}}+\widetilde{W}_j(I_{p,i}N)^{\frac{p}{n-j}}, \quad (44)$$

with equality if and only if M and N are dilates. The inequality (44) is just the left part in Theorem 5.2.

Since M and N are dilates each other if and only if $\rho_M^{n-p-i}(u)=c\rho_N^{n-p-i}(u)$ $(c>0)$ for all $u\in S^{n-1}$. Therefore, for $u\in S^{n-1}$, by Definition 2.1 we get $\alpha\rho_K^{n-p-i}(u)+(1-\alpha)\rho_L^{n-p-i}(u)=c[(1-\alpha)\rho_K^{n-p-i}(u)+\alpha\rho_L^{n-p-i}(u)]$, namely, $(\alpha+c\alpha-c)\rho_K^{n-p-i}(u)$ $=(c\alpha+\alpha-1)\rho_L^{n-p-i}(u)$. Thus M and N are dilates if and only if K and L are dilates. The equality of the left side inequality holds in Theorem 5.2 if and only if K and L are dilates.

On the other hand, according to the inequality in Corollary 5.1, for $i<n-p$, $j\leqslant n-p$, then

$$\begin{aligned}\widetilde{W}_j(I_{p,i}M)^{\frac{p}{n-j}} &= \widetilde{W}_j(I_{p,i}(\alpha\cdot K\widetilde{+}_{n-p-i}(1-\alpha)L))^{\frac{p}{n-j}}\\ &\leqslant \alpha\widetilde{W}_j(I_{p,i}K)^{\frac{p}{n-j}}+(1-\alpha)\widetilde{W}_j(I_{p,i}K)^{\frac{p}{n-j}}, \end{aligned}\quad (45)$$

and

$$\begin{aligned}\widetilde{W}_j(I_{p,i}N)^{\frac{p}{n-j}} &= \widetilde{W}_j(I_{p,i}((1-\alpha)\cdot K\widetilde{+}_{n-p-i}\alpha L))^{\frac{p}{n-j}}\\ &\leqslant (1-\alpha)\widetilde{W}_j(I_{p,i}K)^{\frac{p}{n-j}}+\alpha\widetilde{W}_j(I_{p,i}K)^{\frac{p}{n-j}}, \end{aligned}\quad (46)$$

with equality if and only if K and L are dilates.

Compare with the above inequalities (45) and (46), we obtain that

$$\widetilde{W}_j(I_{p,i}M)^{\frac{p}{n-j}}+\widetilde{W}_j(I_{p,i}N)^{\frac{p}{n-j}}\leqslant \widetilde{W}_j(I_{p,i}K)^{\frac{p}{n-j}}+\widetilde{W}_j(I_{p,i}L)^{\frac{p}{n-j}}, \quad (47)$$

with equality if and only if K and L are dilates. The inequality (47) is just the right inequality in Theorem 5.2.

Similarly, using the above method of proof, we will be easy to prove that for $i<n-p,\ n-p<j<n$, the inequality in Theorem 5.2 is the reverse. The proof of Theorem 5.2 is completed. □

Taking $i=0$ and $j=0$ in Theorem 5.2, we immediately get that

COROLLARY 5.2. *Let* $K,L\in\mathscr{S}_o^n$, $1\leqslant p<n$, *and* $\alpha\in[0,1]$. *Then*

$$\begin{aligned}V(I_p(K\widetilde{+}_{n-p}L))^{\frac{p}{n}} &\leqslant V(I_p(\alpha\cdot K\widetilde{+}_{n-p}(1-\alpha)\cdot L))^{\frac{p}{n}}+V(I_p((1-\alpha)\cdot K\widetilde{+}_{n-p}\alpha\cdot L))^{\frac{p}{n}}\\ &\leqslant V(I_pK)^{\frac{p}{n}}+V(I_pL)^{\frac{p}{n}}, \end{aligned}\quad (48)$$

with equality if and only if K *and* L *are dilates.*

Finally, we will establish a dual Brunn-Minkowski inequality of L_p-mixed intersection body associated with L_p-mixed harmonic Blaschke plus.

THEOREM 5.3. *Let $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$, and $i < n$ is real numbers. Then for $p \leqslant n-i$,*

$$\frac{\widetilde{W}_i(I_{p,i}(K\dot{+}_pL))^{\frac{p}{n-i}}}{\widetilde{W}_i(K\dot{+}_pL)} \leqslant \frac{\widetilde{W}_i(I_{p,i}K)^{\frac{p}{n-i}}}{\widetilde{W}_i(K)} + \frac{\widetilde{W}_i(I_{p,i}L)^{\frac{p}{n-i}}}{\widetilde{W}_i(L)}. \tag{49}$$

If $p > n-i$, The inequality (49) is the reverse. For $p \neq n-i$ the equality holds in inequalities if and only if K and L are dilates.

Proof. From the equation (21), (22) in Definition 2.3 and the formula of dual quermassintegrals $\widetilde{W}_i(K)$, immediately yields $\xi = \widetilde{W}_i(K\dot{+}_pL)$. Therefore, by (22) we have

$$\frac{\rho(K\dot{+}_pL,u)^{n+p-i}}{\widetilde{W}_i(K\dot{+}_pL)} = \frac{\rho(K,u)^{n+p-i}}{\widetilde{W}_i(K)} + \frac{\rho(L,u)^{n+p-i}}{\widetilde{W}_i(L)}. \tag{50}$$

By Definition 1.1 and (50), we have

$$\begin{aligned}
\frac{\rho(I_{p,i}(K\dot{+}_pL),u)^p}{\widetilde{W}_i(K\dot{+}_pL)} &= \frac{1}{(n-1)\omega_{n-1}\widetilde{W}_i(K\dot{+}_pL)}\int_{S^{n-1}\cap u^\perp}\rho_{K\dot{+}_pL}^{n-p-i}(u)dS_{n-2}(u)\\
&= \frac{1}{(n-1)\omega_{n-1}\widetilde{W}_i(K)}\int_{S^{n-1}\cap u^\perp}\rho_K^{n-p-i}(u)dS_{n-2}(u)\\
&\quad + \frac{1}{(n-1)\omega_{n-1}\widetilde{W}_i(L)}\int_{S^{n-1}\cap u^\perp}\rho_L^{n-p-i}(u)dS_{n-2}(u)\\
&= \frac{\rho(I_{p,i}(K,u)^p}{\widetilde{W}_i(K)} + \frac{\rho(I_{p,i}(L,u)^p}{\widetilde{W}_i(L)}.
\end{aligned} \tag{51}$$

Therefore, for $p \leqslant n-i$, $i < n$, by the formula (15), (51) and Minkowski inequality (38), we can get

$$\begin{aligned}
\frac{\widetilde{W}_i(I_{p,i}(K\dot{+}_pL))^{\frac{p}{n-i}}}{\widetilde{W}_i(K\dot{+}_pL)} &= \frac{1}{\widetilde{W}_i(K\dot{+}_pL)}\left(\frac{1}{n}\int_{S^{n-1}}\rho(I_{p,i}(K\dot{+}_pL),u)^{n-i}dS(u)\right)^{\frac{p}{n-i}}\\
&= \left(\frac{1}{n}\int_{S^{n-1}}\left(\frac{\rho(I_{p,i}(K\dot{+}_pL),u)^p}{\widetilde{W}_i(K\dot{+}_pL)}\right)^{\frac{n-i}{p}}dS(u)\right)^{\frac{p}{n-i}}\\
&= \left(\frac{1}{n}\int_{S^{n-1}}\left(\frac{\rho(I_{p,i}K,u)^p}{\widetilde{W}_i(K)} + \frac{\rho(I_{p,i}L,u)^p}{\widetilde{W}_i(L)}\right)^{\frac{n-i}{p}}dS(u)\right)^{\frac{p}{n-i}}\\
&\leqslant \left(\frac{1}{n}\int_{S^{n-1}}\frac{\rho(I_{p,i}K,u)^{n-i}}{\widetilde{W}_i(K)^{\frac{n-i}{p}}}dS(u)\right)^{\frac{p}{n-i}}\\
&\quad + \left(\frac{1}{n}\int_{S^{n-1}}\frac{\rho(I_{p,i}L,u)^{n-i}}{\widetilde{W}_i(L)^{\frac{n-i}{p}}}dS(u)\right)^{\frac{p}{n-i}}\\
&= \frac{\widetilde{W}_i(I_{p,i}K)^{\frac{p}{n-i}}}{\widetilde{W}_i(K)} + \frac{\widetilde{W}_i(I_{p,i}L)^{\frac{p}{n-i}}}{\widetilde{W}_i(L)}.
\end{aligned}$$

From the conditions of equality holds in Minkowski inequality (38), we know that equality holds in inequality (49) if and only if

$$\frac{\rho(I_{p,i}K,u)^p}{\widetilde{W}_i(K)} = \lambda \frac{\rho(I_{p,i}L,u)^p}{\widetilde{W}_i(L)},$$

where $\lambda > 0$ is a constant. Namely

$$\frac{\rho(I_{p,i}K,u)}{\rho(I_{p,i}L,u)} = \left(\frac{\lambda \widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right)^{\frac{1}{p}}.$$

This shows that equality holds in inequality (49) if and only if $I_{p,i}K$ and $I_{p,i}L$ are dilates each other. Let $I_{p,i}K = \mu I_{p,i}L$, together with the concept of dual L_p-mixed quermassintegrals, for any $Q \in \mathscr{S}_o^n$, we have

$$\widetilde{W}_{p,i}(Q,I_{p,i}K) = \widetilde{W}_{p,i}(Q,\mu I_{p,i}L) = \mu^p \widetilde{W}_{p,i}(Q,I_{p,i}L).$$

From Proposition 3.2, the integral expression (18) of dual L_p-mixed quermassintegrals, and definition of the radial function, for any $Q \in \mathscr{S}_o^n$, the above equation can be turned into

$$\widetilde{W}_{p,i}(K,I_{p,i}Q) = \mu^p \widetilde{W}_{p,i}(L,I_{p,i}Q) = \widetilde{W}_{p,i}(\mu^{\frac{p}{n-i-p}}L,I_{p,i}Q).$$

According to Lemma 2.3, we get $K = \mu^{\frac{p}{n-i-p}}L$, this means that K and L are dilates.

Similarly, for $p > n-i$, we can prove the inequality (49) is reversed. □

Acknowledgements. The authors would like to thank Prof. Gangsong Leng for his helps for this paper.

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(Received April 2, 2013)

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