Journal of Mathematical Inequalities
Volume 8, Number 3 (2014), 581–595
doi:10.7153/jmi-08-43

ON EXTENSIONS AND APPLICATIONS OF THE BEESACK INEQUALITY FOR BOUNDING RIEMANN–STIELTJES INTEGRALS

P. CERONE

(*Communicated by A. Aglić Aljinović*)

Abstract. A variety of sharp generalisations and extensions of a Beesack inequality are investigated in the current development. A number of applications are examined, including bounding a renewal integral equation which arises in risk/ruin problems. Sharp and tighter bounds than existing results for the Čebyšev functional involving Riemann-Stieltjes integrals are also determined which is of importance in many applications including in providing bounds for perturbed quadrature rules.

1. Introduction

In 1975, P. R. Beesack [2] showed that, if w,h,v are real valued functions defined on a compact interval $[a,b]$, where w is of bounded variation with total variation $\bigvee_a^b(w)$, and such that the Riemann-Stieltjes integrals $\int_a^b h(t)dv(t)$ and $\int_a^b w(t)h(t)dv(t)$ both exist, then

$$m\int_a^b h(t)\,dv(t)+\bigvee_a^b(w)\cdot s_h[a,b]\leqslant\int_a^b w(t)h(t)\,dv(t) \qquad (1.1)$$
$$\leqslant m\int_a^b h(t)\,dv(t)+\bigvee_a^b(w)\cdot S_h[a,b],$$

where

$$s_h[a,b]:=\inf_{a\leqslant\alpha<\beta\leqslant b}\left[\int_\alpha^\beta h(t)\,dv(t)\right],\quad S_h[a,b]:=\sup_{a\leqslant\alpha<\beta\leqslant b}\left[\int_\alpha^\beta h(t)\,dv(t)\right] \qquad (1.2)$$

and

$$m:=\inf_{t\in[a,b]}\{w(t)\}. \qquad (1.3)$$

The second of the inequalities above extends a result of R. Darst and H. Pollard [7] who dealt with the case $h(t)=1,\ t\in[a,b]$ and $v(t)$ continuous on $[a,b]$.

The inequality (1.1) provides different bounds than that provided by the following more traditional lemma stemming from Theorem 7.21 in [1].

Mathematics subject classification (2010): 26D15, 41A17.
Keywords and phrases: Riemann-Stieltjes integral, Volterra integral equations, integral inequalities, Čebyšev functional, bounds.

LEMMA 1. *Let $h, v : [a,b] \to \mathbb{R}$ be such that v is of bounded variation and the Riemann-Stieltjes integral $\int_a^b h(t)\, dv(t)$, exists then*

$$\left| \int_a^b h(t)\, dv(t) \right| \leqslant \sup_{t \in [a,b]} |h(t)| \cdot \bigvee_a^b (v), \tag{1.4}$$

where $\bigvee_a^b (v)$ is the total variation of v on $[a,b]$.

In [8], S. S. Dragomir has introduced the following *Čebyšev functional for the Riemann-Stieltjes integral*:

$$\mathscr{T}(f,g;u) := \mathscr{M}(f \cdot g;u) - \mathscr{M}(f;u) \cdot \mathscr{M}(g;u), \tag{1.5}$$

where

$$\mathscr{M}(h;u) := \frac{1}{u(b) - u(a)} \int_a^b h(t)\, du(t) \tag{1.6}$$

provided $u(b) \neq u(a)$ and the involved Riemann-Stieltjes integrals exist.

It has been shown in [8] that, if f, g are continuous, $m \leqslant f(t) \leqslant M$ for each $t \in [a,b]$ and u is of bounded variation, then the error in approximating the Riemann-Stieltjes integral of the product in terms of the product of integrals, as described in the definition of the Čebyšev functional (1.5), satisfies the inequality:

$$|\mathscr{T}(f,g;u)| \leqslant \frac{1}{2}(M-m) \cdot \frac{1}{|u(b)-u(a)|} \|g - \mathscr{M}(g;u)\|_\infty \bigvee_a^b (u), \tag{1.7}$$

where the constant $\frac{1}{2}$ is best possible, $\|\cdot\|_\infty$ is the sup-norm and $\mathscr{M}(g;u)$ is as given in (1.6).

Moreover, if f, g are continuous, $m \leqslant f(t) \leqslant M$ for $t \in [a,b]$ and u is monotonic nondecreasing on $[a,b]$, then:

$$|\mathscr{T}(f,g;u)| \leqslant \frac{1}{2}(M-m) \frac{1}{|u(b)-u(a)|} \cdot \int_a^b |g(t) - \mathscr{M}(g;u)|\, du(t) \tag{1.8}$$

and the constant $\frac{1}{2}$ here is also sharp.

Finally, if f, g are Riemann integrable and u is Lipschitzian with the constant $L > 0$ then also

$$|\mathscr{T}(f,g;u)| \leqslant \frac{1}{2}(M-m) \frac{L}{|u(b)-u(a)|} \cdot \int_a^b |g(t) - \mathscr{M}(g;u)|\, du(t). \tag{1.9}$$

The constant $\frac{1}{2}$ is also best possible in (1.9) (see [9] and [10]).

The main aim of the present paper is to develop further inequalities stemming from (1.1) of Beesack to provide bounds for Riemann- Stieltjes integrals. The developments are demonstrated in a variety of applications including in procuring bounds for a renewal equation arising in risk/ruin problems in the actuarial arena and in providing novel bounds for the Čebyšev functional involving Riemann-Stieltjes integrals. Procuring bounds for the Čebyšev functional is an important problem since it plays a crucial role in perturbed quadrature rules (see [11] for example). The main development is provided by Theorem 5 which is applied, in the final section to the Čebyšev functional to produce sharp and tighter bounds than exist in the literature.

2. The results and extensions of Beesack

The following result obtained in Cerone and Dragomir [5] may be stated.

THEOREM 1. *Let $f,g,u:[a,b]\to\mathbb{R}$ be such that f is of bounded variation and the Riemann-Stieltjes integrals $\int_a^b f(t)g(t)\,du(t)$, $\int_a^b f(t)\,du(t)$ and $\int_a^b g(t)\,du(t)$ exist. Then*

$$\begin{aligned}&\bigvee_a^b(f)\cdot\inf_{a\leqslant\alpha<\beta\leqslant b}\int_\alpha^\beta[g(t)-\mathscr{M}(g;u)]\,du(t)\\&\leqslant\int_a^b f(t)g(t)\,du(t)-\frac{1}{u(b)-u(a)}\cdot\int_a^b f(t)\,du(t)\cdot\int_a^b g(t)\,du(t)\\&\leqslant\bigvee_a^b(f)\cdot\sup_{a\leqslant\alpha<\beta\leqslant b}\int_\alpha^\beta[g(t)-\mathscr{M}(g;u)]\,du(t),\end{aligned}\tag{2.1}$$

provided $u(b)\neq u(a)$ and $\mathscr{M}(g;u)$ is as given in (1.6).

The following result involving weighted integrals was also obtained in [5] which is procured from (2.1) by taking $u(t)=\int_a^t w(s)\,ds$.

COROLLARY 1. *Let $f,g,w:[a,b]\to\mathbb{R}$ be such that f is of bounded variation and the Riemann integrals $\int_a^b f(t)g(t)w(t)\,dt$, $\int_a^b f(t)w(t)\,dt$ and $\int_a^b g(t)w(t)\,dt$ exist. Then*

$$\begin{aligned}&\bigvee_a^b(f)\cdot\inf_{a\leqslant\alpha<\beta\leqslant b}\left[\int_\alpha^\beta g(t)w(t)\,dt-\frac{\int_\alpha^\beta w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\int_a^b g(t)w(t)\,dt\right]\\&\leqslant\int_a^b f(t)g(t)w(t)\,dt-\frac{1}{\int_a^b w(s)\,ds}\cdot\int_a^b f(t)w(t)\,dt\cdot\int_a^b g(t)w(t)\,dt\\&\leqslant\bigvee_a^b(f)\cdot\sup_{a\leqslant\alpha<\beta\leqslant b}\left[\int_\alpha^\beta g(t)w(t)\,dt-\frac{\int_\alpha^\beta w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\int_a^b g(t)w(t)\,dt\right],\end{aligned}\tag{2.2}$$

provided $\int_a^b w(s)\,ds\neq 0$.

REMARK 1. Taking $w(t) = 1,\ t \in [a,b]$, in (2.2) or $u(t) = t$ in (2.1):

$$\bigvee_a^b (f) \cdot \inf_{a \leqslant \alpha < \beta \leqslant b} \left[\int_\alpha^\beta g(t)\,dt - \frac{\beta - \alpha}{b-a} \cdot \int_a^b g(t)\,dt \right] \tag{2.3}$$
$$\leqslant \int_a^b f(t)\,g(t)\,dt - \frac{1}{\int_a^b w(s)\,ds} \cdot \int_a^b f(t)\,dt \cdot \int_a^b g(t)\,dt$$
$$\leqslant \bigvee_a^b (f) \cdot \sup_{a \leqslant \alpha < \beta \leqslant b} \left[\int_\alpha^\beta g(t)\,dt - \frac{\beta - \alpha}{b-a} \cdot \int_a^b g(t)\,dt \right],$$

provided f is of bounded variation and the involved Riemann integrals exist.

REMARK 2. It is worthwhile recalling that the Čebyšev functional $\mathscr{T}(f,g;u)$ as given in (1.5) satisfies the following identities

$$\mathscr{T}(f,g;u) = \mathscr{M}[(f-\gamma)\cdot(g-\mathscr{M}(g;u));u] = \mathscr{M}[(f-\mathscr{M}(f;u))\cdot(g-\delta);u] \tag{2.4}$$

where γ and δ can take on any finite real number including zero. The result (2.4) for the Riemann integral is the well known Sonin identity. In particular Theorem 1 may be reformulated as:

THEOREM 2. *Let $f,g,u : [a,b] \to \mathbb{R}$ be such that f is of bounded variation and the Riemann-Stieltjes integrals $\int_a^b f(t)\,g(t)\,du(t)$, $\int_a^b f(t)\,du(t)$ and $\int_a^b g(t)\,du(t)$ exist. Then*

$$m \int_a^b g(t)\,du(t) + \bigvee_a^b (f) \cdot s_g[a,b] \tag{2.5}$$
$$\leqslant \int_a^b f(t)\,g(t)\,du(t) - \frac{1}{u(b)-u(a)} \int_a^b f(t)\,du(t) \cdot \int_a^b g(t)\,du(t)$$
$$\leqslant m \int_a^b g(t)\,du(t) + \bigvee_a^b (f) \cdot S_g[a,b],$$

provided $u(b) \neq u(a)$ and $\mathscr{M}(g;u)$ is as given in (1.6) with m, $s_g[a,b]$ and $S_g[a,b]$ are as given in (1.2) and (1.3).

Proof. We observe that the following identity holds (see also [8])

$$[u(b)-u(a)]\mathscr{T}(f,g;u) = \int_a^b [f(t) - \mathscr{M}(f;u)]g(t)\,du(t). \tag{2.6}$$

Since f is of bounded variation, it follows that f is bounded below and if we denote by m the infimum of f on $[a,b]$, then on applying the Beesack inequality for the choices $w(t) = f(t) - \mathscr{M}(f;u)$, $h(t) = g(t)$ and $v(t) = u(t)$, $t \in [a,b]$, we can write that:

$$(m - \mathscr{M}(f;u)) \int_a^b g(t)\,du(t) + \bigvee_a^b (f) \inf_{a \leqslant \alpha < \beta \leqslant b} \left[\int_\alpha^\beta g(t)\,dv(t) \right] \tag{2.7}$$
$$\leqslant [u(b) - u(a)] \mathscr{T}(f,g;u)$$
$$\leqslant (m - \mathscr{M}(f;u)) \int_a^b g(t)\,du(t) + \bigvee_a^b (f) \sup_{a \leqslant \alpha < \beta \leqslant b} \left[\int_\alpha^\beta g(t)\,dv(t) \right].$$

It may be readily seen that $\mathscr{M}(f;u) \int_a^b g(t)\,du(t)$ may be removed throughout given the identity (2.6). The result (2.5) then readily follows on using the fact that for any constant $\kappa, \bigvee_a^b (f - \kappa) = \bigvee_a^b (f)$ and the Sonin identity given in (2.4) together with the definitions of $s_g[a,b]$ and $S_g[a,b]$. $\square$

REMARK 3. It may be noticed that the result (2.1) may be rewritten in the following form

$$\bigvee_a^b (f) \cdot s_{g(\cdot) - \mathscr{M}(g;u)}[a,b] \tag{2.8}$$
$$\leqslant \int_a^b f(t)\,g(t)\,du(t) - \frac{1}{u(b) - u(a)} \cdot \int_a^b f(t)\,du(t) \cdot \int_a^b g(t)\,du(t)$$
$$\leqslant \bigvee_a^b (f) \cdot S_{g(\cdot) - \mathscr{M}(g;u)}[a,b],$$

provided $u(b) \neq u(a)$ and $\mathscr{M}(g;u)$ is as given in (1.6) with $s_g[a,b]$ and $S_g[a,b]$ are as given in (1.6). This seems like a straight forward application of the Beesack inequality (1.1) however we see that the infimum of $f(\cdot)$ is not present since the integral mean of $g(t) - \mathscr{M}(g;u)$ is equal to zero.

It is not obvious in general to ascertain which of the two forms (2.1) [or equivalently (2.5)] and (2.8) produces tighter bounds. We now turn our attention to procuring extensions of Beesack's inequality but first we represent the Beesack results (1.1) and (1.2) in the following form.

THEOREM 3. (Beesack, 1975 [2]) *Let w,h,v real valued functions defined on a compact interval $[a,b]$, where w is of bounded variation with total variation $\bigvee_a^b(w)$, and such that the Riemann-Stieltjes integrals $\int_a^b h(t)\,dv(t)$ and $\int_a^b w(t)\,h(t)\,dv(t)$ both exist, then*

$$\bigvee_a^b (w) \cdot s_h[a,b] \leqslant \int_a^b (w(t) - m) h(t)\,dv \leqslant \bigvee_a^b (w) \cdot S_h[a,b], \tag{2.9}$$

where $m = \inf_{t \in [a,b]} \{w(t)\}$, $s_h[a,b]$ and $S_h[a,b]$ are as given in (1.2).

A number of complimentary results to Theorem 3 that do not seem to have been presented in the literature will now be investigated.

THEOREM 4. *Let the conditions of Theorem 3 continue to hold. Namely, w, h, v are real valued functions defined on a compact interval $[a,b]$, where w is of bounded variation with total variation $\bigvee_a^b(w)$ and such that the Riemann-Stieltjes integrals $\int_a^b h(t)\,dv(t)$ and $\int_a^b w(t)h(t)\,dv(t)$ both exist, then*

$$\bigvee_a^b(w)\cdot s_h[a,b] \leqslant \int_a^b (M-w(t))h(t)\,dv \leqslant \bigvee_a^b(w)\cdot S_h[a,b], \tag{2.10}$$

where $s_h[a,b]$ and $S_h[a,b]$ are as given in (1.2) and

$$M := \sup_{t\in[a,b]}\{w(t)\}. \tag{2.11}$$

Proof. Straight forward from Theorem 3 since $\inf_{t\in[a,b]}\{M-w(t)\}=0$ and

$$\bigvee_a^b(M-w(t)) = \bigvee_a^b(w). \quad \square$$

REMARK 4. The Beesack result (2.9) or equivalently (1.1) produce upper and lower bounds for $\int_a^b w(t)h(t)\,dv(t)$ *in terms of* $s_h[a,b]$ *and* $S_h[a,b]$ given by (1.2) and m, the infimum of $w(t)$. It should be noted that (2.9) may be obtained by a simple rearrangement of (1.1) or using the facts that $\inf_{t\in[a,b]}\{w(t)-m\}=0$ *and* $\bigvee_a^b(w(t)-m)=\bigvee_a^b(w)$. The result (2.10) consist of M, the supremum of $w(t)$, $s_h[a,b]$ *and* $S_h[a,b]$. It may easily be seen that (2.10) may be rewritten in the form

$$\begin{aligned} M\int_a^b h(t)\,dv(t) - \bigvee_a^b(w)\cdot S_h[a,b] &\leqslant \int_a^b w(t)h(t)\,dv(t) \\ &\leqslant M\int_a^b h(t)\,dv(t) - \bigvee_a^b(w)\cdot s_h[a,b]. \end{aligned} \tag{2.12}$$

The following corollary to Theorem 1 uses both Beesack's inequality (1.1) or equivalently (2.9) and its extension using (2.10) or equivalently (2.12).

COROLLARY 2. *Let $f,g,u:[a,b]\to\mathbb{R}$ be such that f is of bounded variation and the Riemann-Stieltjes integrals $\int_a^b f(t)g(t)\,du(t)$, $\int_a^b f(t)\,du(t)$ and $\int_a^b g(t)\,du(t)$ exist. Then*

$$|[u(b)-u(a)]\mathscr{T}(f,g;u)| \leqslant \frac{\bigvee_a^b(f)}{2}\cdot\left(S_{g(\cdot)-\mathscr{M}(g;u)}[a,b] - s_{g(\cdot)-\mathscr{M}(g;u)}[a,b]\right) \tag{2.13}$$

provided $u(b)\neq u(a)$, $\mathscr{T}(f,g;u)$ is given by (1.5), $\mathscr{M}(g;u)$ by (1.6) and $s_h[a,b]$ and $S_h[a,b]$ are as given in.(1.2).

Proof. From the Sonin identity (2.4) with $\gamma = \frac{M+m}{2}$ where $m \leqslant f(t) \leqslant M$ for $t \in [a,b]$ we have

$$[u(b)-u(a)]\mathscr{T}(f,g;u) = \int_a^b \left(f(t) - \frac{M+m}{2}\right)(g(t) - \mathscr{M}(g;u))\,du(t). \tag{2.14}$$

For the choices $w(t) = f(t) - \frac{M+m}{2}$, $h(t) = g(t) - \mathscr{M}(g;u)$ and $v(t) = u(t)$, $t \in [a,b]$, and applying the Beesack inequality (1.1) or equivalently (2.9) gives

$$\begin{aligned} \bigvee_a^b (f) \cdot s_{g(\cdot)-\mathscr{M}(g;u)}[a,b] &\leqslant [u(b)-u(a)]\mathscr{T}(f,g;u) \\ &\leqslant \bigvee_a^b (f) \cdot S_{g(\cdot)-\mathscr{M}(g;u)}[a,b], \end{aligned} \tag{2.15}$$

where we have used the facts that $\inf_{t\in[a,b]}\{w(t)\} = -\frac{M-m}{2}$, $\bigvee_a^b(w) = \bigvee_a^b(f)$ and $\int_a^b (g(t) - \mathscr{M}(g;u))\,du(t) = 0$.

Similarly, using (2.10) or equivalently (2.12) which uses $\sup_{t\in[a,b]}\{w(t)\} = \frac{M-m}{2}$ produces

$$\begin{aligned} -\bigvee_a^b (f) \cdot S_{g(\cdot)-\mathscr{M}(g;u)}[a,b] &\leqslant [u(b)-u(a)]\mathscr{T}(f,g;u) \\ &\leqslant -\bigvee_a^b (f) \cdot s_{g(\cdot)-\mathscr{M}(g;u)}[a,b]. \end{aligned} \tag{2.16}$$

Here again the supremum of $w(t)$ does not appear since $\int_a^b (g(t) - \mathscr{M}(g;u))\,du(t) = 0$.

Combining the results (2.15) and (2.16) gives

$$\begin{aligned} &-\bigvee_a^b (f)\left(S_{g(\cdot)-\mathscr{M}(g;u)}[a,b] - s_{g(\cdot)-\mathscr{M}(g;u)}[a,b]\right) \\ &\leqslant 2[u(b)-u(a)]\mathscr{T}(f,g;u) \leqslant \bigvee_a^b (f)\left(S_{g(\cdot)-\mathscr{M}(g;u)}[a,b] - s_{g(\cdot)-\mathscr{M}(g;u)}[a,b]\right) \end{aligned}$$

and so the proof is complete. □

REMARK 5. The Corollary provides a symetric bound for the *Čebyšev functional for the Riemann-Stieltjes integral as defined in* (1.5). This was made possible via the extension of Beesack's inequality and the Sonin identity. It could also have been obtained directly from (2.1). If we have $L \leqslant A \leqslant U$ then $-\frac{U-L}{2} \leqslant A - \frac{U+L}{2} \leqslant \frac{U-L}{2}$ and so $\left|A - \frac{U+L}{2}\right| \leqslant \frac{U-L}{2}$ where we have the obvious correspondences $U = S_{g(\cdot)-\mathscr{M}(g;u)}[a,b]$, $L = s_{g(\cdot)-\mathscr{M}(g;u)}[a,b]$ and $A = \frac{[u(b)-u(a)]\mathscr{T}(f,g;u)}{\bigvee_a^b(f)}$. The extra step required here is that $\left|A - \frac{U+L}{2}\right| = |A|$ which is effectively Sonin's identity. The difference between the bounds of the corollary and theorem are the same.

COROLLARY 3. *With the assumptions of Theorem 3 and Theorem 4 the following results hold:*

$$M\int_a^b h(t)\,dv(t) - \bigvee_a^b(w)\cdot S_h[a,b] \leqslant \int_a^b w(t)h(t)\,dv(t) \tag{2.17}$$
$$\leqslant m\int_a^b h(t)\,dv(t) + \bigvee_a^b(w)\cdot S_h[a,b],$$

and

$$m\int_a^b h(t)\,dv(t) + \bigvee_a^b(w)\cdot s_h[a,b] \leqslant \int_a^b w(t)h(t)\,dv(t) \tag{2.18}$$
$$\leqslant M\int_a^b h(t)\,dv(t) - \bigvee_a^b(w)\cdot s_h[a,b].$$

Proof. Taking the lower bound from (2.12) and the upper bound from (1.1) produces (2.17). Similarly, taking the lower bound from (1.1) and the upper bound from (2.12) produces (2.18). □

REMARK 6. It should be noticed that each of the bounds for (1.1), (2.12), (2.17) *and* (2.18) consists of three parameters from a choice of four namely m, M, s_h and S_h. Further, equation (2.17) may be written in the equivalent form

$$\left|\int_a^b \left(w(t) - \frac{M+m}{2}\right)h(t)\,dv(t)\right| \leqslant \bigvee_a^b(w)\cdot S_h[a,b] - \frac{M-m}{2}\int_a^b h(t)\,dv(t), \tag{2.19}$$

and equation (2.18) as

$$\left|\int_a^b \left(w(t) - \frac{M+m}{2}\right)h(t)\,dv(t)\right| \leqslant \frac{M-m}{2}\int_a^b h(t)\,dv(t) - \bigvee_a^b(w)\cdot s_h[a,b]. \tag{2.20}$$

These two results with the specialisation $h(t) = 1$ recapture those obtained by a different method in [3], equations (2.7) and (2.8).

We are now in a position to present the main result of the paper.

THEOREM 5. *Let w, h, v real valued functions defined on a compact interval $[a,b]$, where w is of bounded variation with total variation $\bigvee_a^b(w)$, and such that the Riemann-Stieltjes integrals $\int_a^b h(t)\,dv(t)$ and $\int_a^b w(t)h(t)\,dv(t)$ both exist, then*

$$\left|\int_a^b \left(w(t) - \frac{M+m}{2}\right)h(t)\,dv(t)\right| \tag{2.21}$$
$$\leqslant \frac{1}{2}\bigvee_a^b(w)\cdot(S_h - s_h) - \left|\frac{M-m}{2}\int_a^b h(t)\,dv(t) - \frac{1}{2}\bigvee_a^b(w)\cdot(S_h + s_h)\right|$$
$$\leqslant \frac{1}{2}\bigvee_a^b(w)\cdot(S_h - s_h)$$

where $s_h = s_h[a,b]$ and $S_h = S_h[a,b]$ are as given in (1.2) and M and m are the supremum and the infimum of $w(t)$ for $t \in [a,b]$. The constant $\frac{1}{2}$ is best possible and the inequalities are sharp.

Proof. The proof follows simply by obtaining the maximum of the lower bounds and minimum of the two upper bounds in (2.17) and (2.18). Alternatively, finding the minimum of the bounds in (2.19) and (2.20) produces, using $2\min\{X,Y\} = X+Y-|X-Y|$,

$$2\min\left\{S_h\bigvee_a^b(w) - \frac{M-m}{2}\int_a^b h(t)\,dv(t),\, \frac{M-m}{2}\int_a^b h(t)\,dv(t) - s_h\bigvee_a^b(w)\right\}$$

$$= \bigvee_a^b(w)\cdot(S_h - s_h) - \left|(M-m)\int_a^b h(t)\,dv(t) - \bigvee_a^b(w)\cdot(S_h+s_h)\right|.$$

Finally, using properties of the modulus, the total variation and the supremum and infimum give the result as stated. The second coarser upper bound is obvious.

Now, for the sharpness of the inequality. Assume that $v(t) = t$, $t \in [a,b]$ and $h(t) = 1$ so that $s_1 = s_1[a,b] = \inf_{a\leqslant\alpha<\beta\leqslant b}\left[\int_\alpha^\beta dt\right] = \inf_{a\leqslant\alpha<\beta\leqslant b}[\beta-\alpha] = 0$ and $S_1 = S_1[a,b] = \sup_{a\leqslant\alpha<\beta\leqslant b}[\beta-\alpha] = b-a$. With this choice the inequality (2.21) becomes

$$\left|\int_a^b (w(t) - \frac{M+m}{2})dt\right| \leqslant \frac{(b-a)}{2}\left\{\bigvee_a^b(w) - \left|(M-m) - \bigvee_a^b(w)\right|\right\} \tag{2.22}$$
$$\leqslant \frac{b-a}{2}\bigvee_a^b(w).$$

Further, if we consider

$$w_0(t) = \begin{cases} 0 & \text{if } t \in [a,b), \\ K & \text{if } t = b, \end{cases}$$

where $K > 0$ then $m = 0$, $M = K$, $\int_a^b w_0(t)\,dt = 0$, $\bigvee_a^b(w_0) = K$ producing the quantity $\frac{K(b-a)}{2}$ on both sides of (2.22) demonstrating the sharpness. □

REMARK 7. It should be noted that the coarser upper bound in (2.21) may also be obtained from (2.9) and (2.10). A special case for which $h(t) = 1$ of this coarser bound was obtained by [3]. Thus, Theorem 5 is both a generalisation and a refinement of the result in [3].

3. Application of results to the Renewal equation and other examples

One of the fundamental problems in collective risk theory is the determination of time to ruin. The time to ruin satisfies the renewal type equation [6] given by

$$\psi(t) = \psi(0)[\bar{F}(t) + \int_0^t \psi(t-x)dF(x)] \tag{3.1}$$

where $\psi(t)$ is nonincreasing and $F(t)$ is nondecreasing.

THEOREM 6. *$\psi(t)$ satisfies the following bounds*

$$\frac{\psi(0)\bar{F}(t)}{1-\psi(0)F(t)} \leqslant \psi(t) \leqslant \psi(0)[1-(1-\psi(0))F(t)]. \tag{3.2}$$

Proof. Consider the Riemann-Stieltjes integral $\int_0^t \psi(t-x)dF(x)$ then we have by making the associations $w(t)=\psi(t)$, $h(t)=1$ and $v(t)=F(t)$ in the Beesack inequality (1.1) gives

$$\psi(t)F(t)+\bigvee_0^t(\psi)\, s_F[0,t] \leqslant \int_0^t \psi(t-x)dF(x) \leqslant \psi(t)F(t)+\bigvee_0^t(\psi)\, S_F[0,t] \tag{3.3}$$

where

$$s_F[0,t]=\inf_{0\leqslant\alpha<\beta\leqslant t}\left[\int_\alpha^\beta dF(x)\right]=0, \qquad S_F[0,t]=\sup_{0\leqslant\alpha<\beta\leqslant t}\left[\int_\alpha^\beta dF(x)\right]=F(t)$$

and

$$m=\inf_{x\in[0,t]}\{\psi(t-x)\}=\psi(t).$$

Further since $\psi(t)$ is nonincreasing then $\bigvee_0^t(\psi)=\psi(0)-\psi(t)$ so that we have from (3.3)

$$\psi(t)F(t) \leqslant \int_0^t \psi(t-x)dF(x) \leqslant \psi(t)F(t)+[\psi(0)-\psi(t)]F(t)$$

from which using (3.1) gives

$$\psi(0)[\bar{F}(t)+\psi(t)F(t)] \leqslant \psi(t) \leqslant \psi(0)[\bar{F}(t)+\psi(0)F(t)]$$

which may be simplified to give (3.2). The theorem is thus proven. □

REMARK 8. The same result would have been obtained as (3.2) if Theorem 4 were used instead of result (1.1) where $M=\sup_{x\in[0,t]}\{\psi(t-x)\}=\psi(0)$ is used rather than the infimum, m.

REMARK 9. It is important to note that the above bounds produced from using Beesack related inequalities developed earlier in the paper may also be obtained more simply from using the nonincreasing property of $\psi(t)$. Further, both the upper and lower bounds start at $\psi(0)$ however the lower bound tends to zero for large t whereas the upper bound tends to $\psi^2(0)$. As is well known, improved bounds may be obtained from substitution of the bounds into the integral in (3.1). This however may become restrictive depending on the particular distribution function $F(t)$. These bounds may be obtained from using any of the results in Section 2, as indeed they have been obtained previously in the literature. This is the case because of the underlying monotonicity properties of $\psi(t)$ and $F(t)$ in Theorem 6. It must be remebered however that the results contained in Theorem 5 are much more general and less restrictive than the monotonicity properties inherent in the result (3.2).

EXAMPLE 1. It is well known, see Lucas [12] that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \frac{22}{7} - \pi. \tag{3.4}$$

Since the integrand in (3.4) is positive, bounds may be obtained as $0 < \pi < \frac{22}{7}$. These may be improved upon using the results of Section 2. Let $b(x) = x^4(1-x)^4$ then $0 \leqslant b(x) \leqslant b(\frac{1}{2}) = \frac{1}{256}$ and $\int_0^1 b(x)dx = B(5,5) = \frac{1}{630}$ where $B(\alpha,\beta)$ is the Euler beta function. Further let $\omega(x) = \frac{1}{1+x^2}$ so that $\frac{1}{2} = \omega(1) \leqslant \omega(x) \leqslant \omega(0) = 1$ for $0 \leqslant x \leqslant 1$ and $\bigvee_0^1(\omega) = \int_0^1 \left|\omega'(x)\right| dx = \frac{1}{2}$. Making the following associations $w(t) = \omega(t)$, $h(t) = b(t)$ and $\nu(t) = t$ in the Beesack bounds (1.1) we have $\frac{1}{1260} \leqslant \frac{22}{7} - \pi \leqslant \frac{1}{630}$ which gives upon rearangement $\frac{1979}{630} \leqslant \pi \leqslant \frac{3959}{1260}$.

EXAMPLE 2. Consider finding bounds for the following integral

$$\int_0^1 e^{-x^2} x^p dx, \quad p \geqslant 1. \tag{3.5}$$

It may readily be seen that the following bounds hold $0 \leqslant \int_0^1 e^{-x^2} x^p dx \leqslant \frac{1-e^{-1}}{2}$ and $\frac{e^{-1}}{p+1} \leqslant \int_0^1 e^{-x^2} x^p dx \leqslant \frac{1}{p+1}$ where we have used the bounds $0 \leqslant x^p \leqslant 1$ and $e^{-1} \leqslant e^{-x^2} \leqslant 1$ for $0 \leqslant x \leqslant 1$.

If we let $\omega(x) = xe^{-x^2}$ and $h(x) = x^{p-1}$ then $0 \leqslant \omega(x) \leqslant \frac{e^{-\frac{1}{2}}}{\sqrt{2}} = \gamma$ and $\bigvee_0^1(\omega) = 2\gamma - e^{-1}$, $\int_\alpha^\beta h(x)dx = \frac{(\beta^p - \alpha^p)}{p}$ so that from (1.2) $s_h[0,1] = 0$ and $S_h[0,1] = \frac{1}{p}$. Thus from (1.1) we have

$$0 \leqslant \int_0^1 e^{-x^2} x^p dx \leqslant \frac{\sqrt{2}e^{-\frac{1}{2}} - e^{-1}}{p},$$

from (2.12) we have

$$\frac{1}{\sqrt{2}p} - \frac{\sqrt{2}e^{-\frac{1}{2}} - e^{-1}}{p} \leqslant \int_0^1 e^{-x^2} x^p dx \leqslant \frac{1}{\sqrt{2}p}$$

and from (2.21) we have

$$\frac{1}{\sqrt{2}p} - \frac{\sqrt{2}e^{-\frac{1}{2}} - e^{-1}}{p} \leqslant \int_0^1 e^{-x^2} x^p dx \leqslant \frac{\sqrt{2}e^{-\frac{1}{2}} - e^{-1}}{p}.$$

We note that the first two set of bounds have the same difference of $B := \frac{\sqrt{2}e^{-\frac{1}{2}} - e^{-1}}{p}$ and the last has a difference of $2B - \frac{1}{\sqrt{2}p} = B - (\frac{1}{\sqrt{2}p} - B)$, where the term in the bracket is positive and so the bound interval is smaller for $p \geqslant 1$. The largest difference in the bounds occurs at $p = 1$ giving a maximum difference of the bound intervals of 0.2172 which represents an improvement of 44%. It may be noticed that the last set of inequalities chooses the maximum of the minimums and the minimum of the maximums of the previous two sets of inequalities.

4. Bounds for the Čebyšev functional involving the Riemann-Stieltjes integral

In Section 2, some results from [8] bounding the Čebyšev functional involving Riemann-Stieltjes integrals as defined in (1.5), were presented for f,g,u real valued functions defined on a compact interval $[a,b]$ and f of bounded variation.

The following provides tighter bounds for the Čebyšev functional using the further developments of Beesack type results derived in Section 2. It should be noted that $m \leqslant f(t) \leqslant M$ since f is of bounded variation and the result below does not include these bounds explicitly other than through the total variation $\bigvee_a^b(f)$.

COROLLARY 4. *Let $f,g,u:[a,b]\to\mathbb{R}$ be such that f is of bounded variation and the Riemann-Stieltjes integrals $\int_a^b f(t)\,g(t)\,du(t)$, $\int_a^b f(t)\,du(t)$ and $\int_a^b g(t)\,du(t)$ exist. Then*

$$\begin{aligned}
&|[u(b)-u(a)]\mathscr{T}(f,g;u)| \qquad (4.1)\\
&\leqslant \frac{\bigvee_a^b(f)}{2}\left\{\left(S_{g(\cdot)-\mathscr{M}(g;u)}-s_{g(\cdot)-\mathscr{M}(g;u)}\right)-\left|S_{g(\cdot)-\mathscr{M}(g;u)}+s_{g(\cdot)-\mathscr{M}(g;u)}\right|\right\}\\
&\leqslant \frac{\bigvee_a^b(f)}{2}\cdot\left(S_{g(\cdot)-\mathscr{M}(g;u)}[a,b]-s_{g(\cdot)-\mathscr{M}(g;u)}[a,b]\right)
\end{aligned}$$

provided $u(b)\neq u(a)$, $\mathscr{T}(f,g;u)$ is given by (1.5), $\mathscr{M}(g;u)$ by (1.6) and $s_h=s_h[a,b]$ and $S_h=S_h[a,b]$ are as given (1.2).

Proof. From the Sonin identity (2.4) with $\gamma=\frac{M+m}{2}$ where $m\leqslant f(t)\leqslant M$ for $t\in[a,b]$ we have

$$[u(b)-u(a)]\mathscr{T}(f,g;u)=\int_a^b\left(f(t)-\frac{M+m}{2}\right)(g(t)-\mathscr{M}(g;u))\,du(t). \qquad (4.2)$$

For the choices $w(t)=f(t)-\frac{M+m}{2}$, $h(t)=g(t)-\mathscr{M}(g;u)$ and $v(t)=u(t)$, $t\in[a,b]$, in (2.21) of Theorem 5 produces

$$\begin{aligned}
&|[u(b)-u(a)]\mathscr{T}(f,g;u)|\\
&\leqslant \frac{1}{2}\bigvee_a^b(f)\cdot\left(S_{g(\cdot)-\mathscr{M}(g;u)}-s_{g(\cdot)-\mathscr{M}(g;u)}\right)\\
&\quad-\left|\frac{M-m}{2}\int_a^b(g(t)-\mathscr{M}(g;u))\,du(t)-\frac{1}{2}\bigvee_a^b(f)\cdot\left(S_{g(\cdot)-\mathscr{M}(g;u)}+s_{g(\cdot)-\mathscr{M}(g;u)}\right)\right|
\end{aligned}$$

which simplifies to (4.1) on noting that the integral is zero.

The obvious coarser upper bound (4.1) which was also obtained in Corollary 2 is equivalent to Theorem 1. The proof is thus complete. □

The following lemma provides a further identity for the Čebyšev functional involving Riemann-Stieltjes integrals for a previous more restrictive version in which $P'(t)$ was assumed to be continuous, see [4].

LEMMA 2. *Let $f,g,P:[a,b]\to\mathbb{R}$ be such that f and P are of bounded variation and the Riemann-Stieltjes integrals $\int_a^b f(t)g(t)dP(t)$, $\int_a^b f(t)dP(t)$ and $\int_a^b g(t)dP(t)$ exist. Then the following identy holds*

$$\mathscr{T}(f,g;P)=\frac{1}{P^2(b)}\int_a^b \Psi(t)df(t) \tag{4.3}$$

where $\mathscr{T}(f,g;P)$ is as defined by (1.5),

$$\mathscr{M}(h;P)=\frac{\int_a^b h(t)dP(t)}{P(b)} \tag{4.4}$$

with, for $t\in[a,b]$,

$$\Psi(t)=P(t)G(b)-P(b)G(t) \tag{4.5}$$

and,

$$P(t)=\int_a^t dP(x),\quad G(t)=\int_a^t g(x)dP(x). \tag{4.6}$$

Proof. Firstly it may be noticed that $\Psi(a)=\Psi(b)=0$ and so integration by parts from (4.3) produces on using properties of the Riemann-Stieltjes integral (see for example [1])

$$\begin{aligned}\frac{1}{P^2(b)}\int_a^b \Psi(t)df(t)&=-\frac{1}{P^2(b)}\int_a^b f(t)d\Psi(t) \\ &=-\frac{1}{P^2(b)}\int_a^b f(t)\{G(b)-P(b)g(t)\}dP(t) \\ &=\frac{1}{P(b)}\int_a^b f(t)g(t)dP(t)-\frac{G(b)}{P(b)}\cdot\frac{1}{P(b)}\int_a^b f(t)dP(t) \\ &=\mathscr{T}(f,g;P),\end{aligned} \tag{4.7}$$

where we have used the fact that

$$\frac{G(b)}{P(b)}=\mathscr{M}(g;P). \quad \square$$

The following theorem provides sharp bounds for the the Čebyšev functional using developments from Section 2.

THEOREM 7. *Let $f,g,P:[a,b]\to\mathbb{R}$ be such that f and P are of bounded variation and the Riemann-Stieltjes integrals $\int_a^b f(t)g(t)dP(t)$, $\int_a^b f(t)dP(t)$ and $\int_a^b g(t)dP(t)$ exist. Then*

$$\begin{aligned}&\left|P^2(b)\cdot\mathscr{T}(f,g;P)-\frac{\Psi_M+\Psi_m}{2}(f(b)-f(a))\right| \\ &\leqslant\frac{1}{2}\bigvee_a^b(\Psi)\cdot(S-s)-\left|\frac{\Psi_M-\Psi_m}{2}(f(b)-f(a))-\frac{1}{2}\bigvee_a^b(\Psi)\cdot(S+s)\right|\end{aligned} \tag{4.8}$$

where $S=\sup_{a\leqslant\alpha<\beta\leqslant b}[f(\beta)-f(\alpha)]$, $s=\inf_{a\leqslant\alpha<\beta\leqslant b}[f(\beta)-f(\alpha)]$ *and* $\Psi_m\leqslant\Psi(t)\leqslant\Psi_M$ *for* $t\in[a,b]$, *with* $\Psi_M=\sup_{t\in[a,b]}\{\Psi(t)\}$, $\Psi_m=\inf_{t\in[a,b]}\{\Psi(t)\}$.

Proof. From Lemma 2 and specifically identity (4.3) we have

$$P^2(b)\cdot\mathscr{T}(f,g;P)-\frac{\Psi_M+\Psi_m}{2}(f(b)-f(a))=\int_a^b\left[\Psi(t)-\frac{\Psi_M+\Psi_m}{2}\right]df(t). \quad (4.9)$$

For the choices $w(t)=\Psi(t)$, $h(t)=1$ and $v(t)=f(t)$, $t\in[a,b]$, in (2.21) of Theorem 5 produces from (4.9)

$$\begin{aligned}&\left|\int_a^b\left[\Psi(t)-\frac{\Psi_M+\Psi_m}{2}\right]df(t)\right|\\&\leqslant\frac{1}{2}\bigvee_a^b(\Psi)\cdot(S-s)-\left|\frac{\Psi_M-\Psi_m}{2}(f(b)-f(a))-\frac{1}{2}\bigvee_a^b(\Psi)\cdot(S+s)\right|\end{aligned}$$

where

$$\begin{aligned}s&=\inf_{a\leqslant\alpha<\beta\leqslant b}\left[\int_\alpha^\beta df(t)\right]=\inf_{a\leqslant\alpha<\beta\leqslant b}[f(\beta)-f(\alpha)],\\S&=\sup_{a\leqslant\alpha<\beta\leqslant b}\left[\int_\alpha^\beta df(t)\right]=\sup_{a\leqslant\alpha<\beta\leqslant b}[f(\beta)-f(\alpha)]\end{aligned}$$

and so the result as stated follows. □

COROLLARY 5. *Let the conditions of Theorem 5 continue to hold with the further addition that* f *is increasing then*

$$\begin{aligned}&\left|P^2(b)\cdot\mathscr{T}(f,g;P)-\frac{\Psi_M+\Psi_m}{2}(f(b)-f(a))\right| \quad (4.10)\\&=\left|\int_a^b\left[\Psi(t)-\frac{\Psi_M+\Psi_m}{2}\right]df(t)\right|\\&\leqslant\frac{f(b)-f(a)}{2}\left\{2\bigvee_a^b(\Psi)-(\Psi_M-\Psi_m)\right\},\end{aligned}$$

where $\Psi_m\leqslant\Psi(t)\leqslant\Psi_M$, for $t\in[a,b]$.

Proof. For f increasing we have $S=\sup_{a\leqslant\alpha<\beta\leqslant b}[f(\beta)-f(\alpha)]=f(b)-f(a)$ and $s=\inf_{a\leqslant\alpha<\beta\leqslant b}[f(\beta)-f(\alpha)]=0$ and hence the result follows on using properties of the modulus. □

REMARK 10. Using the identity (4.9) then switching the roles of Ψ and f will produce complimentary reults to Theorem 7 and Corollary 5.

Acknowledgement. The work for this paper was initiated while on sabbatical at The University of Melbourne. Some of the work for this article was undertaken while at Victoria University, Melbourne Australia. I would also like to thank Sever Dragomir for his helpful discussions and comments.

REFERENCES

[1] T. M. APOSTOL, *Mathematical Analysis*, Addison-Wesley Publishing Comp., Reading, Massachusetts, 2^{nd} Ed., 1975.
[2] P. R. BEESACK, *Bounds for Riemann-Stieltjes integrals*, Rocky Mountain J. Math., **5** (1) (1975), 75–78.
[3] N. BARNETT AND S. S. DRAGOMIR, *The Beesack-Darst-Pollard inequalities and approximations of the Riemann-Stieltjes integral*, Applied mathematics Letters, **22** (2009), 58–63.
[4] P. CERONE, *On an Identity for the Chebychev Functional and Some Ramifications*, J. Ineq. Pure and Applied Math., **3** (1) (2002), Art. 4, online `http://www.emis.de/journals/JIPAM/article157.html?sid=157`.
[5] P. CERONE AND S. S. DRAGOMIR, *Bounding the Čebyšev functional for the Riemann-Stieltjes integral via a Beesack inequality and applications*, Computers and Mathematics with Applications, **58** (6) (2009), 1247–1252.
[6] D. C. M. DICKSON, *Insurance Risk and Ruin*, Cambridge University Press, 2005.
[7] R. DARST AND H. POLLARD, *An inequality for the Riemann-Stieltjes integral*, Proc. Amer. Math. Soc., **25** (1970), 912–913.
[8] S. S. DRAGOMIR, *Sharp bounds of Čebyšev functional for Stieltjes integrals and applications*, Bull. Austral. Math. Soc., **67** (2003), 257–266.
[9] S. S. DRAGOMIR, *New estimates of the Čebyšev functional for Stieltjes integrals and applications*, J. Korean Math. Soc., **41** (2) (2004), 249–264.
[10] S. S. DRAGOMIR, *Inequalities of Grüss type for the Stieltjes integral and applications*, Kragujevac J. Math., **26** (2004), 89–122.
[11] S. S. DRAGOMIR AND TH. M. RASSIAS, *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht, 2002.
[12] S. LUCAS, *Approximation to π Derived from Integrals with Nonnegative Integrands*, The American Mathematical Monthly, American Mathematical Society, (2005), 166–172.

(Received April 9, 2013)

P. Cerone
Department of Mathematics and Statistics
La Trobe University
Bundoora VIC 3086, Australia
e-mail: p.cerone@latrobe.edu.au