Journal of
Mathematical
Inequalities
Volume 8, Number 3 (2014), 597–611 doi:10.7153/jmi-08-44

A NEW INEQUALITY FOR A POINT IN THE PLANE OF A TRIANGLE

JIAN LIU

(*Communicated by L. Yang*)

Abstract. In this paper, a new geometric inequality for a point in the plane of a triangle is proved by using mathematical software Maple. Also, a general inequality with one parameter and other three similar interesting inequalities checked by the computer are put forward as conjectures.

1. Introduction

For a given triangle ABC, let a,b,c denote the side lengths BC,CA, AB respectively. Let P be a point in the plane of the triangle. Denote the distances from P to the vertices A,B,C by R_1,R_2,R_3 and the distances from P to the sides BC,CA,AB by r_1,r_2,r_3, respectively.

In my recent paper [4], the author gave the following two inequalities:

$$\frac{R_2^2+R_3^2-2r_1^2}{a^2}+\frac{R_3^2+R_1^2-2r_2^2}{b^2}+\frac{R_1^2+R_2^2-2r_3^2}{c^2}\geqslant\frac{3}{2},\tag{1.1}$$

$$\frac{R_2^2+R_3^2+2r_1^2}{a^2}+\frac{R_3^2+R_1^2+2r_2^2}{b^2}+\frac{R_1^2+R_2^2+2r_3^2}{c^2}\geqslant\frac{5}{2}.\tag{1.2}$$

The equalities in (1.1) and (1.2), hold if and only if P is the circumcenter and the Lhuilier-Lemoine point (for this point see e.g. [5, p. 278]) of $\triangle ABC$, respectively.

The author also proved the unified generalization:

$$\frac{R_2^2+R_3^2+\lambda r_1^2}{a^2}+\frac{R_3^2+R_1^2+\lambda r_2^2}{b^2}+\frac{R_1^2+R_2^2+\lambda r_3^2}{c^2}\geqslant\frac{\lambda+8}{4},\tag{1.3}$$

where λ is a real number such that $-2\leqslant\lambda\leqslant 2$.

In the literature, there are few inequalities involving the segments R_1,R_2,R_3,r_1,r_2,r_3 and side lengths a,b,c (see e.g. the monographs [1] and [6]). In this paper, we shall establish a new inequality similar to (1.1) and (1.2), which is stated as follows:

THEOREM 1. *For any point P in the plane of the triangle ABC, the following inequality holds:*

$$\frac{R_1^2-r_1^2}{b^2+c^2}+\frac{R_2^2-r_2^2}{c^2+a^2}+\frac{R_3^2-r_3^2}{a^2+b^2}\geqslant\frac{3}{8}.\tag{1.4}$$

Equality holds only when $b=c$, $a\neq\sqrt{3}c$ and the barycentric coordinates of P with respect to triangle ABC is $(2c^2/(3c^2-a^2),1,1)$ or any permutation thereof.

Mathematics subject classification (2010): 51M16, 51N20.
Keywords and phrases: triangle, point, difference substitution, conjecture.

The equality condition of the above theorem shows that inequality (1.4) is sharp, and means the following interesting fact: If triangle ABC is an isosceles triangle with 120° top-angle, then inequality (1.4) is strict.

The aim of this paper is to prove Theorem 1. Our proof is based on a great number of calculations. We have to use mathematical software (we used Maple 15). In the last section, we shall also present several related interesting conjectures.

2. Preliminaries

In order to prove Theorem 1, we first give several lemmas.

LEMMA 1. *Let $z>0$, $m \geqslant 0$ and $n \geqslant 0$ be real numbers and let*

$$\begin{aligned}
M_1 &\equiv 44555m^4-125593m^3z-798340m^2z^2+1415488mz^3+19044344z^4,\\
M_2 &\equiv 26979m^6+152952m^5z-421680m^4z^2-5325584m^3z^3-10605706m^2z^4,\\
&\quad +41740408mz^5+300352248z^6,\\
M_3 &\equiv 722046m^5-797331m^4z-17119560m^3z^2-26912392m^2z^3+223896576mz^4\\
&\quad +1396689296z^5,\\
M_4 &\equiv 474139m^4-1231480m^3z-4724964m^2z^2+11167712mz^3+98418928z^4,\\
M_5 &\equiv 26979m^7+198312zm^6+48348z^2m^5-3964592z^3m^4-14665190z^4m^3\\
&\quad -2975032z^5m^2+131398056z^6m+498482560z^7,\\
M_6 &\equiv 897006m^5+970953m^4z-12745032m^3z^2-50438152m^2z^3\\
&\quad +8937856mz^4+592608848z^5,
\end{aligned}$$

then the following inequalities strictly hold:

$$M_i>0, \tag{2.1}$$

where $i=1,2,\cdots,6$.

Proof. We now prove inequality $M_1>0$. Since

$$\begin{aligned}
M_1 &= 44555\left(m^4-\frac{125593}{44555}m^3z-\frac{798340}{44555}m^2z^2+\frac{1415488}{44555}mz^3+\frac{19044344}{44555}z^4\right)\\
&= 44555\left[m^4-(2.81\ldots)zm^3-(17.91\ldots)z^2m^2+(31.76\ldots)z^3m+(427.43\ldots)z^4\right],
\end{aligned}$$

to prove $M_1>0$ we only need to prove that

$$m^4-3zm^3-18z^2m^2+31z^3m+427z^4>0. \tag{2.2}$$

Because of the homogeneity, we may assume that $z=1$ and prove $f(m)>0$, where

$$f(m)=m^4-3m^3-18m^2+31m+427$$

and $m > 0$. Then

$$f'(m) = 4m^3 - 9m^2 - 36m + 31,$$

$$f''(m) = 12m^2 - 18m - 36.$$

Solving $f'(m) = 0$ we obtain

$$m_1 \approx -2.526\ldots, \quad m_2 \approx 0.764\ldots, \quad m_3 \approx 4.011\ldots,$$

and

$$f''(m_1) \approx 86.072\ldots > 0, \quad f''(m_2) \approx -42.747\ldots < 0, \quad f''(m_3) \approx 84.925\ldots > 0.$$

Hence $f(m)$ is strictly increasing on interval $(0, m_2)$ and $(m_3, +\infty)$, and is decreasing on (m_2, m_3). The minimal value of $f(m)$ on interval $(0, +\infty)$ is equal to $f(m_2) \approx 326.994\cdots$. So, we have $f(m) \geqslant f(m_2) > 0$ for $m > 0$ and inequality $M_1 > 0$ is proved.

In the same way, it is easy to prove the rest strict inequalities $M_2 > 0$, $M_3 > 0$, $M_4 > 0$, $M_5 > 0$ and $M_6 > 0$ (We omit details). □

LEMMA 2. *The polynomial inequality*

$$a_3x^3 + a_2x^2 + a_1x + a_0 \geqslant 0 \tag{2.3}$$

with real coefficients a_3, a_2, a_1, a_0 $(a_3 > 0)$ *holds for all positive real numbers* x *if, and only if, one of the following conditions is valid:* (i) $a_0, a_1, a_2 \geqslant 0$; (ii) $a_0 = 0$, $a_2^2 - 4a_1a_3 \leqslant 0$; (iii) $a_0 > 0$, $D_3 \leqslant 0$, *where*

$$D_3 = -27(a_0a_3)^2 + 18a_0a_1a_2a_3 + (a_1a_2)^2 - 4a_2^3a_0 - 4a_1^3a_3. \tag{2.4}$$

REMARK 1. According to [8, p. 159], the above lemma is due to L. Yang. In [12, pp. 53–54], Yang et al also give similar conclusions for quartic polynomials. These are applications of their "Decision Theorems" for polynomials, see [10] and [12].

LEMMA 3. *Let* $x > 0$, $m \geqslant 0$ *and* $n \geqslant 0$ *be real numbers, then*

$$M_7 \equiv k_3m^3 + k_2m^2 + k_1m + k_0 \geqslant 0, \tag{2.5}$$

where

$$k_3 = 30154336x^6 + 3944992x^5n + 211832x^4n^2 - 7544x^3n^3 + 226x^2n^4 + 318xn^5 + 21n^6,$$
$$k_2 = n(3575640x^6 + 311800x^5n - 4706x^4n^2 - 3416x^3n^3 - 140x^2n^4 + 16xn^5 + n^6),$$
$$k_1 = 4(61868x^4 + 3814x^3n - 251x^2n^2 - 60xn^3 - 3n^4)x^2n^2,$$
$$k_0 = 12(832x^2 + 72xn + 3n^2)x^4n^3.$$

Proof. Clearly, (2.5) holds trivially when $m=0$. If $m>0$, note that

$$211832x^4n^2-7544x^3n^3+226x^2n^4=2x^2n^2(105916x^2-3772xn+113n^2)>0,$$

hence $k_3>0$. Also, with the help of famous mathematical software Maple (we used Maple 15 in this paper), it is easy to obtain the following identity:

$$-27(k_0k_3)^2+18k_0k_1k_2k_3+(k_1k_2)^2-4k_2^3k_0-4k_1^3k_3=-32x^6n^6K, \tag{2.6}$$

where

$$\begin{aligned}K=\ &163361142743260657822 72x^{18}+898607790888363524 1472x^{17}n\\ &+2603480913174237230208x^{16}n^2+5094131740706966 01216x^{15}n^3\\ &+7551138373723667 5264x^{14}n^4+929080898290862 2880x^{13}n^5\\ &+1024729896824019240x^{12}n^6+105857417733128296x^{11}n^7\\ &+10270144842212106x^{10}n^8+919660264355296x^9n^9\\ &+75176103555384x^8n^{10}+5516130039344x^7n^{11}\\ &+350793644578x^6n^{12}+18830450220x^5n^{13}+854524830x^4n^{14}\\ &+32314324x^3n^{15}+962017x^2n^{16}+21840xn^{17}+360n^{18}.\end{aligned}$$

Obviously, we have $K>0$, thus the left hand of (2.6) is nonpositive. Hence, inequality $M_7\geqslant 0$ holds for $m>0$ by Lemma 2 and the proof of Lemma 3 is completed. □

LEMMA 4. *For any triangle ABC with sides a,b,c, we have*

$$\begin{aligned}Q_0\equiv\ &2(b-c)^2(b+c)^2a^{12}-(b-c)^2(b+c)^2(b^2+c^2)a^{10}-(3b^4+4b^2c^2\\ &-3c^4)(3b^4-4b^2c^2-3c^4)a^8+(b^2+c^2)(15b^8-13b^6c^2+16b^4c^4\\ &-13b^2c^6+15c^8)a^6+(-9b^{12}+2b^{10}c^2+3b^8c^4+40b^6c^6+3b^4c^8\\ &+2b^2c^{10}-9c^{12})a^4+(b^2+c^2)(2b^{12}-5b^{10}c^2-18b^8c^4\\ &+46b^6c^6-18b^4c^8-5b^2c^{10}+2c^{12})a^2\\ &+2b^2c^2(b-c)^4(b+c)^4(b^4+3b^2c^2+c^4)\geqslant 0,\end{aligned} \tag{2.7}$$

with equality holding if and only if $a:b:c=1:1:\sqrt{3}$.

Next, we shall prove inequality (2.7) by using the method of so-called "Difference Substitution". The proof of Lemma 2 in [4] is just proved by using this method (for more examples, see e.g. [2], [7], [9], [11]).

Proof. Let $b+c-a=2x$, $c+a-b=2y$, $a+b-c=2z$, then $a=y+z$, $b=z+x$, $c=x+y$, and (2.7) becomes the following equivalent inequality:

$$\begin{aligned}Q_0\equiv\ &2(z-y)^2(z+2x+y)^2(y+z)^{12}-(z-y)^2(z+2x+y)^2[(z+x)^2+(x+y)^2]\\ &\cdot(y+z)^{10}-[3(z+x)^4+4(z+x)^2(x+y)^2-3(x+y)^4]\\ &[3(z+x)^4-4(z+x)^2(x+y)^2-3(x+y)^4](y+z)^8+[(z+x)^2+(x+y)^2]\\ &\cdot[15(z+x)^8-13(z+x)^6(x+y)^2+16(z+x)^4(x+y)^4-13(z+x)^2(x+y)^6.\\ &+15(x+y)^8](y+z)^6+[-9(z+x)^{12}+2(z+x)^{10}(x+y)^2\end{aligned}$$

$$+3(z+x)^8(x+y)^4+40(z+x)^6(x+y)^6+3(z+x)^4(x+y)^8+2(z+x)^2(x+y)^{10}$$
$$-9(x+y)^{12}](y+z)^4+[(z+x)^2+(x+y)^2]$$
$$[2(z+x)^{12}-5(z+x)^{10}(x+y)^2-18(z+x)^8(x+y)^4+46(z+x)^6(x+y)^6$$
$$-18(z+x)^4(x+y)^8-5(z+x)^2(x+y)^{10}+2(x+y)^{12}](y+z)^2$$
$$+2(z+x)^2(x+y)^2(z-y)^4(z+2x+y)^4\cdot[(z+x)^4+3(z+x)^2(x+y)^2+(x+y)^4]$$
$$\geqslant 0, \tag{2.8}$$

which involves three positive numbers.

Due to the symmetry of Q_0 with respect to the variables y and z, we assume that $y \geqslant z$ without loss of generality, then we only need to consider three cases: $x \geqslant y \geqslant z$, $y \geqslant x \geqslant z$ and $y \geqslant z \geqslant x$ to prove inequality $Q_0 \geqslant 0$.

Case 1. The positive numbers x,y,z satisfy $x \geqslant y \geqslant z$.

In this case, we put

$$\begin{cases} y = z+m, & (m \geqslant 0) \\ x = z+m+n & (n \geqslant 0). \end{cases} \tag{2.9}$$

Substituting (2.9) into (2.8) and using Maple software for the calculations, one obtains

$$\begin{aligned} Q_0 = n^3(&d_{14}n^{11}+d_{13}n^{10}+d_{12}n^9+d_{11}n^8+d_{10}n^7+d_9n^6+d_8n^5+d_7n^4 \\ &+d_6n^3+d_5n^2+d_4n+d_3)+d_2n^2+d_1n+d_0, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
d_{14} &= 8(m+2z)^2,\\
d_{13} &= 56(3m+4z)(m+2z)^2,\\
d_{12} &= 1660m^4+10496m^3z+27024m^2z^2+30144mz^3+12160z^4,\\
d_{11} &= 16(3m+4z)(213m^4+1206m^3z+3400m^2z^2+4024mz^3+1648z^4),\\
d_{10} &= 43836m^6+323376m^5z+1306352m^4z^2+3049856m^3z^3+3904512m^2z^4\\
&\quad +2530816mz^5+650240z^6,\\
d_9 &= 16(3m+4z)(2883m^6+17763m^5z+74418m^4z^2+192716m^3z^3+266152m^2z^4\\
&\quad +179808mz^5+46912z^6),\\
d_8 &= 331680m^8+2448000m^7z+11690928m^6z^2+42245312m^5z^3\\
&\quad +100168288m^4z^4+144960256m^3z^5+122984960m^2z^6+56190976mz^7\\
&\quad +10670080z^8,\\
d_7 &= 32(3m+4z)(6387m^8+36702m^7z+160686m^6z^2+669496m^5z^3\\
&\quad +1818332m^4z^4+2863040m^3z^5+2552320m^2z^6+1199360mz^7\\
&\quad +230912z^8),\\
d_6 &= 878524m^{10}+5846976m^9z+23354272m^8z^2+109841984m^7z^3\\
&\quad +443794464m^6z^4+1155031168m^5z^5+1879849600m^4z^6\\
&\quad +1922064384m^3z^7+1204111360m^2z^8+423141376mz^9\\
&\quad +63938560z^{10},
\end{aligned}$$

$$
\begin{aligned}
d_5 =\ & 16(3m+4z)(20226m^{10}+98298m^9z+213333m^8z^2+1355640m^7z^3\\
&+8138010m^6z^4+25189976m^5z^5+44522488m^4z^6+47605504m^3z^7\\
&+30632832m^2z^8+10952192mz^9+1674752z^{10}),\\
d_4 =\ & 814460m^{12}+4722544m^{11}z+4398128m^{10}z^2+4979008m^9z^3\\
&+276578448m^8z^4+1678945152m^7z^5+4940702656m^6z^6\\
&+8819303168m^5z^7+10247248384m^4z^8+7844782080m^3z^9\\
&+3834327040m^2z^{10}+1088012288mz^{11}+136675328z^{12},\\
d_3 =\ & 8(3m+4z)(20951m^{12}+89110m^{11}z-251186m^{10}z^2-1596680m^9z^3\\
&+2830976m^8z^4+38088688m^7z^5+128061856m^6z^6+240383552m^5z^7\\
&+286529664m^4z^8+222837760m^3z^9+110180352m^2z^{10}+31576064mz^{11}\\
&+4005888z^{12}),\\
d_2 =\ & 215832m^{14}+1223616m^{13}z-3373440m^{12}z^2-42604672m^{11}z^3\\
&-84845648m^{10}z^4+333923264m^9z^5+2402817984m^8z^6+7052985344m^7z^7\\
&+12845485568m^6z^8+15931738112m^5z^9+13794775040m^4z^{10}\\
&+8259960832m^3z^{11}+3274604544m^2z^{12}+775815168mz^{13}\\
&+83361792z^{14},\\
d_1 =\ & 32(3m+4z)(m+2z)(5m^2+12mz+8z^2)(120m^7+288m^6z-1763m^5z^2\\
&-9416m^4z^3-18604m^3z^4-18768m^2z^5-9696mz^6\\
&-2048z^7)(m^2-4mz-8z^2)(m+z)^2,\\
d_0 =\ & 32(9m^2+20mz+12z^2)(m+2z)^2(5m^2+12mz+8z^2)^2\\
&\cdot(m^2-4mz-8z^2)^2(m+z)^4.
\end{aligned}
$$

Noting that $z>0$, $m\geqslant 0$ and $n\geqslant 0$, hence $d_{14},d_{13},\cdots,d_4$ obviously are all positive. Also, by the identity

$$
\begin{aligned}
&89110m^{11}z-251186m^{10}z^2-1596680m^9z^3+2830976m^8z^4+38088688m^7z^5\\
&=2zm^7M_1
\end{aligned}
$$

and the strict inequality $M_1>0$ of Lemma 1, we see that $d_3>0$. So, it remains to prove that

$$
d_2n^2+d_1n+d_0\geqslant 0. \tag{2.11}
$$

We shall first show that $d_2>0$. It is sufficient to prove that

$$
\begin{aligned}
&215832m^{14}+1223616m^{13}z-3373440m^{12}z^2-42604672m^{11}z^3-84845648m^{10}z^4\\
&+333923264m^9z^5+2402817984m^8z^6>0,
\end{aligned}
$$

which is equivalent to

$$
8m^8M_2>0.
$$

(where M_2 is defined as in Lemma 1). Thus, $d_2>0$ holds true. Since also $d_0\geqslant 0$, we need to prove that the quadratic discriminant of the left hand side of (2.11) is nonnegative, namely

$$F_1 \equiv d_1^2 - 4d_2d_0 \leqslant 0.$$

But, the calculations gives

$$F_1 = -1024(m+2z)^2(5m^2+12mz+8z^2)^2(m^2-4mz-8z^2)^2(m+z)^6Q_1, \quad (2.12)$$

where

$$\begin{aligned} Q_1 = {} & 113211m^{14}+722046m^{13}z-797331m^{12}z^2-17119560m^{11}z^3-26912392m^{10}z^4 \\ & +223896576m^9z^5+1396689296m^8z^6+4056610560m^7z^7+7476201472m^6z^8 \\ & +9465608192m^5z^9+8408941568m^4z^{10}+5184184320m^3z^{11} \\ & +2121891840m^2z^{12}+520159232mz^{13}+57933824z^{14}. \end{aligned}$$

Note that

$$\begin{aligned} & 722046m^{13}z-797331m^{12}z^2-17119560m^{11}z^3-26912392m^{10}z^4+223896576m^9z^5 \\ & +1396689296m^8z^6 = zm^8M_3 \end{aligned}$$

and strict inequality $M_3 > 0$ of Lemma 1, we conclude that $Q_1 > 0$. Hence, $F_1 \leqslant 0$ is valid and then inequality $Q_0 \geqslant 0$ is proved in the first case.

Since inequality $d_3 > 0$ holds strictly, from (2.10) we see that the equality in $Q_0 \geqslant 0$ holds if and only if $n = 0$ and $d_2n^2 + d_1n + d_0 = 0$, namely $n = 0$ and $d_0 = 0$. From (2.9) and $n = 0$, we have $m = x - z$ and $x = y$. Again, $d_0 = 0$ means that

$$m^2 - 4mz - 8z^2 = 0.$$

Hence,

$$(x-z)^2 - 4(x-z)z - 8z^2 = 0,$$

which is equivalent to

$$x^2 - 6zx - 3z^2 = 0.$$

Then it is easy to get that $x : z = (3+2\sqrt{3}) : 1$. Therefore, we conclude that the equality in (2.8) holds if and only if $x : y : z = (3+2\sqrt{3}) : (3+2\sqrt{3}) : 1$ in the first case.

Case 2. The positive numbers x, y, z satisfy $y \geqslant x \geqslant z$.

In this case, we put

$$\begin{cases} x = z+m, & (m \geqslant 0) \\ y = z+m+n. & (n \geqslant 0) \end{cases} \quad (2.13)$$

Substituting (2.13) into (2.8), we obtain that

$$\begin{aligned} Q_0 = {} & n^3(e_{14}n^{11}+e_{13}n^{10}+e_{12}n^9+e_{11}n^8+e_{10}n^7+e_9n^6+e_8n^5+e_7n^4 \\ & +e_6n^3+e_5n^2+e_4n+e_3)+e_2n^2+e_1n+e_0, \end{aligned} \quad (2.14)$$

where

$$
\begin{aligned}
e_{14} &= 8m^2,\\
e_{13} &= 8m(21m^2+40mz+12z^2),\\
e_{12} &= 1660m^4+6288m^3z+7616m^2z^2+3072mz^3+288z^4,\\
e_{11} &= 10224m^5+56496m^4z+117424m^3z^2+110784m^2z^3+46048mz^4+6912z^5,\\
e_{10} &= 43836m^6+307248m^5z+882528m^4z^2+1302592m^3z^3+1037712m^2z^4\\
&\quad +433216mz^5+79872z^6,\\
e_9 &= 138384m^7+1128400m^6z+4013792m^5z^2+7938368m^4z^3+9417344m^3z^4\\
&\quad +6825024m^2z^5+2904704mz^6+583680z^7,\\
e_8 &= 331680m^8+2955744m^7z+12078064m^6z^2+29285888m^5z^3\\
&\quad +46051584m^4z^4+48593024m^3z^5+34148928m^2z^6+14757632mz^7\\
&\quad +2983424z^8,\\
e_7 &= 613152m^9+5683968m^8z+25007456m^7z^2+69441728m^6z^3\\
&\quad +135212128m^5z^4+191616768m^4z^5+195930368m^3z^6+136946688m^2z^7\\
&\quad +58090496mz^8+11165696z^9,\\
e_6 &= 878524m^{10}+8135056m^9z+35959792m^8z^2+106287680m^7z^3\\
&\quad +244394592m^6z^4+452478848m^5z^5+644409536m^4z^6+657908224m^3z^7\\
&\quad +444869120m^2z^8+176879616mz^9+31178752z^{10},\\
e_5 &= 970848m^{11}+8689040m^{10}z+35269728m^9z^2+98093952m^8z^3\\
&\quad +255429984m^7z^4+645808640m^6z^5+1312540672m^5z^6\\
&\quad +1893031168m^4z^7+1836905472m^3z^8+1140426752m^2z^9\\
&\quad +409944064mz^{10}+65011712z^{11},\\
e_4 &= 814460m^{12}+6876208m^{11}z+22233088m^{10}z^2+38445888m^9z^3\\
&\quad +104126592m^8z^4+521140864m^7z^5+1736656832m^6z^6+3542442752m^5z^7\\
&\quad +4657004544m^4z^8+4011388928m^3z^9+2203160576m^2z^{10}\\
&\quad +703381504mz^{11}+99680256z^{12},\\
e_3 &= 502824m^{13}+3954256m^{12}z+7586224m^{11}z^2-19703680m^{10}z^3-75599424m^9z^4\\
&\quad +178683392m^8z^5+1574702848m^7z^6+4532252160m^6z^7+7674584064m^5z^8\\
&\quad +8514465792m^4z^9+6309117952m^3z^{10}+3028811776m^2z^{11}+856096768mz^{12}\\
&\quad +108527616z^{13},\\
e_2 &= 215832m^{14}+1586496m^{13}z+386784m^{12}z^2-31716736m^{11}z^3\\
&\quad -117321520m^{10}z^4-23800256m^9z^5+1051184448m^8z^6+3987860480m^7z^7\\
&\quad +8187181568m^6z^8+11028666368m^5z^9+10226913280m^4z^{10}\\
&\quad +6519652352m^3z^{11}+2745139200m^2z^{12}+690094080mz^{13}+78643200z^{14},\\
e_1 &= 32(3m+4z)(m+2z)(5m^2+12mz+8z^2)(m^2-4mz-8z^2)(120m^7+396m^6z\\
&\quad -961m^5z^2-7096m^4z^3-15348m^3z^4-16560m^2z^5\\
&\quad -9120mz^6-2048z^7)(m+z)^2,
\end{aligned}
$$

$$e_0 = 32(9m^2+20mz+12z^2)(m+2z)^2(5m^2+12mz+8z^2)^2(m^2-4mz-8z^2)^2(m+z)^4.$$

Since $m \geqslant 0$, $n \geqslant 0$ and $z > 0$, we see that $e_{14}, e_{13}, \dots, e_4$ are all positive. Moreover, by the identity:

$$7586224m^{11}z^2 - 19703680m^{10}z^3 - 75599424m^9z^4 + 178683392m^8z^5 \\ +1574702848m^7z^6 = 16m^7z^2M_4$$

and inequality $M_4 > 0$ of Lemma 1 we also know that $e_3 > 0$. Thus, by (2.14), to prove $Q_0 \geqslant 0$ it remains to prove that

$$e_2n^2 + e_1n + e_0 \geqslant 0. \tag{2.15}$$

Note that

$$215832m^{14} + 1586496m^{13}z + 386784m^{12}z^2 - 31716736m^{11}z^3 - 117321520m^{10}z^4 \\ -23800256m^9z^5 + 1051184448m^8z^6 + 3987860480m^7z^7 = 8m^7M_5$$

and inequality $M_5 > 0$ of Lemma 1, we see that $e_2 > 0$. Since also $e_0 \geqslant 0$, to prove (2.15) we only need to prove that

$$F_2 \equiv e_1^2 - 4e_2e_0 \leqslant 0.$$

With the help of Maple software, it is easy to obtain the following identity:

$$F_2 = -1024(m+2z)^2(5m^2+12mz+8z^2)^2(m^2-4mz-8z^2)^2(m+z)^6Q_2, \tag{2.16}$$

where

$$\begin{aligned} Q_2 = {} & 113211m^{14} + 897006m^{13}z + 970953m^{12}z^2 - 12745032m^{11}z^3 - 50438152m^{10}z^4 \\ & +8937856m^9z^5 + 592608848m^8z^6 + 2197773056m^7z^7 + 4551321088m^6z^8 \\ & +6234726400m^5z^9 + 5903295488m^4z^{10} + 3854295040m^3z^{11} \\ & +1666842624m^2z^{12} + 431685632mz^{13} + 50855936z^{14}. \end{aligned}$$

Also, by the following identity

$$897006m^{13}z + 970953m^{12}z^2 - 12745032m^{11}z^3 - 50438152m^{10}z^4 + 8937856m^9z^5 \\ +592608848m^8z^6 = zm^8M_6$$

and inequality $M_6 > 0$ of Lemma 1 we see that $Q_2 > 0$ holds strictly. Hence $F_2 \leqslant 0$ and (2.15) are proved. This completes the proof of $Q_0 \geqslant 0$ in the second case. Furthermore, it is easy to know that (as done in the above case) the equality condition of (2.8) is the same as that of the first case, namely the equality holds if and only if $x : y : z = (3+2\sqrt{3}) : (3+2\sqrt{3}) : 1$.

Case 3. The positive numbers x, y, z satisfy $y \geqslant z \geqslant x$.

In this case, we put

$$\begin{cases} z = x+m, & (m \geqslant 0) \\ y = x+m+n. & (n \geqslant 0) \end{cases} \tag{2.17}$$

Substituting it into (2.8), we obtain

$$Q_0 = f_{12}n^{12} + f_{11}n^{11} + f_{10}n^{10} + f_9n^9 + f_8n^8 + f_7n^7 + f_6n^6 \\ + f_5n^5 + f_4n^4 + f_3n^3 + f_2n^2 + f_1n + f_0 + 8n^7M_7, \tag{2.18}$$

where M_7 is defined as in Lemma 3, and

$$\begin{aligned}
f_{12} &= 1660m^4,\\
f_{11} &= 144(71m+173x)m^4,\\
f_{10} &= 12(3653m^2+13064mx+11108x^2)m^4,\\
f_9 &= 138384m^7+694880m^6x+1289968m^5x^2+1331072m^4x^3+583680x^7,\\
f_8 &= 331680m^8+2256768m^7x+6799744m^6x^2+13660160m^5x^3+23121168m^4x^4\\
&\quad +14362880mx^7+2983424x^8,\\
f_7 &= 613152m^9+5466336m^8x+23204416m^7x^2+65756352m^6x^3\\
&\quad +139915872m^5x^4+220770816m^4x^5+168078336m^2x^7+66268160mx^8\\
&\quad +11165696x^9,\\
f_8 &= 878524m^{10}+9931024m^9x+54567856m^8x^2+195580800m^7x^3\\
&\quad +501198944m^6x^4+929600640m^5x^5+1222265024m^4x^6+1098245632m^3x^7\\
&\quad +636337664m^2x^8+213067776mx^9+31178752x^{10},\\
f_5 &= 970848m^{11}+13477552m^{10}x+90590112m^9x^2+388274880m^8x^3\\
&\quad +1162263936m^7x^4+2502831744m^6x^5+3888553344m^5x^6\\
&\quad +4304334080m^4x^7+3299851264m^3x^8+1660557312m^2x^9\\
&\quad +492257280mx^{10}+65011712x^{11},\\
f_4 &= 814460m^{12}+13461568m^{11}x+106298352m^{10}x^2+526098624m^9x^3\\
&\quad +1798191840m^8x^4+4424602368m^7x^5+7967363200m^6x^6\\
&\quad +10512597504m^5x^7+10045245952m^4x^8+6761459712m^3x^9\\
&\quad +3038466048m^2x^{10}+817840128mx^{11}+99680256x^{12},\\
f_3 &= 8(2x+m)(62853m^{12}+1074818m^{11}x+8651516m^{10}x^2+42935256m^9x^3\\
&\quad +145119808m^8x^4+349750432m^7x^5+613656256m^6x^6+787605120m^5x^7\\
&\quad +732701696m^4x^8+481435648m^3x^9+212027392m^2x^{10}\\
&\quad +56201216mx^{11}+6782976x^{12}),\\
f_2 &= 8(26979m^{12}+470880m^{11}x+3799470m^{10}x^2+18645688m^9x^3+61772264m^8x^4\\
&\quad +145241216m^7x^5+248220320m^6x^6+310501248m^5x^7+282107520m^4x^8\\
&\quad +181570560m^3x^9+78603264m^2x^{10}+20557824mx^{11}+2457600x^{12})(2x+m)^2,\\
f_1 &= 32(3m+4x)(x+m)(5m^2+12mx+8x^2)(120m^4+567m^3x+1020m^2x^2\\
&\quad +828mx^3+256x^4)(2x+m)^7,\\
f_0 &= 32(9m^2+20mx+12x^2)(x+m)^2(5m^2+12mx+8x^2)^2(2x+m)^8.
\end{aligned}$$

Clearly, $f_{12}, f_{11}, \cdots, f_0$ are all positive for $x > 0$, $m \geqslant 0$, $n \geqslant 0$. By the identity (2.18) and the inequality $M_7 \geqslant 0$ of Lemma 3, we deduce that $Q_0 > 0$ strictly hold in the third case.

Combining the arguments of the three cases above, we finish the proof of inequality (2.8) and know that the equality holds if and only if $x : y : z = (3+2\sqrt{3}) : (3+2\sqrt{3}) : 1$. The inequality (2.7) equivalent with (2.8) is also proved, while the equality holds if and only if

$$(b+c-a) : (c+a-b) : (a+b-c) = (3+2\sqrt{3}) : (3+2\sqrt{3}) : 1,$$

wherefrom we easily obtain that $a : b : c = 1 : 1 : \sqrt{3}$. This completes the proof of Lemma 4. □

By applying barycentric coordinates, the inequality (1.4) in Theorem 1 can be transformed into a ternary quadratic inequality involving three sides of the triangle. For general ternary quadratic inequality, we have the following known important result:

LEMMA 5. *Let p_1, p_2, p_3, q_1, q_2, q_3 be real numbers such that $p_1 > 0$, $p_2 > 0$, $p_3 > 0$, $4p_2p_3 - q_1^2 > 0$, $4p_3p_1 - q_2^2 > 0$, $4p_1p_2 - q_3^2 > 0$ and*

$$D_0 \equiv 4p_1p_2p_3 - (q_1q_2q_3 + p_1q_1^2 + p_2q_2^2 + p_3q_3^2) \geqslant 0. \tag{2.19}$$

Then the inequality

$$p_1x^2 + p_2y^2 + p_3z^2 \geqslant q_1yz + q_2zx + q_3xy \tag{2.20}$$

holds for all real numbers x, y, z. If $x, y, z \neq 0$, then the equality in (2.20) holds if and only if $D_0 = 0$ and

$$(2p_1q_1 + q_2q_3)x = (2p_2q_2 + q_3q_1)y = (2p_3q_3 + q_1q_2)z. \tag{2.21}$$

For the elementary proofs of Lemma 5, see e.g. [4] and [5].

3. Proof of Theorem 1

We are now in a position to prove our theorem.

Proof. Let (x, y, z) be the barycentric coordinates of point P with respect to triangle ABC (where x, y, z are real numbers such that $x + y + z \neq 0$), then we have the following well known formulas (which have been used by the author in [3], [4] and [5]):

$$(x+y+z)^2R_1^2 = (x+y+z)(yc^2 + zb^2) - (yza^2 + zxb^2 + xyc^2) \tag{3.1}$$

and

$$r_1 = \left| \frac{2xS}{(x+y+z)a} \right|, \tag{3.2}$$

etc., where S is the area of $\triangle ABC$. If we denote cyclic sum by $\sum$, then inequality (1.4) can be written as $\sum \frac{R_1^2-r_1^2}{b^2+c^2} \geqslant \frac{3}{8}$ and we have that

$$\begin{aligned}
&\sum \frac{R_1^2-r_1^2}{b^2+c^2} \\
&= \sum \frac{1}{b^2+c^2}\left[\frac{yc^2+zb^2}{x+y+z} - \frac{yza^2+zxb^2+xyc^2}{(x+y+z)^2}\right] - \frac{4S^2}{(\sum x)^2}\sum \frac{x^2}{a^2(b^2+c^2)} \\
&= \frac{1}{\sum x}\sum \frac{yc^2+zb^2}{b^2+c^2} - \frac{\sum yza^2}{(\sum x)^2}\sum \frac{1}{b^2+c^2} - \frac{4S^2}{(\sum x)^2}\sum \frac{x^2}{a^2(b^2+c^2)}.
\end{aligned}$$

Further, using the equivalent form of Heron's area formula:

$$16S^2 = 2\sum b^2c^2 - \sum a^4, \tag{3.3}$$

we see that inequality (1.4) is equivalent to

$$\begin{aligned}
&8(abc)^2\sum x\sum (c^2+a^2)(a^2+b^2)(yc^2+zb^2) - 8(abc)^2\sum (c^2+a^2)(a^2+b^2)\sum yza^2 \\
&-2\left(2\sum b^2c^2 - \sum a^4\right)\sum b^2c^2(c^2+a^2)(a^2+b^2)x^2 \\
&-3(abc)^2(b^2+c^2)(c^2+a^2)(a^2+b^2)\left(\sum x\right)^2 \geqslant 0.
\end{aligned} \tag{3.4}$$

Expansion and simplification give the following equivalent inequality required to prove:

$$m_1x^2+m_2y^2+m_3z^2 \geqslant n_1yz+n_2zx+n_3xy, \tag{3.5}$$

where

$$\begin{aligned}
m_1 &= b^2c^2\Big[2a^8 - 5(b^2+c^2)a^6 + 3(b^4+c^4)a^4 \\
&\qquad +(2b^2+c^2)(2c^2+b^2)(b^2+c^2)a^2 + 2b^2c^2(b-c)^2(b+c)^2\Big], \\
m_2 &= c^2a^2\Big[2b^8 - 5(a^2+c^2)b^6 + 3(c^4+a^4)b^4 \\
&\qquad +(2c^2+a^2)(2a^2+c^2)(c^2+a^2)b^2 + 2c^2a^2(a-c)^2(c+a)^2\Big], \\
m_3 &= a^2b^2\Big[2c^8 - 5(a^2+b^2)c^6 + 3(a^4+b^4)c^4 \\
&\qquad +(2a^2+b^2)(2b^2+a^2)(a^2+b^2)c^2 + 2a^2b^2(a-b)^2(a+b)^2\Big], \\
n_1 &= 2a^2b^2c^2(c^2+a^2)(a^2+b^2)(4a^2-b^2-c^2), \\
n_2 &= 2a^2b^2c^2(a^2+b^2)(b^2+c^2)(4b^2-c^2-a^2), \\
n_3 &= 2a^2b^2c^2(b^2+c^2)(c^2+a^2)(4c^2-a^2-b^2).
\end{aligned}$$

Next, we shall consider two cases to finish the proof of inequality (3.5).

Case 1. $a:b:c = 1:1:\sqrt{3}$.

In this case, to prove (3.5) we may assume that $a=b=1$, $c=\sqrt{3}$, then it is easy to get that $m_1=m_2=528$, $m_3=0$, $n_1=n_2=0$, $n_3=960$, and (3.5) becomes

$$528(x^2+y^2) > 960xy,$$

which is an obvious strict inequality for all real numbers x and y.

Case 2. $a:b:c \neq 1:1:\sqrt{3}$.

Firstly, by using software Maple we obtain the following identity:

$$4m_2m_3 - n_1^2 = 8b^2c^2a^4Q_0, \tag{3.6}$$

where Q_0 is defined as in Lemma 4. By Lemma 4 and the hypothesis, we have strict inequality $Q_0 > 0$. Thus, the strict inequality $4m_2m_3 - n_1^2 > 0$ follows from identity (3.6) and similar inequalities $4m_3m_1 - n_2^2 > 0$ and $4m_1m_2 - n_3^2 > 0$ also hold.

On the other hand, one can check the following identity by using Maple:

$$\begin{aligned}
&4m_1m_2m_3 - (n_1n_2n_3 + m_1n_1^2 + m_2n_2^2 + m_3n_3^2)\\
&= 32(abc)^4(b+c)^2(c+a)^2(a+b)^2(b+c-a)^2(c+a-b)^2(a+b-c)^2\\
&\quad \cdot(a^4+b^4+c^4+3b^2c^2+3c^2a^2+3a^2b^2)(b-c)^2(c-a)^2(a-b)^2,
\end{aligned} \tag{3.7}$$

which shows the left hand is nonnegative. According to Lemma 5, we have proved inequality (3.5) in the second case. This completes the proof of (3.5) for any triangle ABC.

We now discuss the equality conditions of (1.4). By means of Lemma 5 and identity (3.7), we know that the equality in (3.5) occurs if and only if

$$(b-c)^2(c-a)^2(a-b)^2 = 0, \tag{3.8}$$

and

$$(2m_1n_1 + n_2n_3)x = (2m_2n_2 + n_3n_1)y = (2m_3n_3 + n_1n_2)z. \tag{3.9}$$

Again, it is easy to check the following identities:

$$2m_1n_1 + n_2n_3 = 8a^2b^4c^4(c^2+a^2)(a^2+b^2)t_1, \tag{3.10}$$

$$2m_2n_2 + n_3n_1 = 8b^2c^4a^4(a^2+b^2)(b^2+c^2)t_2, \tag{3.11}$$

$$2m_3n_3 + n_1n_2 = 8c^2a^4b^4(b^2+c^2)(c^2+a^2)t_3, \tag{3.12}$$

where

$$\begin{aligned}
t_1 &= 4a^{10} - 11(b^2+c^2)a^8 + (9b^4+6b^2c^2+9c^4)a^6 + (b^2+c^2)(b^4+7b^2c^2+c^4)a^4\\
&\quad -(3b^4+2b^2c^2+3c^4)(b-c)^2(b+c)^2a^2 - b^2c^2(b^2+c^2)(b-c)^2(b+c)^2,\\
t_2 &= 4b^{10} - 11(c^2+a^2)b^8 + (9c^4+6c^2a^2+9a^4)b^6 + (c^2+a^2)(c^4+7c^2a^2+a^4)b^4\\
&\quad -(3c^4+2c^2a^2+3a^4)(c-a)^2(c+a)^2b^2 - c^2a^2(c^2+a^2)(c-a)^2(c+a)^2,\\
t_3 &= 4c^{10} + 11(a^2+b^2)c^8 + (9a^4+6a^2b^2+9b^4)c^6 + (a^2+b^2)(a^4+7a^2b^2+b^4)c^4\\
&\quad -(3a^4+2a^2b^2+3b^4)(a-b)^2(a+b)^2c^2 - a^2b^2(a^2+b^2)(a-b)^2(a+b)^2.
\end{aligned}$$

Thus, it follows from (3.9) that

$$\frac{xt_1}{a^2(b^2+c^2)} = \frac{yt_2}{b^2(c^2+a^2)} = \frac{zt_3}{c^2(a^2+b^2)}. \tag{3.13}$$

By (3.8) we conclude that triangle ABC is isosceles. If $b = c$, then it is easy to get

$$t_1 = 2a^4(c^2 + 2a^2)(a^2 - 3c^2)^2, \tag{3.14}$$

$$t_2 = t_3 = -2c^2a^2(a^2 - 3c^2)(c^2 + 2a^2)(c^2 + a^2). \tag{3.15}$$

Furthermore, if $a^2 \neq 3c^2$ then it follows from (3.13), (3.14) and (3.15) that

$$x(3c^2 - a^2) = 2c^2y = 2c^2z. \tag{3.16}$$

When $b = c$ and $a^2 = 3c^2$, it is easily known that inequality (3.5) holds strictly. Therefore, the equality of (3.5) holds if and only if $b = c$, $a^2 \neq 3c^2$ and (3.16) is valid or any permutation thereof. Moreover, (3.16) shows that the barycentric coordinates of P is $(2c^2/(3c^2 - a^2), 1, 1)$, hence we know that the statements of the theorem for the equality condition is true. This completes the proof of Theorem 1. $\square$

4. Several conjectures

In this section, we present several interesting open problems (conjectures).

It is known that the inequality (1.4) can not be generalized as the form (1.3). However, from another point of view, we propose the following generalization with one parameter:

CONJECTURE 1. *Let $0 < \lambda < 1$ be a real number, then for any point P in the plane of $\triangle ABC$ the following inequality*

$$\frac{R_1^2 - r_1^2}{b^2 + c^2 + \lambda a^2} + \frac{R_2^2 - r_2^2}{c^2 + a^2 + \lambda b^2} + \frac{R_3^2 - r_3^2}{a^2 + b^2 + \lambda c^2} \geqslant \frac{3}{4(\lambda + 2)} \tag{4.1}$$

holds.

REMARK 2. If we take $\lambda = 1$, then (4.1) becomes

$$R_1^2 + R_2^2 + R_3^2 - r_1^2 - r_2^2 - r_3^2 \geqslant \frac{1}{4}(a^2 + b^2 + c^2), \tag{4.2}$$

which can be derived by $R_2^2 + R_3^2 - 2r_1^2 \geqslant \frac{1}{2}a^2$ (appeared in [4]). Therefore, if Conjecture 1 is true, then (4.1) would be valid for $0 \leqslant \lambda \leqslant 1$ by inequalities (1.4) and (4.2).

The author also found three inequalities similar to our main result which could not be proved for the moment. We introduce them as follows:

CONJECTURE 2. *For any interior point P of $\triangle ABC$, we have*

$$\frac{R_1 - r_1}{b + c} + \frac{R_2 - r_2}{c + a} + \frac{R_3 - r_3}{a + b} \geqslant \frac{\sqrt{3}}{4}, \tag{4.3}$$

with equality holding if and only if $\triangle ABC$ is equilateral and P is its center.

CONJECTURE 3. *For any point P in the plane of $\triangle ABC$, the following inequality*

$$\frac{2R_1^2-r_2^2-r_3^2}{(b+c)^2}+\frac{2R_2^2-r_3^2-r_1^2}{(c+a)^2}+\frac{2R_3^2-r_1^2-r_2^2}{(a+b)^2}\geqslant\frac{3}{8} \tag{4.4}$$

holds.

CONJECTURE 4. *For any point P in the plane of $\triangle ABC$, the following inequality*

$$\frac{R_2^2+R_3^2-r_2^2-r_3^2}{(b+c)^2}+\frac{R_3^2+R_1^2-r_3^2-r_1^2}{(c+a)^2}+\frac{R_1^2+R_2^2-r_1^2-r_2^2}{(a+b)^2}\geqslant\frac{3}{8} \tag{4.5}$$

holds.

REMARK 3. The author has known that the equality conditions of (4.4) and (4.5) are both special ($\triangle ABC$ may not be equilateral). We tried to prove (4.4) and (4.5) by using the method which proves Theorem 1, but failed when we proved the corresponding inequalities which are similar to that of Lemma 4.

Acknowledgement. The authors wishes to thank the anonymous referees for their careful reading of the manuscript and their suggestions.

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(Received April 12, 2013)

Jian Liu
East China Jiaotong University
Jiangxi, Nanchang 330013
P.R. China
e-mail: China99jian@163.com