Journal of
Mathematical
Inequalities
Volume 8, Number 3 (2014), 613–634 doi:10.7153/jmi-08-45

SOME BMO ESTIMATES FOR VECTOR–VALUED MULTILINEAR OPERATORS

XIN-SHENG YUAN AND ZHI-GANG WANG

(*Communicated by J. Pečarić*)

Abstract. In this paper, some BMO endpoint estimates for certain multilinear integral operators are obtained. The operators include Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

1. Introduction and preliminaries

As the development of singular integral operators, the boundedness of their commutators and multilinear operators have been well studied (see [2–8]). In [3–5], [7], [8], the authors proved that the commutators and multilinear operators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$; Chanillo (see [1]) proved a similar result when singular integral operators are replaced by the fractional integral operators. In [2], [10], the boundedness properties of the commutators and multilinear operators for the extreme values of p are obtained. The main purpose of this paper is to establish the BMO endpoint estimates for some vector-valued multilinear integral operators. The operators include Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

First, let us introduce some notations (see [9], [20]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function f, let $f_Q = |Q|^{-1}\int_Q f(x)dx$, $f(Q) = \int_Q f(x)dx$ and $f^{\#}(x) = \sup_{Q\ni x}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. Moreover, for a non-negative weight function w, f is said to belong to $BMO(w)$ if $f^{\#} \in L^\infty(w)$ and define $||f||_{BMO(w)} = ||f^{\#}||_{L^\infty(w)}$. We also define the central BMO space by $CMO(w)$, which is the space of those functions $f \in L_{loc}(R^n)$ such that

$$||f||_{CMO(w)} = \sup_{d>1} |w(Q(0,d))|^{-1}\int_Q |f(y) - f_Q|w(y)dy < \infty.$$

It is well-known that

$$||f||_{BMO(w)} \approx \sup_{d>0}\inf_{c\in C} |w(Q(x_0,d))|^{-1}\int_{Q(x_0,d)} |f(x) - c|w(x)dx.$$

Mathematics subject classification (2010): 42B20, 42B25.
Keywords and phrases: Vector-valued multilinear operator, Littlewood-Paley operator, Marcinkiewicz operator, Bochner-Riesz operator, BMO space.

We write that $BMO(w) = BMO(R^n)$ and $CMO(w) = CMO(R^n)$ if $w \equiv 1$.

DEFINITION. (1). Let $0 < \delta < n$ and $1 < p < n/\delta$. We shall call $B_p^\delta(R^n)$ the space of those functions f on R^n such that

$$||f||_{B_p^\delta} = \sup_{r>1} r^{-n(1/p-\delta/n)} ||f\chi_{Q(0,d)}||_{L^p} < \infty.$$

(2). Let $1 < p < \infty$ and w be a non-negative weight functions on R^n. We shall call $B_p(w)$ the space of those functions f on R^n such that

$$||f||_{B_p(w)} = \sup_{d>1} [w(Q(0,d))]^{-1/p} ||f\chi_{Q(0,d)}||_{L^p(w)} < \infty.$$

2. Main results

In this paper, we will study a class of vector-valued multilinear integral operators, whose definition are following.

Suppose m_j are the positive integers $(j = 1, \cdots, l)$, $m_1 + \cdots + m_l = m$ and A_j are the functions on R^n $(j = 1, \cdots, l)$. Let $F_t(x,y)$ be the function defined on $R^n \times R^n \times [0, +\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F_t(x,y) f(y) dy$$

and

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x-y|^m} F_t(x,y) f(y) dy$$

for every bounded and compactly supported function f, where

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leqslant m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha.$$

Let H be the Banach space $H = \{h : ||h|| < \infty\}$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H. For $1 < r < \infty$, the vector-valued multilinear operator related to F_t is defined by

$$|T_\delta^A(f)(x)|_r = \left(\sum_{i=1}^\infty (T_\delta^A(f_i)(x))^r \right)^{1/r},$$

where

$$T_\delta^A(f_i)(x) = ||F_t^A(f_i)(x)||,$$

and F_t satisfies: for fixed $\varepsilon > 0$ and $\delta \geqslant 0$,

$$||F_t(x,y)|| \leqslant C|x-y|^{-n+\delta}$$

and

$$||F_t(y,x) - F_t(z,x)|| \leqslant C|y-z|^\varepsilon |x-z|^{-n-\varepsilon+\delta}$$

if $2|y-z| \leqslant |x-z|$. Set $T(f)(x) = ||F_t(f)(x)||$, we also denote that

$$|T_\delta(f)(x)|_r = \left(\sum_{i=1}^{\infty} |T_\delta(f_i)(x)|^r\right)^{1/r} \text{ and } |f|_r = \left(\sum_{i=1}^{\infty} |f_i(x)|^r\right)^{1/r}.$$

We write that $T_\delta = T$, $|T_\delta|_r = |T|_r$ and $|T_\delta^A|_r = |T^A|_r$ if $\delta = 0$.

Note that when $m = 0$, T_δ^A is just the multilinear commutators of T_δ and A (see [1], [10], [12]). It is well-known that multilinear operator, as a non-trivial extension of commutator, is of great interest in harmonic analysis and have been widely studied by many authors (see [2–6]). In [8], the weighted L^p $(p > 1)$-boundedness of the multilinear operator related to some singular integral operator are obtained. In [2], the weak (H^1, L^1)-boundedness of the multilinear operator related to some singular integral operator are obtained. In this paper, we will prove the BMO estimates for the vector-valued multilinear operators $|T_\delta^A|_r$ and $|T^A|_r$.

Now we state our results as following.

THEOREM 1. *Let $1 < r < \infty$, $0 < \delta < n$, $1 < p < n/\delta$ and $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Suppose that $|T_\delta|_r$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for any $p, q \in (1, +\infty]$ with $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then*

(a). *$|T_\delta^A|_r$ is bounded from $L^{n/\delta}(R^n)$ to $BMO(R^n)$, that is*

$$|||T_\delta^A(f)|_r||_{BMO} \leqslant C|||f|_r||_{L^{n/\delta}};$$

(b). *$|T_\delta^A|_r$ is bounded from $B_p^\delta(R^n)$ to $CMO(R^n)$, that is*

$$|||T_\delta^A(f)|_r||_{CMO} \leqslant C|||f|_r||_{B_p^\delta}.$$

THEOREM 2. *Let $1 < r < \infty$, $1 < p < \infty$, $w \in A_1$ and $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Suppose that $|T|_r$ is bounded on $L^p(w)$ for any $1 < p \leqslant \infty$ and $w \in A_1$. Then*

(i). *$|T^A|_r$ is bounded from $L^\infty(w)$ to $BMO(w)$, that is*

$$|||T^A(f)|_r||_{BMO(w)} \leqslant C|||f|_r||_{L^\infty(w)};$$

(ii). *$|T^A|_r$ is bounded from $B_p(w)$ to $CMO(w)$, that is*

$$|||T^A(f)|_r||_{CMO(w)} \leqslant C|||f|_r||_{L^\infty(w)}.$$

3. Proofs of Theorems

To prove the theorems, we need the following lemma.

LEMMA 2. (see [5]) *Let A be a function on R^n and $D^\beta A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A;x,y)| \leqslant C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Proof of Theorem 1(a). It is only to prove that there exists a constant C_Q such that

$$\frac{1}{|Q|} \int_Q ||T_\delta^A(f)(x)|_r - C_Q| dx \leqslant C|||f|_r||_{L^{n/\delta}}$$

holds for any cube Q. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}} x^\alpha$, then $R_{m_j}(A_j;x,y) = R_{m_j}(\tilde{A}_j;x,y)$ and $D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We split $f = g + h = \{g_i\} + \{h_i\}$ for $g_i = f_i \chi_{\tilde{Q}}$ and $h_i = f_i \chi_{R^n \setminus \tilde{Q}}$. Write

$$\begin{aligned}
F_t^A(f_i)(x) &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) f_i(y) dy \\
&= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) h_i(y) dy \\
&\quad + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) g_i(y) dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{A}_1(y) F_t(x,y) g_i(y) dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{A}_2(y) F_t(x,y) g_i(y) dy \\
&\quad + \sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m} F_t(x,y) g_i(y) dy,
\end{aligned}$$

then, by the Minkowski' inequality,

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q \left| |T_\delta^A(f)(x)|_r - |T_\delta^{\tilde{A}}(h)(x_0)|_r \right| dx \\
\leqslant\ & \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^\infty |T_A(f_i)(x) - T_{\tilde{A}}(h_i)(x_0)|^r \right)^{1/r} dx
\end{aligned}$$

$$\leqslant \frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left\|\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}F_t(x,y)g_i(y)dy\right\|^r\right)^{1/r}dx$$
$$+\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left\|\sum_{|\alpha_1|=m_1}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)F_t(x,y)g_i(y)dy\right\|^r\right)^{1/r}dx$$
$$+\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left\|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)F_t(x,y)g_i(y)dy\right\|^r\right)^{1/r}dx$$
$$+\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left\|\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}F_t(x,y)g_i(y)dy\right\|^r\right)^{1/r}dx$$
$$+\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|T_\delta^{\tilde{A}}(h_i)(x)-T_\delta^{\tilde{A}}(h_i)(x_0)\right|^r\right)^{1/r}dx$$
$$:= I_1+I_2+I_3+I_4+I_5.$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x\in Q$ and $y\in\tilde{Q}$, by Lemma 1, we get

$$R_{m_j}(\tilde{A}_j;x,y)\leqslant C|x-y|^{m_j}\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO},$$

thus, by the $(L^{n/\delta},L^\infty)$-boundedness of $|T_\delta|_r$, we get

$$\begin{aligned}I_1 &\leqslant C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\frac{1}{|Q|}\int_Q|T_\delta(g)(x)|_r dx\\ &\leqslant C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||T_\delta(g)|_r||_{L^\infty}\\ &\leqslant C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{L^{n/\delta}}.\end{aligned}$$

For I_2, by the (L^p,L^q)-boundedness of T_δ for $1/q=1/p-\delta/n$, $n/\delta>p>1$ and Hölder' inequality, we get

$$\begin{aligned}I_2 &\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|}\int_Q|T_\delta(D^{\beta_1}\tilde{A}_1g)(x)|_r dx\\ &\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T_\delta(D^{\alpha_1}\tilde{A}_1g)(x)|_r^q dx\right)^{1/q}\\ &\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1/q}\left(\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)g(x)|_r^p dx\right)^{1/p}\end{aligned}$$

$$\leqslant C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_1}A_1(x)-(D^{\alpha_1}A_1)_{\tilde{Q}}|^q dx\right)^{1/q} |||f|_r||_{L^{n/\delta}}$$

$$\leqslant C \prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{BMO}\right) |||f|_r||_{L^{n/\delta}}.$$

For I_3, similar to the proof of I_2, we get

$$I_3 \leqslant C \prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{BMO}\right) |||f|_r||_{L^{n/\delta}}.$$

Similarly, for I_4, choose $1<p<n/\delta$ and $q,s_1,s_2>1$ such that $1/q=1/p-\delta/n$ and $1/s_1+1/s_2+p\delta/n=1$, we obtain, by Hölder' inequality,

$$I_4 \leqslant C \sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}} \frac{1}{|Q|}\int_Q |T_\delta(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)(x)|_r dx$$

$$\leqslant C \sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}} \left(\frac{1}{|Q|}\int_{R^n} |T_\delta(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)(x)|_r^q dx\right)^{1/q}$$

$$\leqslant C \sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}} |Q|^{-1/q}\left(\int_{R^n} |D^{\alpha_1}\tilde{A}_1(x) D^{\alpha_2}\tilde{A}_2(x) g(x)|_r^p dx\right)^{1/p}$$

$$\leqslant C \sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(x)|^{ps_1} dx\right)^{1/ps_1} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_2}\tilde{A}_2(x)|^{ps_2} dx\right)^{1/ps_2} |||f|_r||_{L^{n/\delta}}$$

$$\leqslant C \prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{BMO}\right) |||f|_r||_{L^{n/\delta}}.$$

For I_5, we write

$$\begin{aligned}
&F_t^{\tilde{A}}(h_i)(x)-F_t^{\tilde{A}}(h_i)(x_0)\\
=&\int_{R^n}\left(\frac{F_t(x,y)}{|x-y|^m}-\frac{F_t(x_0,y)}{|x_0-y|^m}\right)\prod_{j=1}^{2} R_{m_j}(\tilde{A}_j;x,y)h_i(y)dy\\
&+\int_{R^n}\left(R_{m_1}(\tilde{A}_1;x,y)-R_{m_1}(\tilde{A}_1;x_0,y)\right)\frac{R_{m_2}(\tilde{A}_2;x,y)}{|x_0-y|^m}F_t(x_0,y)h_i(y)dy\\
&+\int_{R^n}\left(R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)\right)\frac{R_{m_1}(\tilde{A}_1;x_0,y)}{|x_0-y|^m}F_t(x_0,y)h_i(y)dy\\
&-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}F_t(x,y)-\frac{R_{m_2}(\tilde{A}_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m}F_t(x_0,y)\right]
\end{aligned}$$

$$\times D^{\alpha_1}\tilde{A}_1(y)h_i(y)dy$$
$$-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}F_t(x,y)-\frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m}F_t(x_0,y)\right]$$
$$\times D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy$$
$$+\sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}F_t(x,y)-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}F_t(x_0,y)\right]$$
$$\times D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy$$
$$=I_5^{(1)}+I_5^{(2)}+I_5^{(3)}+I_5^{(4)}+I_5^{(5)}+I_5^{(6)}.$$

By Lemma 1 and the following inequality (see [20])

$$|b_{Q_1}-b_{Q_2}|\leqslant C\log(|Q_2|/|Q_1|)||b||_{BMO} \text{ for } Q_1\subset Q_2,$$

we know that, for $x\in Q$ and $y\in 2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}$,

$$|R_{m_j}(\tilde{A}_j;x,y)|\leqslant C|x-y|^{m_j}\sum_{|\alpha|=m_j}(||D^\alpha A_j||_{BMO}+|(D^\alpha A_j)_{\tilde{Q}(x,y)}-(D^\alpha A_j)_{\tilde{Q}}|)$$
$$\leqslant Ck|x-y|^{m_j}\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}.$$

Note that $|x-y|\sim|x_0-y|$ for $x\in Q$ and $y\in R^n\setminus\tilde{Q}$, we obtain, by the condition on F_t,

$$||I_5^{(1)}||\leqslant C\int_{R^n}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1-\delta}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon-\delta}}\right)\prod_{j=1}^2|R_{m_j}(\tilde{A}_j;x,y)||h_i(y)|dy$$
$$\leqslant C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)$$
$$\times\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}}\right)|f_i(y)|dy,$$

thus, by the Minkowski' inequality,

$$\left(\sum_{i=1}^\infty||I_5^{(1)}||^r\right)^{1/r}$$
$$\leqslant C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)$$
$$\times\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}}\right)|f(y)|_r dy$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})|||f|_r||_{L^{n/\delta}}$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{L^{n/\delta}}.$$

For $I_5^{(2)}$, by the formula (see [5]):

$$R_{m_j}(\tilde{A}_j;x,y)-R_{m_j}(\tilde{A}_j;x_0,y)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^{\beta}\tilde{A}_j;x,x_0)(x-y)^{\beta}$$

and Lemma 1, we have

$$|R_{m_j}(\tilde{A}_j;x,y)-R_{m_j}(\tilde{A}_j;x_0,y)|\leqslant C\sum_{|\beta|<m_j}\sum_{|\alpha|=m_j}|x-x_0|^{m_j-|\beta|}|x-y|^{|\beta|}||D^{\alpha}A_j||_{BMO},$$

thus

$$\left(\sum_{i=1}^{\infty}||I_5^{(2)}||^r\right)^{1/r}\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{BMO}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}|f(y)|_r dy$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{L^{n/\delta}}.$$

Similarly,

$$\left(\sum_{i=1}^{\infty}||I_5^{(3)}||^r\right)^{1/r}\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{L^{n/\delta}}.$$

For $I_5^{(4)}$, taking $s>1$ such that $1/s+\delta/n=1$, then

$$\left(\sum_{i=1}^{\infty}||I_5^{(4)}||^r\right)^{1/r}$$

$$\leqslant C\sum_{|\alpha_1|=m_1}\int_{R^n}\left|\left|\frac{(x-y)^{\alpha_1}F_t(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}F_t(x_0,y)}{|x_0-y|^m}\right|\right||R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||h(y)|_r dy$$

$$+C\sum_{|\alpha_1|=m_1}\int_{R^n}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)|\frac{||(x_0-y)^{\alpha_1}F_t(x_0,y)||}{|x_0-y|^m}|D^{\alpha_1}\tilde{A}_1(y)||h(y)|_r dy$$

$$\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^s dy\right)^{1/s}|||f|_r||_{L^{n/\delta}}$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{L^{n/\delta}}.$$

Similarly,

$$\left(\sum_{i=1}^{\infty}||I_5^{(5)}||^r\right)^{1/r} \leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{L^{n/\delta}}.$$

For $I_5^{(6)}$, taking $s_1,s_2>1$ such that $\delta/n+1/s_1+1/s_2=1$, then

$$\begin{aligned}
&\left(\sum_{i=1}^{\infty}||I_5^{(6)}||^r\right)^{1/r}\\
&\leqslant C\sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}}\int_{R^n}\left|\left|\frac{(x-y)^{\alpha_1+\alpha_2}F_t(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}F_t(x_0,y)}{|x_0-y|^m}\right|\right|\\
&\quad\times|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||h(y)|_r dy\\
&\leqslant C\sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}}\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})|||f|_r||_{L^{n/\delta}}\\
&\quad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{s_1}dy\right)^{1/s_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_2}\tilde{A}_2(y)|^{s_2}dy\right)^{1/s_2}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{L^{n/\delta}}.
\end{aligned}$$

Thus

$$I_5\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{L^{n/\delta}}.$$

(b). It suffices to prove that there exists a constant C_Q such that

$$\frac{1}{|Q|}\int_Q||T_\delta^A(f)(x)|_r-C_Q|dx\leqslant C|||f|_r||_{B_p^\delta}$$

holds for any cube $Q=Q(0,d)$ with $d>1$. Without loss of generality, we may assume $l=2$. Fix a cube $Q=Q(0,d)$ with $d>1$. Let $\tilde{Q}$ and $\tilde{A}_j(x)$ be the same as the proof of (a). Write, for $f=g+h=\{g_i\}+\{h_i\}$ with $g_i=f_i\chi_{\tilde{Q}}$ and $h_i=f_i\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q\left||T_\delta^A(f)(x)|_r-|T_\delta^{\tilde{A}}(h)(0)|_r\right|dx\\
&\leqslant\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}|T_A(f_i)(x)-T_{\tilde{A}}(h_i)(0)|^r\right)^{1/r}dx\\
&\leqslant\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|\int_{R^n}\frac{\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}dx
\end{aligned}$$

$$+\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|\sum_{|\alpha_1|=m_1}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}dx$$

$$+\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}dx$$

$$+\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}dx$$

$$+\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|T_\delta^{\tilde{A}}(h_i)(x)-T_\delta^{\tilde{A}}(h_i)(0)\right|^r\right)^{1/r}dx$$

$$:=J_1+J_2+J_3+J_4+J_5.$$

Similar to the proof of (a), we get, for $1/u=1/s-\delta/n$, $1<s<p$, $1<u_1,u_2<\infty$ and $1/u_1+1/u_2+s/p=1$,

$$\begin{aligned}
J_1 &\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_Q|T_\delta(g)(x)|_r^q dx\right)^{1/q}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)d^{-n(1/p-\delta/n)}|||f|_r\chi_{\tilde{Q}}||_{L^p}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p^\delta};\\
J_2 &\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T_\delta(D^{\alpha_1}\tilde{A}_1 g)(x)|_r^u dx\right)^{1/u}\\
&\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1/u}|||D^{\alpha_1}\tilde{A}_1 g|_r||_{L^s}\\
&\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{ps/(p-s)}dy\right)^{(p-s)/(ps)}\\
&\quad\times|Q|^{\delta/n-1/p}|||f|_r\chi_{\tilde{Q}}||_{L^p}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p^\delta};\\
J_3 &\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p^\delta};
\end{aligned}$$

$$
\begin{aligned}
J_4 &\leqslant C \sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}} \left(\frac{1}{|Q|}\int_{R^n} |T_\delta(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 f_1)(x)|_r^q dx\right)^{1/q} \\
&\leqslant C \sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}} |Q|^{-1/s}\left(\int_{R^n} |D^{\alpha_1}\tilde{A}_1(x) D^{\alpha_2}\tilde{A}_2(x) g(x)|_r^s dx\right)^{1/s} \\
&\leqslant C \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(x)|^{su_1} dx\right)^{1/su_1} \sum_{|\alpha_2|=m_2} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_2}\tilde{A}_2(x)|^{su_2} dx\right)^{1/su_2} \\
&\quad \times |Q|^{\delta/n-1/p} || |f|_r \chi_{\tilde{Q}} ||_{L^p} \\
&\leqslant C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO}\right) || |f|_r ||_{B_p^\delta};
\end{aligned}
$$

For J_5, we write, for $x \in Q$,

$$
\begin{aligned}
& F_t^{\tilde{A}}(h_i)(x) - F_t^{\tilde{A}}(h_i)(0) \\
&= \int_{R^n} \left(\frac{F_t(x,y)}{|x-y|^m} - \frac{F_t(0,y)}{|y|^m}\right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j;x,y) h_i(y) dy \\
&\quad + \int_{R^n} \left(R_{m_1}(\tilde{A}_1;x,y) - R_{m_1}(\tilde{A}_1;0,y)\right) \frac{R_{m_2}(\tilde{A}_2;x,y)}{|y|^m} F_t(0,y) h_i(y) dy \\
&\quad + \int_{R^n} \left(R_{m_2}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;0,y)\right) \frac{R_{m_1}(\tilde{A}_1;0,y)}{|y|^m} F_t(0,y) h_i(y) dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m} F_t(x,y) - \frac{R_{m_2}(\tilde{A}_2;0,y)(-y)^{\alpha_1}}{|y|^m} F_t(0,y)\right] \\
&\quad \times D^{\alpha_1}\tilde{A}_1(y) h_i(y) dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m} F_t(x,y) - \frac{R_{m_1}(\tilde{A}_1;0,y)(-y)^{\alpha_2}}{|y|^m} F_t(0,y)\right] \\
&\quad \times D^{\alpha_2}\tilde{A}_2(y) h_i(y) dy \\
&\quad + \sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} F_t(x,y) - \frac{(-y)^{\alpha_1+\alpha_2}}{|y|^m} F_t(0,y)\right] \\
&\quad \times D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y) h_i(y) dy \\
&= J_5^{(1)} + J_5^{(2)} + J_5^{(3)} + J_5^{(4)} + J_5^{(5)} + J_5^{(6)}.
\end{aligned}
$$

Similar to the proof of Theorem 1, we get, for $1 < s_1, s_2 < \infty$ and $1/s_1 + 1/s_2 + 1/p = 1$,

$$
\left(\sum_{i=1}^{\infty} ||J_5^{(1)}||^r\right)^{1/r} \leqslant C \int_{R^n} \left(\frac{|x|}{|y|^{m+n+1-\delta}} + \frac{|x|^\varepsilon}{|y|^{m+n+\varepsilon-\delta}}\right) \prod_{j=1}^{2} |R_{m_j}(\tilde{A}_j;x,y)| |h(y)|_r dy
$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\left(\frac{|x|}{|y|^{n+1-\delta}}+\frac{|x|^{\varepsilon}}{|y|^{n+\varepsilon-\delta}}\right)|f(y)|_r dy$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})(2^k d)^{-n(1/p-\delta/n)}|||f|_r\chi_{2^k\tilde{Q}}||_{L^p}$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p^{\delta}};$$

$$\left(\sum_{i=1}^{\infty}||J_5^{(2)}||^r\right)^{1/r}\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x|}{|y|^{n+1-\delta}}|f(y)|_r dy$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p^{\delta}};$$

$$\left(\sum_{i=1}^{\infty}||J_5^{(3)}||^r\right)^{1/r}\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p^{\delta}};$$

$$\left(\sum_{i=1}^{\infty}||J_5^{(4)}||^r\right)^{1/r}\leqslant C\sum_{|\alpha_1|=m_1}\int_{R^n}\left|\left|\frac{(x-y)^{\alpha_1}F_t(x,y)}{|x-y|^m}-\frac{(-y)^{\alpha_1}F_t(0,y)}{|y|^m}\right|\right|$$

$$\times|R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||h(y)|_r dy$$

$$+C\sum_{|\alpha_1|=m_1}\int_{R^n}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;0,y)|\frac{||(-y)^{\alpha_1}F_t(0,y)||}{|y|^m}|D^{\alpha_1}\tilde{A}_1(y)||h(y)|_r dy$$

$$\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})(2^k d)^{-n(1/p-\delta/n)}|||f|_r\chi_{2^k\tilde{Q}}||_{L^p}$$

$$\times\sum_{|\alpha_1|=m_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{p'}dy\right)^{1/p'}$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p^{\delta}};$$

$$\left(\sum_{i=1}^{\infty}||J_5^{(5)}||^r\right)^{1/r}\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p^{\delta}};$$

$$\left(\sum_{i=1}^{\infty}||J_5^{(6)}||^r\right)^{1/r}\leqslant C\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})(2^k d)^{-n(1/p-\delta/n)}|||f|_r\chi_{2^k\tilde{Q}}||_{L^p}$$

$$\times\sum_{|\alpha_1|=m_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{s_1}dy\right)^{1/s_1}\sum_{|\alpha_2|=m_2}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_2}\tilde{A}_2(y)|^{s_2}dy\right)^{1/s_2}$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p^\delta};$$

Thus

$$J_5 \leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p^\delta}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2(i). It is only to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)}\int_Q||T^A(f)(x)|_r - C_Q|w(x)dx \leqslant C|||f|_r||_{L^\infty(w)}$$

holds for any cube Q. Without loss of generality, we may assume $l=2$. Fix a cube $Q=Q(x_0,d)$. Let $\tilde{Q}$ and $\tilde{A}_j(x)$ be the same as the proof of Theorem 1. Write, for $f=g+h=\{g_i\}+\{h_i\}$ with $g_i=f_i\chi_{\tilde{Q}}$ and $h_i=f_i\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned}
&\frac{1}{w(Q)}\int_Q\Big||T^A(f)(x)|_r-|T^{\tilde{A}}(h)(x_0)|_r\Big|w(x)dx\\
\leqslant\ &\frac{1}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}|T_A(f_i)(x)-T_{\tilde{A}}(h_i)(x_0)|^r\right)^{1/r}w(x)dx\\
\leqslant\ &\frac{1}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}w(x)dx\\
&+\frac{C}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|\sum_{|\alpha_1|=m_1}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}\\
&\times w(x)dx\\
&+\frac{C}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}\\
&\times w(x)dx\\
&+\frac{C}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}\\
&\times w(x)dx\\
&+\frac{1}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}\left|T_\delta^{\tilde{A}}(h_i)(x)-T_\delta^{\tilde{A}}(h_i)(x_0)\right|^r\right)^{1/r}w(x)dx\\
:=\ &L_1+L_2+L_3+L_4+L_5.
\end{aligned}$$

Similar to the proof of Theorem 1, by the $L^\infty(w)$-boundedness of $|T|_r$, we get

$$\begin{aligned}L_1 &\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\frac{1}{w(Q)}\int_Q|T(g)(x)|_r w(x)dx\\&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||T(g)|_r||_{L^\infty(w)}\\&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{L^\infty(w)}.\end{aligned}$$

For L_2, since $w\in A_1$, w satisfies the reverse of Hölder's inequality:

$$\left(\frac{1}{|Q|}\int_Q w(x)^l dx\right)^{1/l}\leqslant\frac{C}{|Q|}\int_Q w(x)dx$$

for all cube Q and some $1<l<\infty$ (see [9], [20]). Thus, by the $L^s(w)$-boundedness of $|T|_r$ for $s>1$ and Hölder' inequality, we get

$$\begin{aligned}L_2 &\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\frac{1}{w(Q)}\int_Q|T(D^{\alpha_1}\tilde{A}_1g)(x)|_r w(x)dx\\&\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{w(Q)}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1g)(x)|_r^s w(x)dx\right)^{1/s}\\&\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{w(Q)}\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)g(x)|_r^s w(x)dx\right)^{1/s}\\&\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^{\alpha_1}A_1(x)-(D^{\alpha_1}A_1)_{\tilde{Q}}|^{sl'}dx\right)^{1/sl'}\\&\quad\times w(Q)^{-1/s}|Q|^{1/s}\left(\frac{1}{|Q|}\int_{\tilde{Q}}w(x)^l dx\right)^{1/sl}|||f|_r||_{L^\infty(w)}\\&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{L^\infty(w)};\\L_3 &\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{L^\infty(w)}.\end{aligned}$$

Similarly, for L_4, choosing $1<s,u_1,u_2<\infty$ such that $1/u_1+1/u_2+1/l=1$, we obtain, by Hölder' inequality,

$$L_4\leqslant C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\frac{1}{w(Q)}\int_Q|T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2g)(x)|_r w(x)dx$$

$$\leqslant C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\left(\frac{1}{w(Q)}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2g)(x)|_r^s w(x)dx\right)^{1/s}$$

$$\leqslant C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}} w(Q)^{-1/s}\left(\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)D^{\alpha_2}\tilde{A}_2(x)g(x)|_r^s w(x)dx\right)^{1/s}$$

$$\leqslant C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(x)|^{su_1}dx\right)^{1/su_1}\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^{\alpha_2}\tilde{A}_2(x)|^{su_2}dx\right)^{1/su_2}$$

$$\times w(Q)^{-1/s}|Q|^{1/s}\left(\frac{1}{|Q|}\int_{\tilde{Q}}w(x)^l dx\right)^{1/sl}|||f|_r||_{L^\infty(w)}$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{L^\infty(w)}.$$

For L_5, similar to the proof of Theorem 1, we get

$$|L_5|\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{L^\infty(w)}.$$

(ii). It suffices to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)}\int_Q||T^A(f)(x)|_r-C_Q|w(x)dx\leqslant C|||f|_r||_{B_p(w)}$$

holds for any cube $Q=Q(0,d)$ with $d>1$. Without loss of generality, we may assume $l=2$. Fix a cube $Q=Q(0,d)$ with $d>1$. Let $\tilde{Q}$ and $\tilde{A}_j(x)$ be the same as the proof of Theorem 1. Write, for $f=g+h=\{g_i\}+\{h_i\}$ with $g_i=f_i\chi_{\tilde{Q}}$ and $h_i=f_i\chi_{R^n\setminus\tilde{Q}}$,

$$\frac{1}{w(Q)}\int_Q\Big||T^A(f)(x)|_r-|T^{\tilde{A}}(h)(0)|_r\Big|w(x)dx$$

$$\leqslant\frac{1}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}|T_A(f_i)(x)-T_{\tilde{A}}(h_i)(0)|^r\right)^{1/r}w(x)dx$$

$$\leqslant\frac{1}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|\int_{R^n}\frac{\prod_{j=1}^2R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}w(x)dx$$

$$+\frac{C}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|\sum_{|\alpha_1|=m_1}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}$$

$$\times w(x)dx$$

$$+\frac{C}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}\left|\left|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)F_t(x,y)g_i(y)dy\right|\right|^r\right)^{1/r}$$

$$\times w(x)dx$$

$$+\frac{C}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}\left\|\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}F_t(x,y)g_i(y)dy\right\|^r\right)^{1/r}$$

$$\times w(x)dx+\frac{1}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}\left|T_\delta^{\tilde{A}}(h_i)(x)-T_\delta^{\tilde{A}}(h_i)(0)\right|^r\right)^{1/r}w(x)dx$$

$$:=M_1+M_2+M_3+M_4+M_5.$$

Similar to the proof of Theorem 1, we get

$$\begin{aligned}M_1&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\left(\frac{1}{w(Q)}\int_Q|T(g)(x)|_r^pw(x)dx\right)^{1/p}\\&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)w(\tilde{Q})^{-1/p}|||f|_r\chi_{\tilde{Q}}||_{L^p(w)}\\&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p(w)}.\end{aligned}$$

For M_2 and M_3, taking $s,u>1$ such that $su<p$ and $l=(pu-su)/(p-su)$, then, by the reverse of Hölder's inequality,

$$\begin{aligned}M_2&\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{w(Q)}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1g)(x)|_r^sw(x)dx\right)^{1/s}\\&\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}w(Q)^{-1/s}||D^{\alpha_1}\tilde{A}_1|g|_r||_{L^s(w)}\\&\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}w(Q)^{-1/s}\sum_{|\alpha_1|=m_1}\left(\int_{\tilde{Q}}|D^{\alpha}\tilde{A}_1(y)|^{su'}dy\right)^{1/su'}\\&\quad\times\left(\int_{\tilde{Q}}|f(x)|_r^{su}w(x)^udx\right)^{1/su}\\&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|Q|^{1/su'}w(Q)^{-1/s}\left(\int_{\tilde{Q}}|f(x)|_r^pw(x)dx\right)^{1/p}\\&\quad\times\left(\int_{\tilde{Q}}w(x)^ldx\right)^{(p-s)/pls}\\&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)w(\tilde{Q})^{-1/p}|||f|_r\chi_{\tilde{Q}}||_{L^p(w)}\\&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p(w)};\end{aligned}$$

$$M_3 \leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p(w)}.$$

For M_4, taking $s,u_1,u_2,u_3>1$ such that $1/u_1+1/u_2+1/u_3=1$, $su_3<p$ and $l=(pu_3-su_3)/(p-su_3)$, then, by the reverse of Hölder's inequality,

$$\begin{aligned}
M_4 &\leqslant C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\left(\frac{1}{w(Q)}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2g)(x)|_r^s w(x)dx\right)^{1/s}\\
&\leqslant C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}w(Q)^{-1/s}\left(\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)D^{\alpha_2}\tilde{A}_2(x)g(x)|_r^s w(x)dx\right)^{1/s}\\
&\leqslant C\sum_{|\alpha_1|=m_1}\left(\int_{\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(x)|^{su_1}dx\right)^{1/su_1}\sum_{|\alpha_2|=m_2}\left(\int_{\tilde{Q}}|D^{\alpha_2}\tilde{A}_2(x)|^{su_2}dx\right)^{1/su_2}\\
&\quad\times w(Q)^{-1/s}\left(\int_{\tilde{Q}}|f(x)|_r^{su_3}w(x)^{u_3}dx\right)^{1/su_3}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)w(\tilde{Q})^{-1/p}|||f|_r\chi_{\tilde{Q}}||_{L^p(w)}\\
&\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p(w)}.
\end{aligned}$$

For M_5, we write, for $x\in Q$,

$$\begin{aligned}
&F_t^{\tilde{A}}(h_i)(x)-F_t^{\tilde{A}}(h_i)(0)\\
&=\int_{R^n}\left(\frac{F_t(x,y)}{|x-y|^m}-\frac{F_t(0,y)}{|y|^m}\right)\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)h_i(y)dy\\
&\quad+\int_{R^n}\left(R_{m_1}(\tilde{A}_1;x,y)-R_{m_1}(\tilde{A}_1;0,y)\right)\frac{R_{m_2}(\tilde{A}_2;x,y)}{|y|^m}F_t(0,y)h_i(y)dy\\
&\quad+\int_{R^n}\left(R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;0,y)\right)\frac{R_{m_1}(\tilde{A}_1;0,y)}{|y|^m}F_t(0,y)h_i(y)dy\\
&\quad-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}F_t(x,y)-\frac{R_{m_2}(\tilde{A}_2;0,y)(-y)^{\alpha_1}}{|y|^m}F_t(0,y)\right]\\
&\quad\times D^{\alpha_1}\tilde{A}_1(y)h_i(y)dy\\
&\quad-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}F_t(x,y)-\frac{R_{m_1}(\tilde{A}_1;0,y)(-y)^{\alpha_2}}{|y|^m}F_t(0,y)\right]\\
&\quad\times D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy
\end{aligned}$$

$$+\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}F_t(x,y)-\frac{(-y)^{\alpha_1+\alpha_2}}{|y|^m}F_t(0,y)\right]$$
$$\times D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy$$
$$=M_5^{(1)}+M_5^{(2)}+M_5^{(3)}+M_5^{(4)}+M_5^{(5)}+M_5^{(6)}.$$

Similar to the proof of Theorem 1 and notice that $w\in A_1\subset A_p$, we get

$$\left(\sum_{i=1}^{\infty}||M_5^{(1)}||^r\right)^{1/r}\leqslant C\int_{R^n}\left(\frac{|x|}{|y|^{m+n+1}}+\frac{|x|^{\varepsilon}}{|y|^{m+n+\varepsilon}}\right)\prod_{j=1}^{2}|R_{m_j}(\tilde{A}_j;x,y)||h(y)|_r dy$$
$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})w(2^k\tilde{Q})^{-1/p}$$
$$\times\left(\int_{2^k\tilde{Q}}|f(y)|_r^p w(y)dy\right)^{1/p}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)dy\right)^{1/p}$$
$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)^{-1/(p-1)}dy\right)^{(p-1)/p}$$
$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p(w)};$$
$$\left(\sum_{i=1}^{\infty}||M_5^{(2)}||^r\right)^{1/r}\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x|}{|y|^{n+1}}|f(y)|_r dy$$
$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p(w)};$$
$$\left(\sum_{i=1}^{\infty}||M_5^{(3)}||^r\right)^{1/r}\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p(w)}.$$

For $M_5^{(4)}$ and M_5, choose $1<s<p$, notice that $w\in A_1\subset A_{p/s}$, we get

$$\left(\sum_{i=1}^{\infty}||M_5^{(4)}||^r\right)^{1/r}\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\left(\frac{|x|}{|y|^{n+1}}+\frac{|x|^{\varepsilon}}{|y|^{n+\varepsilon}}\right)$$
$$\times|D^{\alpha_1}\tilde{A}_1(y)||f(y)|_r dy$$
$$\leqslant C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{k=0}^{\infty}\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^{\varepsilon}}{(2^kd)^{n+\varepsilon}}\right)$$
$$\times\left(\int_{2^{k+1}\tilde{Q}}|f(y)|_r^s dy\right)^{1/s}dy\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{s'}dy\right)^{1/s'}$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})w(2^k\tilde{Q})^{-1/p}$$

$$\times\left(\int_{2^k\tilde{Q}}|f(y)|_r^p w(y)dy\right)^{1/p}\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)dy\right)^{1/p}$$

$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)^{-s/(p-s)}dy\right)^{(p-s)/ps}$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p(w)};$$

$$\left(\sum_{i=1}^{\infty}||M_5^{(5)}||^r\right)^{1/r}\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p(w)}.$$

For $L_5^{(6)}$, choose $1<u_1,u_2,u_3<\infty$ such that $u_3<p$ and $1/u_1+1/u_2+1/u_3=1$, notice that $w\in A_1\subset A_{p/u_3}$, we get

$$\left(\sum_{i=1}^{\infty}||M_5^{(6)}||^r\right)^{1/r}$$

$$\leqslant C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\int_{R^n}\left\|\frac{(x-y)^{\alpha_1+\alpha_2}K(x,y)}{|x-y|^m}-\frac{(-y)^{\alpha_1+\alpha_2}F_t(0,y)}{|y|^m}\right\||D^{\alpha_1}\tilde{A}_1(y)|$$

$$\times|D^{\alpha_2}\tilde{A}_2(y)||h(y)|_r dy$$

$$\leqslant C\sum_{k=0}^{\infty}\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^{\varepsilon}}{(2^kd)^{n+\varepsilon}}\right)\left(\int_{2^{k+1}\tilde{Q}}|f(y)|_r^{u_3}dy\right)^{1/u_3}dy$$

$$\times\sum_{|\alpha_1|=m_1}\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{u_1}dy\right)^{1/u_1}\sum_{|\alpha_2|=m_2}\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_2}\tilde{A}_2(y)|^{u_2}dy\right)^{1/u_2}$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})w(2^k\tilde{Q})^{-1/p}\left(\int_{2^k\tilde{Q}}|f(y)|_r^p w(y)dy\right)^{1/p}$$

$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)dy\right)^{1/p}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)^{-u_3/(p-u_3)}dy\right)^{(p-u_3)/pu_3}$$

$$\leqslant C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p(w)}.$$

Thus

$$M_5\leqslant C\prod_{j=1}^{2}\left(\sum_{\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|||f|_r||_{B_p(w)}.$$

This completes the proof of Theorem 2. □

4. Applications

Now we give some applications of Theorems in this paper.

APPLICATION 1. *Littlewood-Paley operator.*

Fixed $0 \leqslant \delta < n$ and $\varepsilon > 0$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \leqslant C(1+|x|)^{-(n+1-\delta)}$,

(3) $|\psi(x+y) - \psi(x)| \leqslant C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$.

The Littlewood-Paley multilinear operator are defined by

$$g_{\psi}^{A}(f)(x) = \left(\int_0^{\infty} |F_t^A(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j;x,y)}{|x-y|^m} \psi_t(x-y) f(y) dy$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_{\psi}(f)(x) = \left(\int_0^{\infty} |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

which are the Littlewood-Paley operator (see [21]). Let H be the space

$$H = \left\{ h : ||h|| = \left(\int_0^{\infty} |h(t)|^2 dt/t \right)^{1/2} < \infty \right\},$$

then, for each fixed $x \in R^n$, $F_t^A(f)(x)$ may be viewed as the mapping from $[0,+\infty)$ to H, and it is clear that

$$g_{\psi}^{A}(f)(x) = ||F_t^A(f)(x)||, \quad g_{\psi}(f)(x) = ||F_t(f)(x)||.$$

It is easily to see that g_{ψ} satisfies the conditions of Theorems 1 and 2 (see [11], [13], [15–17]), thus Theorems 1 and 2 hold for g_{ψ}^{A}.

APPLICATION 2. *Marcinkiewicz operator.*

Fixed $0 \leqslant \delta < n$ and $0 < \gamma \leqslant 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_{\gamma}(S^{n-1})$. The Marcinkiewicz multilinear operators are defined by

$$\mu_{\Omega}^{A}(f)(x) = \left(\int_0^{\infty} |F_t^A(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\leqslant t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy.$$

Set

$$F_t(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy.$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which are the Marcinkiewicz operator (see [22]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}.$$

Then, it is clear that

$$\mu_\Omega^A(f)(x) = ||F_t^A(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||.$$

It is easily to see that μ_Ω satisfies the conditions of Theorems 1 and 2 (see [12–14], [16], [17]), thus Theorems 1 and 2 hold for μ_Ω^A.

APPLICATION 3. *Bochner-Riesz operator.*

Let $\delta > (n-1)/2$, $B_t^\delta(f)\hat{}(\xi) = (1-t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} B_t^\delta(x-y)f(y)dy,$$

The maximal Bochner-Riesz multilinear operator are defined by

$$B_{\delta,*}^A(f)(x) = \sup_{t>0} |B_{\delta,t}^A(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|$$

which is the maximal Bochner-Riesz operator (see [18]). Let H be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\delta,*}^A(f)(x) = ||B_{\delta,t}^A(f)(x)||, \;\; B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||.$$

It is easily to see that $B_{\delta,*}^A$ satisfies the conditions of Theorems 1 and 2 (see [13, 16, 17]), thus Theorems 1 and 2 hold for $B_{\delta,*}^A$.

Acknowledgements. The present investigation was supported by the National Natural Science Foundation under Grant nos. 11301008, 11226088 and 11101053, the Foundation for Excellent Youth Teachers of Colleges and Universities of Henan Province under Grant no. 2013GGJS-146, and the Natural Science Foundation of Educational Committee of Henan Province under Grant 14B110012 of the People's Republic of China.

REFERENCES

[1] S. CHANILLO, *A note on commutators*, Indiana Univ. Math. J., **31** (1982), 7–16.

[2] W. CHEN AND G. HU, *Weak type (H^1 , L^1) estimate for multilinear singular integral operator*, Adv. in Math. (China), **30** (2001), 63–69.

[3] J. COHEN, *A sharp estimate for a multilinear singular integral on R^n*, Indiana Univ. Math. J., **30** (1981), 693–702.

[4] J. COHEN AND J. GOSSELIN, *On multilinear singular integral operators on R^n*, Studia Math., **72** (1982), 199–223.

[5] J. COHEN AND J. GOSSELIN, *A BMO estimate for multilinear singular integral operators*, Illinois J. Math., **30** (1986), 445–465.

[6] R. COIFMAN AND Y. MEYER, *Wavelets, Calderón-Zygmund and multilinear operators*, Cambridge Studies in Advanced Math., 48, Cambridge University Press, Cambridge, 1997.

[7] R. COIFMAN, R. ROCHBERG AND G. WEISS, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103** (1976), 611–635.

[8] Y. DING AND S. Z. LU, *Weighted boundedness for a class rough multilinear operators*, Acta Math. Sinica, **17** (2001), 517–526.

[9] J. GARCIA-CUERVA AND J. L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*, North-Holland Math., 116, Amsterdam, 1985.

[10] E. HARBOURE, C. SEGOVIA AND J. L. TORREA, *Boundedness of commutators of fractional and singular integrals for the extreme values of p*, Illinois J. Math., **41** (1997), 676–700.

[11] L. Z. LIU, *Weighted weak type (H^1, L^1) estimates for commutators of Littlewood-Paley operator*, Indian J. of Math., **45** (2003), 71–78.

[12] L. Z. LIU, *Endpoint estimates for multilinear Marcinkiewicz integral operators*, East J. on Approximations, **9** (2003), 339–350.

[13] L. Z. LIU, *Weighted Herz spaces continuity of multilinear opeartos for the extreme cases*, Siberia Math. J., **45** (2004), 940–955.

[14] L. Z. LIU, *Weighted continuity of multilinear Marcinkiewicz operators for the extreme cases of p*, Commun. Korean Math. Soc., **19** (2004), 435–452.

[15] L. Z. LIU, *Weighted endpoint estimates for multilinear Littlewood-Paley operators*, Acta Math. Univ. Comenianae, **73** (2004), 55–67.

[16] L. Z. LIU, *Weighted boundedness of multilinear operators for the extreme cases*, Taiwanese J. Math., **10** (2006), 669–690.

[17] L. Z. LIU, *Endpoint estimates for multilinear operators of some sublinear operators on Herz and Herz type Hardy spaces*, Studia Sci. Math. Hungarica, **42** (2005), 131–151.

[18] S. Z. LU, *Four lectures on real H^p spaces*, World Scientific, River Edge, NI, 1995.

[19] C. PÉREZ AND R. TRUJILLO-GONZALEZ, *Sharp weighted estimates for vector-valued singular integral operators and commutators*, Tohoku Math. J., **55** (2003), 109–129.

[20] E. M. STEIN, *Harmonic analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.

[21] A. TORCHINSKY, *Real variable methods in harmonic analysis*, Pure and Applied Math., 123, Academic Press, New York, 1986.

[22] A. TORCHINSKY AND S. WANG, *A note on the Marcinkiewicz integral*, Colloq. Math., **60/61** (1990), 235–243.

(Received April 25, 2013)

Xin-Sheng Yuan
School of Mathematics and Statistics
Anyang Normal University
Anyang 455002, Henan, P.R. China

Zhi-Gang Wang
School of Mathematics and Statistics
Anyang Normal University
Anyang 455002, Henan, P.R. China
e-mail: zhigangwang@foxmail.com