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A NOTE ON GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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Abstract. In this paper, we prove that the function $x \to \log(x/\sin_p(x))/\log(\sinh_p(x)/x)$ $(p \in [2,\infty))$ is strictly increasing on $(0, \pi_p/2)$, where $\pi_p/2 = \int_0^1 (1-t^p)^{-1/p}dt$, and $\sin_p(x)$ and $\sinh_p(x)$ denote the generalized trigonometric sine and generalized hyperbolic sine functions, respectively. As application, a conjecture due to Klén, Vuorinen and Zhang [J. Math. Anal. Appl. 409 (2014), 521–529] is proved, and the best positive constants α and β such that

$$\left(\frac{\sinh_p(x)}{x}\right)^{\alpha} < \frac{x}{\sin_p(x)} < \left(\frac{\sinh_p(x)}{x}\right)^{\beta}$$

are determined.

1. Introduction

It is well known from basic calculus that

$$\frac{\pi}{2} = \int_0^1 \frac{1}{\sqrt{1-t^2}}dt$$

and

$$\arcsin(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}dt, \quad 0 \leqslant x \leqslant 1.$$

Since the function $\arcsin(x)$ is a differentiable function on $[0,1]$ and $t \to 1/\sqrt{1-t^2}$ is strictly increasing on $[0,1)$, we can define sin on $[0,\pi/2]$ as the inverse function of arcsin. By standard extension procedures we can define the sin function on $(-\infty,\infty)$.

For $p > 1$, let

$$F_p(x) = \int_0^x (1-t^p)^{-1/p}dt, \quad x \in [0,1],$$

$$\frac{\pi_p}{2} = \int_0^1 (1-t^p)^{-1/p}dt.$$

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Then $F_p : [0,1] \to [0,\pi_p/2]$ is an increasing homeomorphism, denoted by $\arcsin_p$. Thus its inverse

$$\sin_p = F_p^{-1}$$

is defined on the interval $[0,\pi_p/2]$. By the similar extension as the sine function, we can get a differentiable function $\sin_p$ defined on $\mathbb{R}$. We call $\sin_p$ the generalized trigonometric sine function.

Similarly, the generalized inverse hyperbolic sine function

$$\operatorname{arcsinh_p}(x) = \begin{cases} \int_0^x (1+t^p)^{-1/p}, \ x \in [0,\infty), \\ -\operatorname{arcsinh_p}(-x), \ x \in (-\infty,0) \end{cases}, \quad p > 1$$

generalized the classical inverse hyperbolic sine function. The inverse of $\operatorname{arcsinh_p}$ is named the generalized hyperbolic sine function and denoted by $\sinh_p$.

Recently, the generalized trigonometric and hyperbolic functions have been found many important applications in differential equations, the theory of operator, approximation theory and other related fields [5, 10, 12]. In particular, many remarkable properties and inequalities can be found in the literatures [2–9, 11, 14].

Klén, Vuorinen and Zhang [9] generalized some classical inequalities for trigonometric and hyperbolic functions, such as Mitrinović-Adamović inequality and Lazarević's inequality. Moreover, they also raised the following conjecture.

CONJECTURE 1.1. *For $p \in [2,\infty)$, the function*

$$f(x) = \frac{\log(x/\sin_p(x))}{\log(\sinh_p(x)/x)}$$

is strictly increasing on $(0,\pi_p/2)$.

The main purpose of this paper is to give a positive answer to the Conjecture 1.1. Our main result is the following Theorem 1.1.

THEOREM 1.1. *If $p \in [2,\infty)$, then the function*

$$f(x) = \frac{\log(x/\sin_p(x))}{\log(\sinh_p(x)/x)}$$

is strictly increasing from $(0,\pi_p/2)$ onto $(1, \log(\pi_p/2)/\log[2\sinh_p(\pi_p/2)/\pi_p])$. In particular, inequality

$$\left(\frac{\sinh_p(x)}{x}\right)^{\alpha} < \frac{x}{\sin_p(x)} < \left(\frac{\sinh_p(x)}{x}\right)^{\beta} \tag{1.1}$$

holds for all $x \in (0,\pi_p/2)$ with the best possible constants $\alpha = 1$ and $\beta = \log(\pi_p/2)/\log[2\sinh_p(\pi_p/2)/\pi_p]$.

REMARK 1.1. If $p = 2$, then Theorem 1.1 reduces to Theorem 1.1 in [13].

2. Lemmas

In order to establish our main results we need some basic knowledge and Lemmas, which we present in this section.

The generalized cosine function $\cos_p$, generalized tangent function $\tan_p$, generalized hyperbolic cosine function $\cosh_p$ and generalized hyperbolic tangent function $\tanh_p$ are defined as

$$\cos_p(x) \equiv \frac{d}{dx}\sin_p(x), \quad \tan_p(x) \equiv \frac{\sin_p(x)}{\cos_p(x)}, \quad x \in (0, \pi_p/2),$$

$$\cosh_p(x) \equiv \frac{d}{dx}\sinh_p(x), \quad \tanh_p(x) \equiv \frac{\sinh_p(x)}{\cosh_p(x)}, \quad x \in [0, \infty),$$

respectively. And the following formulas can be found in [8, 9]:

$$\cos_p(x) = (1 - \sin_p(x)^p)^{1/p}, \quad x \in (0, \pi_p/2),$$

$$\frac{d}{dx}\cos_p(x) = -\cos_p(x)^{2-p}\sin_p(x)^{p-1}, \quad x \in (0, \pi_p/2),$$

$$\frac{d}{dx}\tan_p(x) = 1 + \tan_p(x)^p, \quad x \in (0, \pi_p/2),$$

$$\cosh_p(x) = (1 + \sinh_p(x)^p)^{1/p}, \quad x \in [0, \infty),$$

$$\frac{d}{dx}\cosh_p(x) = \cosh_p(x)^{2-p}\sinh_p(x)^{p-1}, \quad x \in [0, \infty),$$

$$\frac{d}{dx}\tanh_p(x) = 1 - \tanh_p(x)^p, \quad x \in [0, \infty).$$

LEMMA 2.1. (See [1, Theorem 1.25]) *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b), let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

LEMMA 2.2. *If $p \in [2, \infty)$, then*

(1) *$\xi(x) = [p+2-2(p+1)(p-2)x-(p+1)(p-2)x^2]/(p^2+2p+2px)$ is strictly decreasing in $[0, \infty)$;*

(2) *$\eta(x) = [p+2+2(p+1)(p-2)x-(p+1)(p-2)x^2]/(p^2+2p-2px)$ is strictly increasing in $[0, 1)$.*

Proof. Parts (1) and (2) follows easily from

$$\xi'(x) = -\frac{2[(p+2)(p^2-p-1)+(p+1)(p+2)(p-2)x+(p+1)(p-2)x^2]}{p(p+2+2x)^2}$$
$$< 0$$

and

$$\eta'(x) = \frac{2[(p+2)(p^2-p-1)-(p+1)(p+2)(p-2)x+(p+1)(p-2)x^2]}{p(p+2-2x)^2}$$
$$> \frac{2[(p+2)+(p+1)(p-2)x^2]}{p(p+2-2x)^2}$$
$$> 0. \quad \square$$

LEMMA 2.3. *If* $p \in [2,\infty)$, *then inequality*

$$\log\left(\frac{x}{\sin_p(x)}\right) < \frac{\sin_p(x) - x\cos_p(x)}{p\sin_p(x)}$$

holds for all $x \in (0, \pi_p/2)$.

Proof. Let

$$\phi(x) = \frac{\sin_p(x) - x\cos_p(x)}{p\sin_p(x)} - \log\left(\frac{x}{\sin_p(x)}\right), \quad x \in (0, \pi_p/2), \tag{2.1}$$

then simple computation leads to

$$\phi(0^+) = 0, \tag{2.2}$$

$$\begin{aligned}\phi'(x) =& \frac{x\cos_p(x)^{2-p}\sin_p(x)^p - \cos_p(x)\sin_p(x) + x\cos_p(x)^2}{p\sin_p(x)^2} \\ &- \frac{\sin_p(x)^2 - x\cos_p(x)\sin_p(x)}{x\sin_p(x)^2} \\ =& \frac{x^2\cos_p(x)^2[\cos_p(x)^{-p}\sin_p(x)^p + 1] - x\cos_p(x)\sin_p(x)}{px\sin_p(x)^2} \\ &+ \frac{p(x\cos_p(x)\sin_p(x) - \sin_p(x)^2)}{px\sin_p(x)^2} \\ =& \frac{x^2\cos_p(x)^{2-p} - x\cos_p(x)\sin_p(x) + p[x\cos_p(x)\sin_p(x) - \sin_p(x)^2]}{px\sin_p(x)^2} \\ =& \frac{1}{px}\left(1 - \frac{x}{\tan_p(x)}\right)\left[\frac{1}{\phi_1(x)} - p\right],\end{aligned} \tag{2.3}$$

where

$$\phi_1(x) = \frac{x\cos_p(x)\sin_p(x) - \sin_p(x)^2}{x\cos_p(x)\sin_p(x) - x^2\cos_p(x)^{2-p}}. \tag{2.4}$$

Letting $\phi_2(x) = x\cos_p(x)^{p-1}\sin_p(x) - x^2$, $\phi_3(x) = x\cos_p(x)^{p-1}\sin_p(x) - \sin_p(x)^2\cos_p(x)^{p-2}$, $\phi_4(x) = p^2x + 2p\tan_p(x)$, $\phi_5(x) = p^2x - (p-2)\tan_p(x)^{p+1} - (p+1)(p-2)\tan_p(x)$. Then $\phi_1(x) = \phi_3(x)/\phi_2(x)$,

$$\phi_2(0) = \phi_3(0) = 0, \tag{2.5}$$

$$\phi_2'(x) = \cos_p(x)^{p-1}\sin_p(x) - (p-1)x\sin_p(x)^p + x\cos_p(x)^p - 2x, \tag{2.6}$$

$$\begin{aligned}\phi_3'(x) = &-\cos_p(x)^{p-1}\sin_p(x) - (p-1)x\sin_p(x)^p + x\cos_p(x)^p \\ &+ (p-2)\sin_p(x)^p\tan_p(x),\end{aligned} \tag{2.7}$$

$$\phi_2'(0) = \phi_3'(0) = 0 \tag{2.8}$$

$$\phi_2''(x) = -\sin_p(x)^p\left[2p + p^2\frac{x}{\tan_p(x)}\right], \tag{2.9}$$

$$\phi_3''(x) = -\sin_p(x)^p\left[p^2\frac{x}{\tan_p(x)} - (p-2)\tan_p(x)^p - (p+1)(p-2)\right], \tag{2.10}$$

$$\frac{\phi_3''(x)}{\phi_2''(x)} = \frac{\phi_5(x)}{\phi_4(x)}, \quad \phi_4(0) = \phi_5(0) = 0, \tag{2.11}$$

and

$$\frac{\phi_5'(x)}{\phi_4'(x)} = \frac{p+2-2(p+1)(p-2)\tan_p(x)^p - (p+1)(p-2)\tan_p(x)^{2p}}{p^2+2p+2p\tan_p(x)^p}. \tag{2.12}$$

It follows from (2.4)–(2.12) and Lemmas 2.1 and 2.2 (1) that $\phi_1(x)$ is strictly decreasing in $(0, \pi_p/2)$. Then

$$\phi_1(x) < \phi_1(0^+) = \lim_{x\to 0}\frac{\phi_5'(x)}{\phi_4'(x)} = \frac{1}{p}. \tag{2.13}$$

From (2.3) and (2.13) we clearly see that $\phi'(x) > 0$, then from equation (2.2) one has

$$\phi(x) > 0. \tag{2.14}$$

.

Therefore, Lemma 2.3 follows from (2.1) and (2.14). □

LEMMA 2.4. *If* $p \in [2,\infty)$, *then inequality*

$$\log\left(\frac{x}{\sinh_p(x)}\right) < \frac{\sinh_p(x) - x\cosh_p(x)}{p\sinh_p(x)}$$

holds for all $x \in (0,+\infty)$.

Proof. Let

$$\varphi(x) = \frac{\sinh_p(x) - x\cosh_p(x)}{p\sinh_p(x)} - \log\left(\frac{x}{\sinh_p(x)}\right), \quad x \in (0,+\infty), \tag{2.15}$$

then simple computation leads to

$$\varphi(0^+) = 0, \tag{2.16}$$

$$\begin{aligned}\varphi'(x) =&\frac{-x\cosh_p(x)^{2-p}\sinh_p(x)^p-\cosh_p(x)\sinh_p(x)+x\cosh_p(x)^2}{p\sinh_p(x)^2}\\&-\frac{\sinh_p(x)^2-x\cosh_p(x)\sinh_p(x)}{x\sinh_p(x)^2}\\=&\frac{x^2\cosh_p(x)^2[-\cosh_p(x)^{-p}\sinh_p(x)^p+1]-x\cosh_p(x)\sinh_p(x)}{px\sinh_p(x)^2}\\&+\frac{p(x\cosh_p(x)\sinh_p(x)-\sinh_p(x)^2)}{px\sinh_p(x)^2}\\=&\frac{x^2\cosh_p(x)^{2-p}-x\cosh_p(x)\sinh_p(x)+p[x\cosh_p(x)\sinh_p(x)-\sinh_p(x)^2]}{px\sinh_p(x)^2}\\=&\frac{1}{px}\left(\frac{x}{\tanh_p(x)}-1\right)\left[p-\frac{1}{\varphi_1(x)}\right],\end{aligned}\tag{2.17}$$

where

$$\varphi_1(x)=\frac{x\cosh_p(x)\sinh_p(x)-\sinh_p(x)^2}{x\cosh_p(x)\sinh_p(x)-x^2\cosh_p(x)^{2-p}}.\tag{2.18}$$

Letting $\varphi_2(x)=x\cosh_p(x)^{p-1}\sinh_p(x)-x^2$, $\varphi_3(x)=x\cosh_p(x)^{p-1}\sinh_p(x)-\sinh_p(x)^2\cosh_p(x)^{p-2}$, $\varphi_4(x)=p^2x+2p\tanh_p(x)$, $\varphi_5(x)=p^2x+(p-2)\tanh_p(x)^{p+1}-(p+1)(p-2)\tanh_p(x)$. Then $\varphi_1(x)=\varphi_3(x)/\varphi_2(x)$,

$$\varphi_2(0)=\varphi_3(0)=0,\tag{2.19}$$

$$\varphi_2'(x)=\cosh_p(x)^{p-1}\sinh_p(x)+(p-1)x\sinh_p(x)^p+x\cosh_p(x)^p-2x,\tag{2.20}$$

$$\begin{aligned}\varphi_3'(x)=&-\cosh_p(x)^{p-1}\sinh_p(x)+(p-1)x\sinh_p(x)^p+x\cosh_p(x)^p\\&-(p-2)\sinh_p(x)^p\tanh_p(x),\end{aligned}\tag{2.21}$$

$$\varphi_2'(0)=\varphi_3'(0)=0\tag{2.22}$$

$$\varphi_2''(x)=\sinh_p(x)^p\left[2p+p^2\frac{x}{\tanh_p(x)}\right],\tag{2.23}$$

$$\varphi_3''(x)=\sinh_p(x)^p\left[p^2\frac{x}{\tanh_p(x)}+(p-2)\tanh_p(x)^p-(p+1)(p-2)\right],\tag{2.24}$$

$$\frac{\varphi_3''(x)}{\varphi_2''(x)}=\frac{\varphi_5(x)}{\varphi_4(x)},\quad \varphi_4(0)=\varphi_5(0)=0,\tag{2.25}$$

and

$$\frac{\varphi_5'(x)}{\varphi_4'(x)}=\frac{p+2+2(p+1)(p-2)\tanh_p(x)^p-(p+1)(p-2)\tanh_p(x)^{2p}}{p^2+2p-2p\tanh_p(x)^p}.\tag{2.26}$$

It follows from (2.18)–(2.26) and Lemmas 2.1 and 2.2 (2) that $\varphi_1(x)$ is strictly increasing in $(0,+\infty)$. Then

$$\varphi_1(x) > \varphi_1(0^+) = \lim_{x\to 0}\frac{\varphi_5'(x)}{\varphi_4'(x)} = \frac{1}{p}. \tag{2.27}$$

From (2.17) and (2.27) we clearly see that $\varphi'(x) > 0$, then from equation (2.16) one has

$$\varphi(x) > 0. \tag{2.28}$$

Therefore, Lemma 2.4 follows from (2.15) and (2.28). □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $f_1(x) = \log(x/\sin_p(x))$ and $f_2(x) = \log(\sinh_p(x)/x)$. Then simple computations lead to

$$f_1'(x) = \frac{\sin_p(x) - x\cos_p(x)}{x\sin_p(x)}, \quad f_2'(x) = \frac{x\cosh_p(x) - \sinh_p(x)}{x\sinh_p(x)}$$

and

$$\begin{aligned}\frac{xf_2(x)^2 f'(x)}{p} =& \frac{\sin_p(x) - x\cos_p(x)}{p\sin_p(x)}\log\left(\frac{\sinh_p(x)}{x}\right)\\ &- \frac{x\cosh_p(x) - \sinh_p(x)}{p\sinh_p(x)}\log\left(\frac{x}{\sin_p(x)}\right).\end{aligned} \tag{3.1}$$

It follows from Lemmas 2.3 and 2.4 together with (3.1) that $f'(x) > 0$ for $x \in (0,\pi_p/2)$. This implies

$$\alpha \equiv \lim_{x\to 0} f(x) < f(x) < f\left(\frac{\pi_p}{2}\right) = \frac{\log(\pi_p/2)}{\log[2\sinh_p(\pi_p/2)/\pi_p]} \equiv \beta.$$

By l'Hoptial's rule we have $\alpha = 1$, and the remaining results are clear. □

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