

## FEKETE–SZEGÖ INEQUALITY FOR GENERALIZED SUBCLASSES OF UNIVALENT FUNCTIONS

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*Abstract.* Let  $\mathcal{P}_\varphi(n, b, \lambda)$  denote the class of normalized univalent functions  $f(z) = z + a_2z^2 + \dots$ , which are defined in the unit disk  $\Delta$  and satisfying  $1 + [(\lambda D^{n+2}f(z) + (1 - \lambda)D^{n+1}f(z)) / (\lambda D^{n+1}f(z) + (1 - \lambda)D^n f(z)) - 1] / b \prec \varphi(z)$ , where  $\varphi(z)$  is the function with positive real part,  $D^n f$  denotes the sălăgean operator,  $n \geq 0$ ,  $0 \leq \lambda \leq 1$ ,  $b \in \mathbb{C}$ . In this paper, for the class  $\mathcal{P}_\varphi(n, b, \lambda)$ , the Fekete-Szegő inequalities are completely solved. A more general class  $\mathcal{K}(\beta, n, \lambda, g(z))$  related  $\mathcal{P}_\varphi(n, b, \lambda)$  is also considered with same subject, which extends the earlier corresponding results for the class of strongly close-to-convex functions of order  $\beta$ .

### 1. Introduction

Let  $\mathcal{A}$  be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic univalent in the open unit disk  $\Delta = \{z : |z| < 1\}$  and normalized by the conditions  $f(0) = 0$ ,  $f'(0) = 1$ . It is well-known that  $f(z) \in \mathcal{A}$ ,  $|a_3 - a_2^2| \leq 1$ . If  $f$  and  $g$  are analytic in  $\Delta$ , we say that  $f$  is subordinate to  $g$ , written  $f(z) \prec g(z)$ , provided there exists a analytic function  $\omega(z)$  defined on  $\Delta$  with  $\omega(0) = 0$  and  $|\omega(z)| < 1$  satisfying  $f(z) = g(\omega(z))$ .

For  $f(z) \in \mathcal{A}$ , Sălăgean [20] defined the following operator:

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z), \dots, \quad D^n f(z) = D(D^{n-1}f(z)),$$

where  $n \in N = \{1, 2, \dots\}$ . We note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad n \in N_0 = \{0\} \cup N. \tag{1.2}$$

Let  $S^*$ ,  $C$  and  $\mathcal{K}$  denote the usual starlike function, convex function and close-to-convex function, respectively. Ma and Minda [12] unified various subclasses of

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starlike and convex functions for which either one of the quantities  $zf'(z)/f(z)$  or  $1 + zf''(z)/f'(z)$  is subordinate to a more general superordinate function. The classes  $S^*(\varphi)$  and  $C(\varphi)$  of Ma-Minda starlike and Ma-Minda convex functions, are respectively characterized by  $zf'(z)/f(z) \prec \varphi(z)$  and  $1 + zf''(z)/f'(z) \prec \varphi(z)$ , where function  $\varphi$  with positive real part in  $\Delta$ ,  $\varphi(0) = 0$ ,  $\varphi'(0) > 1$ . The coefficient functional  $\rho_\mu(f) = a_3 - \mu a_2^2$  on the normalized analytic functions  $f$  in  $\Delta$  plays an important role in function theorem. The problem of maximizing the absolute value of the functional  $\rho_\mu(f)$  is called the Fekete-Szegő problem. A classical theorem of Fekete-Szegő (see [5]) states that for  $f \in \mathcal{A}$  given by (1.1),

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0, \\ 1 + 2 \exp\left(\frac{-2\mu}{1-\mu}\right), & \text{if } 0 \leq \mu < 1, \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases}$$

Later, Pfluger [17] considered the problem when  $\mu$  is complex. In the case of  $C, S^*$  and  $\mathcal{H}$ , the above inequalities can be improved [9, 10]. Actually, many authors have considered the Fekete-Szegő problem for various subclasses of  $\mathcal{A}$ , the upper bound for  $|a_3 - \mu a_2^2|$  was investigated by many different authors (see [3, 6, 19, 22]). Recently, some results on this subject were improved (see [2, 4, 8, 13, 14, 15, 16, 21, 23]).

We denote by  $\mathcal{P}$  a class of analytic function in  $\Delta$  with  $p(0) = 0$  and  $\Re p(z) > 0$ . Here we assume that  $\varphi \in \mathcal{P}$ , satisfying  $\varphi(0) = 0$ ,  $\varphi'(0) > 0$ , and  $\varphi(\Delta)$  is symmetric with respect to the real axis. Also,  $\varphi(z)$  has a series expansion of the form

$$\varphi(z) = 1 + A_1z + A_2z^2 + A_3z^3 + \dots, (A_1 > 0). \tag{1.3}$$

With the aid of salagean operator, we introduce the class  $\mathcal{P}_\varphi(n, b, \lambda)$  as follows:

A function  $f \in \mathcal{A}$  is said to be in the class  $\mathcal{P}_\varphi(n, b, \lambda)$  if and only if

$$1 + \frac{1}{b} \left( \frac{\lambda D^{n+2}f(z) + (1-\lambda)D^{n+1}f(z)}{\lambda D^{n+1}f(z) + (1-\lambda)D^n f(z)} - 1 \right) \prec \varphi(z), z \in \Delta, \tag{1.4}$$

where  $b$  is nonzero complex number,  $\varphi$  is defined as (1.3),  $n \geq 0, 0 \leq \lambda \leq 1$ .

By giving specific values to the parameters  $b, \lambda$  and  $\varphi$ , we obtain the following important subclasses studied by various authors in earlier works, for instance,

$$\mathcal{P}_\varphi(0, 1, 0) \equiv S^*(\varphi), \quad \mathcal{P}_\varphi(0, 1, 1) \equiv C(\varphi),$$

and

$$\mathcal{P}_{\frac{1+Az}{1+Bz}}(0, 1, 0) \equiv S^*[A, B], \quad \mathcal{P}_{\frac{1+Az}{1+Bz}}(0, 1, 1) \equiv C[A, B],$$

where  $-1 \leq A < B \leq 1$ . The  $S^*(\varphi), C(\varphi)$  were introduced by Ma-Minda [12]. The  $S^*[A, B], C[A, B]$  were defined by Janowski [7].

By taking  $b = 1, \varphi(z) = \left(\frac{1+z}{1-z}\right)^\beta$  ( $0 \leq \beta \leq 1$ ), we want to extend the  $\mathcal{P}_\varphi(n, b, \lambda)$  to a more general class  $\mathcal{H}(\beta, n, \lambda, g(z))$ .

A function  $f \in \mathcal{A}$  is said to be in the class  $\mathcal{H}(\beta, n, \lambda, g(z))$  if and only if

$$\left| \arg \left( \frac{\lambda D^{n+2} f(z) + (1 - \lambda) D^{n+1} f(z)}{\lambda D^{n+1} g(z) + (1 - \lambda) D^n g(z)} \right) \right| \leq \frac{\pi}{2} \beta, \quad z \in \Delta, \tag{1.5}$$

where  $n \geq 0, 0 \leq \lambda \leq 1, 0 \leq \beta \leq 1, g(z) = z + b_2 z^2 + b_3 z^3 + \dots \in S^*$ .

We note that  $\mathcal{H}(\beta, 0, 0, g(z)) \equiv \mathcal{H}(\beta)$ , where  $\mathcal{H}(\beta)$  is the class of strongly close-to-convex functions of order  $\beta$  defined by Koepf [11] and Abdel-Gawad [1]. Koepf [11] considered the Fekete-Szegő problem for  $\mathcal{H}(\beta)$  with some particular values of  $\mu$ . Later, Abdel-Gawad [1] improved the results for any  $\mu \in \mathbb{R}$  without  $\mu = 1$ .

In this paper, we concentrate on the Fekete-Szegő problem for the subclass  $\mathcal{P}_\phi(n, b, \lambda)$ , which is discussed with four different cases as: (i)  $\mu$  is special number,  $b \in \mathbb{C}$ . (ii)  $\mu \in \mathbb{C}, b \in \mathbb{C}$ . (iii)  $b > 0, \mu \in \mathbb{R}$ . (IV)  $\mu \in \mathbb{R}, b \in \mathbb{C}$ . Also, a more general class  $\mathcal{H}(\beta, n, \lambda, g(z))$  is considered on the same subject, which extends the corresponding earlier results for the class  $\mathcal{H}(\beta)$  of strongly close-to-convex functions of order  $\beta$ .

### 2. Main results

In order to derive our main results, we have to recall here the following Lemmas.

LEMMA 1. ([10]) *Let  $g \in S^*$  with  $g(z) = z + b_2 z^2 + b_3 z^3 + \dots$ , then for real  $\mu$ ,*

$$|b_3 - \mu b_2^2| \leq \max\{1, |3 - 4\mu|\}.$$

*The result is sharp.*

LEMMA 2. ([18]) *Let  $p \in \mathcal{P}$  with  $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ , then  $|c_n| \leq 2$  for  $n \geq 1$ . If  $|c_1| = 2$  then  $p(z) \equiv p_1(z) = (1 + \gamma_1 z)/(1 - \gamma_1 z)$  with  $\gamma_1 = c_1/2$ . Conversely, if  $p(z) \equiv p_1(z)$  for some  $|\gamma_1| = 1$ , then  $c_1 = 2\gamma_1$  and  $|c_1| = 2$ . Furthermore we have*

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

*If  $|c_1| < 2$  and  $|c_2 - \frac{c_1^2}{2}| = 2 - \frac{|c_1|^2}{2}$ , then  $p(z) \equiv p_2(z)$ , where*

$$p_2(z) = \frac{1 + z \frac{\gamma_2 z + \gamma_1}{1 + \gamma_1 \gamma_2 z}}{1 - z \frac{\gamma_2 z + \gamma_1}{1 + \gamma_1 \gamma_2 z}}$$

*and  $\gamma_1 = c_1/2, \gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$ . Conversely if  $p(z) = p_2(z)$  for some  $|\gamma_1| < 1$  and  $|\gamma_2| = 1$ , then  $\gamma_1 = c_1/2, \gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$  and  $|c_2 - \frac{c_1^2}{2}| = 2 - \frac{|c_1|^2}{2}$ .*

**THEOREM 1.** *Let  $n \geq 0$ ,  $0 \leq \lambda \leq 1$ , and Let  $b$  be nonzero complex number. If  $f(z) \in \mathcal{P}_\varphi(n, b, \lambda)$ , where  $\varphi(z) = 1 + A_1z + A_2z^2 + \dots + A_nz^n + \dots$ , ( $A_1 > 0$ ), then*

$$|a_2| \leq \frac{1}{2^n} \frac{1}{1 + \lambda} |b| A_1, \tag{2.1}$$

$$|a_3| \leq \frac{1}{2} \cdot \frac{1}{1 + 2\lambda} \frac{1}{3^n} |b| A_1 \max \left\{ 1, \left| bA_1 + \frac{A_2}{A_1} \right| \right\}, \tag{2.2}$$

$$\left| a_3 - \frac{2 \cdot 4^{n-1} \cdot (1 + \lambda)^2 \cdot (bA_1 + \frac{A_2}{A_1} - 1)}{3^n(1 + 2\lambda)bA_1} a_2^2 \right| \leq \frac{A_1|b|}{(2 + 4\lambda)3^n}. \tag{2.3}$$

*These results are sharp.*

*Proof.* Let  $f(z) \in \mathcal{P}_\varphi(n, b, \lambda)$ . Then there is a function  $w(z)$ , such that

$$1 + \frac{1}{b} \left( \frac{\lambda D^{n+2}f(z) + (1 - \lambda)D^{n+1}f(z)}{\lambda D^{n+1}f(z) + (1 - \lambda)D^n f(z)} - 1 \right) = \varphi(w(z)), \quad z \in \Delta.$$

Define the function  $p(z)$  by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + r_1z + r_2z^2 + \dots < \frac{1 + z}{1 - z}, \quad z \in \Delta. \tag{2.4}$$

We can note that  $p(0) = 1$  and  $p(z)$  is a function with positive real part. In fact, using the (2.4), it is easy to know that

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left( r_1z + \left( r_2 - \frac{r_1^2}{2} \right) z^2 + \dots \right),$$

So

$$\begin{aligned} 1 + \frac{1}{b} \left( \frac{\lambda D^{n+2}f(z) + (1 - \lambda)D^{n+1}f(z)}{\lambda D^{n+1}f(z) + (1 - \lambda)D^n f(z)} - 1 \right) &= \varphi(w(z)) \\ &= 1 + \frac{1}{2} A_1 r_1 z + \left( \frac{1}{2} A_1 \left( r_2 - \frac{r_1^2}{2} \right) + \frac{1}{4} A_2 r_1^2 \right) z^2 + \dots \end{aligned} \tag{2.5}$$

Actually, a computation shows that

$$\begin{aligned} 1 + \frac{1}{b} \left( \frac{\lambda D^{n+2}f(z) + (1 - \lambda)D^{n+1}f(z)}{\lambda D^{n+1}f(z) + (1 - \lambda)D^n f(z)} - 1 \right) \\ = 1 + \frac{1}{b} 2^n (1 + \lambda) a_2 z + \frac{1}{b} [(2 + 4\lambda) \cdot 3^n a_3 - (1 + \lambda)^2 4^n a_2^2] z^2 + \dots \end{aligned} \tag{2.6}$$

The equations (2.5) and (2.6) yield

$$\frac{1}{b} 2^n (1 + \lambda) a_2 = \frac{1}{2} A_1 r_1, \quad \frac{1}{b} [(2 + 4\lambda) \cdot 3^n a_3 - (1 + \lambda)^2 4^n a_2^2] = \frac{1}{2} A_1 \left( r_2 - \frac{r_1^2}{2} \right) + \frac{1}{4} A_2 r_1^2. \tag{2.7}$$

Taking into account (2.7) and Lemma 2, we obtain

$$|a_2| = \left| \frac{1}{2^{n+1}} \frac{1}{1+\lambda} bA_1 r_1 \right| \leq \frac{1}{2^n} \frac{1}{1+\lambda} |b|A_1, \tag{2.8}$$

and

$$\begin{aligned} |a_3| &= \left| \frac{1}{4} \cdot \frac{b}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ r_2 + \left( \frac{1}{2} bA_1 + \frac{1}{2} \frac{A_2}{A_1} - \frac{1}{2} \right) r_1^2 \right] \right| \\ &= \left| \frac{1}{4} \cdot \frac{b}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ r_2 - \frac{1}{2} r_1^2 + \frac{r_1^2}{2} (bA_1 + \frac{A_2}{A_1}) \right] \right| \\ &\leq \frac{1}{4} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ |r_2 - \frac{1}{2} r_1^2| + \frac{|r_1^2|}{2} |bA_1 + \frac{A_2}{A_1}| \right] \\ &\leq \frac{1}{4} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ 2 - \frac{1}{2} |r_1|^2 + \frac{|r_1|^2}{2} |bA_1 + \frac{A_2}{A_1}| \right] \\ &= \frac{1}{4} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ 2 + \frac{1}{2} |r_1|^2 (|bA_1 + \frac{A_2}{A_1}| - 1) \right] \\ &\leq \frac{1}{2} \cdot \frac{1}{1+2\lambda} \frac{1}{3^n} |b|A_1 \max \left\{ 1, |bA_1 + \frac{A_2}{A_1}| \right\}. \end{aligned}$$

Furthermore,

$$\left| a_3 - \frac{2 \cdot 4^{n-1} \cdot (1+\lambda)^2 \cdot (bA_1 + \frac{A_2}{A_1} - 1)}{3^n(1+2\lambda)bA_1} a_2^2 \right| = \left| \frac{1}{4} \cdot \frac{b}{3^n} \cdot \frac{1}{1+2\lambda} A_1 r_2 \right| \leq \frac{A_1 |b|}{(2+4\lambda)3^n}. \tag{2.9}$$

An examination of the proof shows the first equality holds if  $c_1 = 2$ . Equivalently, we have  $p(z) = p_1(z) = (1+z)/(1-z)$ . Therefore, the extremal function in  $\mathcal{P}_\varphi(n, b, \lambda)$  is defined by

$$1 + \frac{1}{b} \left( \frac{\lambda D^{n+2} f(z) + (1-\lambda) D^{n+1} f(z)}{\lambda D^{n+1} f(z) + (1-\lambda) D^n f(z)} - 1 \right) = \varphi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right). \tag{2.10}$$

Next, in (2.2), for first case, the equality holds if  $c_1 = 0, c_2 = 2$ . Equivalently, we have  $p(z) = p_2(z) = \frac{1+z^2}{1-z^2}$ . Therefore, the extremal functions in  $\mathcal{P}_\varphi(n, b, \lambda)$  is given by

$$1 + \frac{1}{b} \left( \frac{\lambda D^{n+2} f(z) + (1-\lambda) D^{n+1} f(z)}{\lambda D^{n+1} f(z) + (1-\lambda) D^n f(z)} - 1 \right) = \varphi \left( \frac{p_2(z) - 1}{p_2(z) + 1} \right). \tag{2.11}$$

In (2.2), for the second case, the equality holds if  $c_1 = 2, c_2 = 2$ . Therefore, the extremal function in  $\mathcal{P}_\varphi(n, b, \lambda)$  is given by (2.10).

Finally, in (2.3), the equality holds. Obtained extremal function for (2.1) is also valid for (2.3).

In fact, Theorem 1 gives a special case of Fekete-Szegő problem with

$$\mu = \frac{2 \cdot 4^{n-1} \cdot (1+\lambda)^2 \cdot (bA_1 + \frac{A_2}{A_1} - 1)}{3^n(1+2\lambda)bA_1},$$

which obtain the naturally and simple estimate. Thus the proof is completed.  $\square$

Now, we consider the Fekete-Szegő problem with complex  $\mu$ .

**THEOREM 2.** *Let  $n \geq 0$ ,  $0 \leq \lambda \leq 1$  and Let  $b$  be a nonzero complex number. If  $f(z) \in \mathcal{P}_\varphi(n, b, \lambda)$ , then for any complex  $\mu$ ,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1, & \mathcal{U} \leq 1, \\ \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \mathcal{U}, & \mathcal{U} > 1. \end{cases}$$

where  $\varphi(z) = 1 + A_1 z + A_2 z^2 + \dots, (A_1 > 0)$ ,  $\mathcal{U} = \left| bA_1 + \frac{A_2}{A_1} - 2 \left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 \right|$ .  
 The results are sharp.

*Proof.* Following (2.7), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{4} \cdot \frac{b}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ r_2 + \left( \frac{1}{2} bA_1 + \frac{1}{2} \frac{A_2}{A_1} - \frac{1}{2} \right) r_1^2 \right] - \mu \frac{1}{(1+\lambda)^2} \frac{b^2 A_1^2 r_1^2}{4^{n+1}} \\ &= \frac{1}{4} \cdot \frac{b}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ r_2 + \left( \frac{1}{2} bA_1 + \frac{1}{2} \frac{A_2}{A_1} - \frac{1}{2} \right) r_1^2 - \left( \frac{3}{4} \right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 r_1^2 \right] \\ &= \frac{1}{4} \cdot \frac{b}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ r_2 - \frac{1}{2} r_1^2 + \frac{1}{2} r_1^2 \left( bA_1 + \frac{A_2}{A_1} - 2 \left( \frac{3}{4} \right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 \right) \right]. \end{aligned}$$

Again, using the Lemma 2, it has

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{4} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left| r_2 - \frac{1}{2} r_1^2 + \frac{1}{2} r_1^2 \left( bA_1 + \frac{A_2}{A_1} - 2 \left( \frac{3}{4} \right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 \right) \right| \\ &\leq \frac{1}{4} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ 2 + \frac{1}{2} |r_1|^2 \left[ \left| bA_1 + \frac{A_2}{A_1} - 2 \left( \frac{3}{4} \right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 \right| - 1 \right] \right] \\ &\leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \max \left\{ 1, \left| bA_1 + \frac{A_2}{A_1} - 2 \left( \frac{3}{4} \right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 \right| \right\}. \end{aligned}$$

Equality holds for each  $\mu$  with the first case if functions in (2.11) and the second case if functions in (2.10). Thus the proof is completed.  $\square$

Next, we want to consider the Fekete-Szegő problem with real  $\mu$  and real  $b$ .

**THEOREM 3.** *Let  $n \geq 0$ ,  $0 \leq \lambda \leq 1$  and Let  $b > 0$ . If  $f(z) \in \mathcal{P}_\varphi(n, b, \lambda)$ , then for any real  $\mu$ ,*

(1) If  $bA_1 \geq 1$ , we have  $|a_3 - \mu a_2^2|$

$$\leq \begin{cases} \frac{1}{2} \frac{b}{3^n} \frac{1}{1+2\lambda} A_1 \left[ \mathcal{N}_1 - 2 \left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 \right], & \mu < \frac{A_2}{A_1} \mathcal{N}_1, \\ \frac{1}{2} \frac{b^2}{3^n} \frac{1}{1+2\lambda} A_1^2, & \frac{A_2}{A_1} \mathcal{N}_1 \leq \mu < (\mathcal{N}_1 - 1) \mathcal{N}_2, \\ \frac{1}{2} \frac{b}{3^n} \frac{1}{1+2\lambda} A_1, & (\mathcal{N}_1 - 1) \mathcal{N}_2 \leq \mu < (\mathcal{N}_1 + 1) \mathcal{N}_2, \\ \frac{1}{2} \frac{b}{3^n} \frac{1}{1+2\lambda} A_1 \left[ 2 \left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 - \mathcal{N}_1 \right], & \mu \geq (\mathcal{N}_1 + 1) \mathcal{N}_2. \end{cases}$$

(2) If  $bA_1 < 1$ , we have  $|a_3 - \mu a_2^2|$

$$\leq \begin{cases} \frac{1}{2} \frac{b}{3^n} \frac{1}{1+2\lambda} A_1 \left[ \mathcal{N}_1 - 2 \left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 \right], & \mu < (\mathcal{N}_1 - 1) \mathcal{N}_2, \\ \frac{1}{2} \frac{b}{3^n} \frac{1}{1+2\lambda} A_1, & (\mathcal{N}_1 - 1) \mathcal{N}_2 \leq \mu < (\mathcal{N}_1 + 1) \mathcal{N}_2, \\ \frac{1}{2} \frac{b}{3^n} \frac{1}{1+2\lambda} A_1 \left[ 2 \left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 - \mathcal{N}_1 \right], & \mu \geq (\mathcal{N}_1 + 1) \mathcal{N}_2. \end{cases}$$

where  $\varphi(z) = 1 + A_1z + A_2z^2 + \dots, (A_1 > 0, A_2 > 0)$ ,

$$\mathcal{N}_1 = bA_1 + \frac{A_2}{A_1}, \quad \mathcal{N}_2 = \frac{4^n(1+\lambda)^2}{2A_1b3^n(1+2\lambda)}.$$

For each  $\mu$  there is a function  $f \in \mathcal{P}_\varphi(n, b, \lambda)$  such that equality holds.

*Proof.* It follows from (2.7) that

$$a_3 - \mu a_2^2 = \frac{1}{4} \cdot \frac{b}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ r_2 - \frac{1}{2} r_1^2 + \frac{1}{2} r_1^2 \left( bA_1 + \frac{A_2}{A_1} - 2 \left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 \right) \right], \tag{2.12}$$

As the Lemma 2, we have

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} \cdot \frac{b}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ 2 + \frac{1}{2} |r_1|^2 \left[ \left| bA_1 + \frac{A_2}{A_1} - 2 \left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 \right| - 1 \right] \right]. \tag{2.13}$$

Firstly, we want to consider the case with  $bA_1 \geq 1$ . Several possible cases need to further analyze.

Case 1. If  $\mu \leq \frac{4^n A_2 (1+\lambda)^2}{2A_1^2 b 3^n (1+2\lambda)}$ , using (2.13), we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{4} \cdot \frac{b}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ 2 + \frac{1}{2} |r_1|^2 \left[ bA_1 + \frac{A_2}{A_1} - 2 \left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 - 1 \right] \right] \\ &\leq \frac{1}{2} \frac{b}{3^n} \frac{1}{1+2\lambda} A_1 \left[ bA_1 + \frac{A_2}{A_1} - 2 \left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 \right]. \end{aligned}$$

Case 2. If  $\frac{4^n A_2 (1 + \lambda)^2}{2A_1^2 b 3^n (1 + 2\lambda)} \leq \mu \leq \left(bA_1 + \frac{A_2}{A_1} - 1\right) \frac{4^n (1 + \lambda)^2}{2A_1 b 3^n (1 + 2\lambda)}$ , using (2.13),

we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{4} \cdot \frac{b}{3^n} \cdot \frac{1}{1 + 2\lambda} A_1 \left[ 2 + \frac{1}{2} |r_1|^2 \left[ bA_1 + \frac{A_2}{A_1} - 2 \left(\frac{3}{4}\right)^n \frac{1 + 2\lambda}{(1 + \lambda)^2} \mu bA_1 - 1 \right] \right] \\ &\leq \frac{1}{2} \frac{b}{3^n} \frac{1}{1 + 2\lambda} A_1 \left[ bA_1 + \frac{A_2}{A_1} - 2 \left(\frac{3}{4}\right)^n \frac{1 + 2\lambda}{(1 + \lambda)^2} \mu bA_1 \right] \leq \frac{1}{2} \frac{b^2}{3^n} \frac{1}{1 + 2\lambda} A_1^2. \end{aligned}$$

Case 3. If  $\left(bA_1 + \frac{A_2}{A_1} - 1\right) \frac{4^n (1 + \lambda)^2}{2A_1 b 3^n (1 + 2\lambda)} \leq \mu \leq \left(bA_1 + \frac{A_2}{A_1}\right) \frac{4^n (1 + \lambda)^2}{2A_1 b 3^n (1 + 2\lambda)}$ ,

we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{4} \cdot \frac{b}{3^n} \cdot \frac{1}{1 + 2\lambda} A_1 \left[ 2 + \frac{1}{2} |r_1|^2 \left[ bA_1 + \frac{A_2}{A_1} - 2 \left(\frac{3}{4}\right)^n \frac{1 + 2\lambda}{(1 + \lambda)^2} \mu bA_1 - 1 \right] \right] \\ &\leq \frac{1}{2} \frac{b}{3^n} \frac{1}{1 + 2\lambda} A_1. \end{aligned}$$

Case 4. If  $\left(bA_1 + \frac{A_2}{A_1}\right) \frac{4^n (1 + \lambda)^2}{2A_1 b 3^n (1 + 2\lambda)} \leq \mu \leq \left(bA_1 + \frac{A_2}{A_1} + 1\right) \frac{4^n (1 + \lambda)^2}{2A_1 b 3^n (1 + 2\lambda)}$ ,

we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{4} \cdot \frac{b}{3^n} \cdot \frac{1}{1 + 2\lambda} A_1 \left[ 2 + \frac{1}{2} |r_1|^2 \left[ 2 \left(\frac{3}{4}\right)^n \frac{1 + 2\lambda}{(1 + \lambda)^2} \mu bA_1 - bA_1 - \frac{A_2}{A_1} - 1 \right] \right] \\ &\leq \frac{1}{2} \frac{b}{3^n} \frac{1}{1 + 2\lambda} A_1. \end{aligned}$$

Case 5. If  $\mu \geq \left(bA_1 + \frac{A_2}{A_1} + 1\right) \frac{4^n (1 + \lambda)^2}{2A_1 b 3^n (1 + 2\lambda)}$ , we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{4} \cdot \frac{b}{3^n} \cdot \frac{1}{1 + 2\lambda} A_1 \left[ 2 + \frac{1}{2} |r_1|^2 \left[ 2 \left(\frac{3}{4}\right)^n \frac{1 + 2\lambda}{(1 + \lambda)^2} \mu bA_1 - bA_1 - \frac{A_2}{A_1} - 1 \right] \right] \\ &\leq \frac{1}{2} \frac{b}{3^n} \frac{1}{1 + 2\lambda} A_1 \left[ 2 \left(\frac{3}{4}\right)^n \frac{1 + 2\lambda}{(1 + \lambda)^2} \mu bA_1 - bA_1 - \frac{A_2}{A_1} \right]. \end{aligned}$$

Finally, if  $bA_1 < 1$ , the similar discussions can readily yield the desired results.

If  $bA_1 \geq 1$ , equality is attained for the second and third case on choosing  $c_1 = 0, c_2 = 2$  in (2.11). Also, equality is attained for the first and fourth case on choosing  $c_1 = 2, c_2 = 2$  and  $c_1 = 2i, c_2 = -2$  in (2.10), respectively.

If  $bA_1 \leq 1$ , equality is attained for the second case on choosing  $c_1 = 0, c_2 = 2$  in (2.11). Also, equality is attained for the first and third case on choosing  $c_1 = 2, c_2 = 2$  and  $c_1 = 2i, c_2 = -2$  in (2.10), respectively. Thus the proof is completed.  $\square$

Here, we discuss the Fekete-Szegő problem with complex  $b$  and real  $\mu$ .

**THEOREM 4.** *Let  $n \geq 0, 0 \leq \lambda \leq 1$  and Let  $b$  be a nonzero complex number. If  $f(z) \in \mathcal{P}_\phi(n, b, \lambda)$ , then for any real  $\mu$ ,*



(1) If  $|\frac{A_2}{A_1} \sin \theta| < 1$ , we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} (\Re(\mathcal{X}_1) - \mu) + \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} |A_2| |\sin \theta|, & \mu \leq \mathcal{M}_1, \\ \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1, & \mathcal{M}_1 < \mu \leq \mathcal{M}_2, \\ \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} (\mu - \Re(\mathcal{X}_1)) + \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} |A_2| |\sin \theta|, & \mu > \mathcal{M}_2. \end{cases}$$

(2) If  $|\frac{A_2}{A_1} \sin \theta| > 1$ , we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} (\Re(\mathcal{X}_1) - \mu) + \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} |A_2| |\sin \theta|, & \mu \leq \Re(\mathcal{X}_1), \\ \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} (\mu - \Re(\mathcal{X}_1)) + \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} |A_2| |\sin \theta|, & \mu > \Re(\mathcal{X}_1). \end{cases}$$

where  $\varphi(z) = 1 + A_1 z + A_2 z^2 + \dots$ ,  $b = |b|e^{-i\theta}$ ,  $\mathcal{X}_1 = \frac{4^n(1+\lambda)^2}{2 \cdot 3^n \cdot (1+2\lambda)} + \frac{4^{n+1}(1+\lambda)^2 A_2 e^{i\theta}}{8 \cdot 3^n (1+2\lambda) A_1^2 |b|}$ ,  $\mathcal{J}_1 = \frac{4^{n+1}(1+\lambda)^2}{8 \cdot 3^n \cdot (1+2\lambda) |b| A_1}$ ,  $\mathcal{M}_1 = \Re(\mathcal{X}_1) - \mathcal{J}_1(1 - \frac{|A_2}{A_1} |\sin \theta|)$ ,  $\mathcal{M}_2 = \Re(\mathcal{X}_1) + \mathcal{J}_1(1 - \frac{|A_2}{A_1} |\sin \theta|)$ . For each  $\mu$  there is a function in  $\mathcal{P}_\varphi(n, b, \lambda)$ , such that the equality holds.

*Proof.* Suppose  $f(z) = z + \sum_{k=2}^\infty a_k z^k \in \mathcal{P}_\varphi(n, b, \lambda)$ , using (2.7), then we have

$$\begin{aligned} |a_3 - \mu a_2^2| &= \frac{1}{4} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left| r_2 - \frac{1}{2} r_1^2 + \frac{1}{2} r_1^2 (bA_1 + \frac{A_2}{A_1} - 2 \left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1) \right| \\ &\leq \frac{1}{4} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ 2 + \frac{1}{2} |r_1|^2 \left[ \left| bA_1 + \frac{A_2}{A_1} - 2 \left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 \right| - 1 \right] \right] \\ &= \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{1}{8} \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ \left| bA_1 + \frac{A_2}{A_1} - 2 \left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 \right| - 1 \right] |r_1|^2 \\ &= \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{1}{8} \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ \left| 2 \left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu bA_1 - bA_1 - \frac{A_2}{A_1} \right| - 1 \right] |r_1|^2 \\ &= \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{1}{8} \frac{|b|^2}{3^n} \cdot \frac{1}{1+2\lambda} A_1 \left[ \left| 2 \left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2} \mu A_1 - A_1 - \frac{A_2}{bA_1} \right| - \frac{1}{|b|} \right] |r_1|^2 \\ &= \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} \left[ \left| \mu - \frac{4^n(1+\lambda)^2}{2 \cdot 3^n \cdot (1+2\lambda)} - \frac{4^{n+1}(1+\lambda)^2 A_2}{8 \cdot 3^n (1+2\lambda) A_1^2} \frac{1}{b} \right. \right. \\ &\quad \left. \left. - \frac{4^{n+1}(1+\lambda)^2}{8 \cdot 3^n \cdot (1+2\lambda) |b| A_1} \right] |r_1|^2. \end{aligned} \tag{2.14}$$

Taking  $b = |b|e^{-i\theta}$ ,  $\frac{4^n(1+\lambda)^2}{2 \cdot 3^n \cdot (1+2\lambda)} + \frac{4^{n+1}(1+\lambda)^2 A_2 e^{i\theta}}{8 \cdot 3^n (1+2\lambda) A_1^2 |b|} = \mathcal{X}_1$ ,  $\frac{4^{n+1}(1+\lambda)^2}{8 \cdot 3^n \cdot (1+2\lambda) |b| A_1} = \mathcal{J}_1$ , a direct calculation with (2.14) shows that

$$\begin{aligned}
 & |a_3 - \mu a_2^2| \\
 & \leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} \left[ |\mu - \mathfrak{X}_1| - \mathcal{J}_1 \right] |r_1|^2 \\
 & \leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} \left[ |\mu - \Re(\mathfrak{X}_1) - i(\text{Im}(\mathfrak{X}_1))| - \mathcal{J}_1 \right] |r_1|^2 \\
 & \leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} \left[ |\mu - \Re(\mathfrak{X}_1)| + \mathcal{J}_1 \left| \frac{A_2}{A_1} \right| |\sin \theta| - \mathcal{J}_1 \right] |r_1|^2 \\
 & = \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} \left[ |\mu - \Re(\mathfrak{X}_1)| - \mathcal{J}_1 \left( 1 - \left| \frac{A_2}{A_1} \right| |\sin \theta| \right) \right] |r_1|^2.
 \end{aligned}
 \tag{2.15}$$

Here, for later convenience as well, we set  $\Re(\mathfrak{X}_1) - \mathcal{J}_1(1 - \frac{|A_2}{A_1}|\sin \theta) = \mathcal{M}_1$ ,  $\Re(\mathfrak{X}_1) + \mathcal{J}_1(1 - \frac{|A_2}{A_1}|\sin \theta) = \mathcal{M}_2$ . Now we make some discussions for several different cases.

Firstly, if  $\frac{|A_2}{A_1}|\sin \theta \leq 1$ , we can note that  $\mathcal{M}_1 \leq \Re(\mathfrak{X}_1) \leq \mathcal{M}_2$ . Thus, it gives

(i) Let  $\mu \leq \mathcal{M}_1$ . Then from (2.15) we have

$$\begin{aligned}
 & |a_3 - \mu a_2^2| \\
 & \leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} \left[ \Re(\mathfrak{X}_1) - \mathcal{J}_1 \left( 1 - \left| \frac{A_2}{A_1} \right| |\sin \theta| \right) - \mu \right] |r_1|^2 \\
 & = \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} [\mathcal{M}_1 - \mu] |r_1|^2 \\
 & \leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} [\mathcal{M}_1 - \mu] \\
 & = \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} \left[ \Re(\mathfrak{X}_1) - \frac{4^{n+1}(1+\lambda)^2}{8 \cdot 3^n \cdot (1+2\lambda)} \left( 1 - \left| \frac{A_2}{A_1} \right| |\sin \theta| \right) - \mu \right] \\
 & = \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} (\Re(\mathfrak{X}_1) - \mu) + \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} |A_2| |\sin \theta|.
 \end{aligned}$$

(ii) Let  $\mathcal{M}_1 < \mu \leq \Re(\mathfrak{X}_1)$ , we have

$$\begin{aligned}
 |a_3 - \mu a_2^2| & \leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} \left[ \Re(\mathfrak{X}_1) - \mathcal{J}_1 \left( 1 - \left| \frac{A_2}{A_1} \right| |\sin \theta| \right) - \mu \right] |r_1|^2 \\
 & = \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} [\mathcal{M}_1 - \mu] |r_1|^2 \leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1.
 \end{aligned}$$

(iii) Let  $\Re(\mathfrak{X}_1) < \mu \leq \mathcal{M}_2$ . Then from (2.15) we have

$$|a_3 - \mu a_2^2| \leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} \left[ \mu - \Re(\mathfrak{X}_1) - \mathcal{J}_1 \left( 1 - \left| \frac{A_2}{A_1} \right| |\sin \theta| \right) \right] |r_1|^2$$

$$= \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} [\mu - \mathcal{M}_2] |r_1|^2 \leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1.$$

(iii) Let  $\mu > \mathcal{M}_2$ , we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} [\mu - \mathcal{M}_2] |r_1|^2 \\ &\leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} [\mu - \mathcal{M}_2] \\ &= \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} \left[ \mu - \Re(\mathcal{X}_1) - \frac{4^{n+1}(1+\lambda)^2}{8 \cdot 3^n \cdot (1+2\lambda) |b| A_1} \left( 1 - \left| \frac{A_2}{A_1} \right| |\sin \theta| \right) \right] \\ &= \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} (\mu - \Re(\mathcal{X}_1)) + \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} |A_2| |\sin \theta|. \end{aligned}$$

Moreover, if  $\left| \frac{A_2}{A_1} \right| |\sin \theta| > 1$ , we can note that  $\mathcal{M}_2 \leq \Re(\mathcal{X}_1) \leq \mathcal{M}_1$ . Thus, it gives

(i) Let  $\mu \leq \mathcal{M}_2$ . Then from (2.15), we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} \left[ \Re(\mathcal{X}_1) - \mathcal{I}_1 \left( 1 - \left| \frac{A_2}{A_1} \right| |\sin \theta| \right) - \mu \right] |r_1|^2 \\ &= \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} [\mathcal{M}_1 - \mu] |r_1|^2 \\ &\leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} \left[ \Re(\mathcal{X}_1) - \frac{4^{n+1}(1+\lambda)^2}{8 \cdot 3^n \cdot (1+2\lambda) |b| A_1} \left( 1 - \left| \frac{A_2}{A_1} \right| |\sin \theta| \right) - \mu \right] \\ &= \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} (\Re(\mathcal{X}_1) - \mu) + \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} |A_2| |\sin \theta|. \end{aligned}$$

(ii) Let  $\mathcal{M}_2 < \mu \leq \Re(\mathcal{X}_1)$ . Then

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} \left[ \Re(\mathcal{X}_1) - \mathcal{I}_1 \left( 1 - \left| \frac{A_2}{A_1} \right| |\sin \theta| \right) - \mu \right] |r_1|^2 \\ &\leq \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} (\Re(\mathcal{X}_1) - \mu) + \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} |A_2| |\sin \theta|. \end{aligned}$$

(iii) Let  $\Re(\mathcal{X}_1) < \mu \leq \mathcal{M}_1$ . Then we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} \left[ \mu - \Re(\mathcal{X}_1) - \mathcal{I}_1 \left( 1 - \left| \frac{A_2}{A_1} \right| |\sin \theta| \right) \right] |r_1|^2 \\ &= \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} [\mu - \mathcal{M}_2] |r_1|^2 \\ &\leq \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} (\mu - \Re(\mathcal{X}_1)) + \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} |A_2| |\sin \theta|. \end{aligned}$$

(iii) Let  $\mu > \mathcal{M}_1$ . Then we have

$$\begin{aligned} & |a_3 - \mu a_2^2| \\ \leq & \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} \left[ \mu - \Re(\mathcal{X}_1) - \mathcal{J}_1 \left( 1 - \left| \frac{A_2}{A_1} \right| |\sin \theta| \right) \right] |r_1|^2 \\ = & \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} A_1 + \frac{|b|^2 A_1^2}{4^{n+1}(1+\lambda)^2} [\mu - \mathcal{M}_2] |r_1|^2 \\ \leq & \frac{|b|^2 A_1^2}{4^n(1+\lambda)^2} (\mu - \Re(\mathcal{X}_1)) + \frac{1}{2} \cdot \frac{|b|}{3^n} \cdot \frac{1}{1+2\lambda} |A_2| |\sin \theta|. \end{aligned}$$

Thus the proof is completed.  $\square$

Motivated essentially by the earlier works of Abdel-Gawad [1], Koepf [11], we extend the corresponding results by investigating the class  $\mathcal{K}(\beta, n, \lambda, g(z))$  in the next Theorem.

**THEOREM 5.** *If  $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{K}(\beta, n, \lambda, g(z))$ , then for  $0 \leq \beta \leq 1$ ,  $0 \leq \lambda \leq 1$ ,  $\mu \in \mathbb{R}$ , we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \Lambda(x_0)|_{\mu=\mathcal{W}_1} + (\mathcal{W}_1 - \mu) \left[ \frac{\beta}{2^n(1+\lambda)} + 1 \right]^2, & \text{if } \mu \leq \mathcal{W}_1, \\ \Lambda(x_0), & \text{if } \mathcal{W}_1 \leq \mu \leq \mathcal{W}_2 \\ \frac{(\beta+1)(\mu-\mathcal{W}_2)}{\mathcal{W}_2} \Theta(1,1) + \frac{\beta+1}{\mathcal{W}_2} (\mathcal{W}_2 \frac{\beta+2}{\beta+1} - \mu) \cdot \Lambda(x_0)|_{\mu=\mathcal{W}_2}, & \text{if } \mathcal{W}_2 \leq \mu \leq \mathcal{W}_2 \frac{\beta+2}{\beta+1}, \\ \Theta(1,1) + \left( \mu - \mathcal{W}_2 \frac{\beta+2}{\beta+1} \right) \left[ \frac{\beta}{2^n(1+\lambda)} + 1 \right]^2, & \text{if } \mu \geq \mathcal{W}_2 \frac{\beta+2}{\beta+1}. \end{cases}$$

where

$$\mathcal{W}_1 = \frac{8^{n+1}(1+\lambda)^3 - 2 \cdot 4^{n+1}(1+\lambda)^2(1-\beta)}{4\beta \cdot 3^{n+1}(1+2\lambda) + 2 \cdot 6^{n+1}(1+\lambda)(1+2\lambda)}, \quad \mathcal{W}_2 = \frac{1}{2} \left( \frac{4}{3} \right)^{n+1} \cdot \frac{(1+\lambda)^2}{1+2\lambda},$$

$$\begin{aligned} \Lambda(x_0) = & 1 - \mu + \frac{\beta}{3^{n+1}(1+2\lambda)} \left( 2 - \frac{1}{2} x_0^2 \right) + \frac{\beta^2 [(1+\lambda)^2 4^{n+1} - 2 \cdot 3^{n+1}(1+2\lambda)\mu]}{2 \cdot 12^{n+1}(1+\lambda)^2(1+2\lambda)} x_0^2 \\ & + \beta \frac{4^{n+1}(1+\lambda)^2 - 2 \cdot 3^{n+1}(1+2\lambda)\mu}{6^{n+1}(1+\lambda)(1+2\lambda)} x_0, \end{aligned}$$

$$\begin{aligned} \Theta(1,1) = & \frac{2^{n+3}\beta(1+\lambda) - 4\beta - 8 \cdot 3^n(1+2\lambda)(\beta+1) + 4^{n+1}(1+\lambda)^2(\beta+2)}{2 \cdot 3^{n+1}(1+2\lambda)(\beta+1)} \\ & + \frac{1}{3} + \frac{2\beta}{3^{n+1}(1+2\lambda)}. \end{aligned}$$

$$x_0 = \frac{2^{n+1}(1+\lambda)[4^{n+1}(1+\lambda)^2 - 2 \cdot 3^{n+1}(1+2\lambda)\mu]}{4^{n+1}(1+\lambda)^2(1-\beta) + 2\beta \cdot 3^{n+1}(1+2\lambda)\mu}.$$

For each  $\mu$  there are functions in  $\mathcal{K}(\beta, n, \lambda, g(z))$  such that equality holds in all cases.

*Proof.* If  $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{K}(\beta, n, \lambda, g(z))$ , then there are analytic functions  $g(z)$  and  $h(z)$ , such that

$$\frac{\lambda D^{n+2}f(z) + (1-\lambda)D^{n+1}f(z)}{\lambda D^{n+1}g(z) + (1-\lambda)D^n g(z)} = (h(z))^\beta, \tag{2.16}$$

where  $g(z) = z + b_2 z^2 + b_3 z^3 + \dots \in S^*$ ,  $h(z) = 1 + c_1 z + c_2 z^2 + \dots \in \mathcal{P}$ . Equating coefficients of the power series in the relation with (2.16) we find that

$$2^{n+1}(1+\lambda)a_2 = c_1\beta + 2^n(1+\lambda)b_2, \tag{2.17}$$

and

$$3^{n+1}(2\lambda+1)a_3 = \frac{\beta(\beta-1)}{2}c_1^2 + 2^n(1+\lambda)b_2c_1\beta + \beta c_2 + 3^n(2\lambda+1)b_3. \tag{2.18}$$

From (2.17) and (2.18), it follows that

$$a_3 - \mu a_2^2 = \frac{1}{3}(b_3 - \frac{3}{4}\mu b_2^2) + \beta \left( \frac{2^n(1+\lambda)}{3^{n+1}(1+2\lambda)} - \frac{\mu}{2^{n+1}(1+\lambda)} \right) b_2 c_1 + \frac{\beta}{3^{n+1}(1+2\lambda)} \left\{ c_2 + \left[ \frac{\beta[(1+\lambda)^2 4^{n+1} - 2 \cdot 3^{n+1}(1+2\lambda)\mu]}{2(1+\lambda)^2 \cdot 4^{n+1}} - \frac{1}{2} \right] c_1^2 \right\}. \tag{2.19}$$

Suppose

$$\mathcal{W}_1 = \frac{8^{n+1}(1+\lambda)^3 - 2 \cdot 4^{n+1}(1+\lambda)^2(1-\beta)}{4\beta \cdot 3^{n+1}(1+2\lambda) + 2 \cdot 6^{n+1}(1+\lambda)(1+2\lambda)}, \quad \mathcal{W}_2 = \frac{1}{2} \left( \frac{4}{3} \right)^{n+1} \cdot \frac{(1+\lambda)^2}{1+2\lambda},$$

where  $0 \leq \beta \leq 1$ ,  $0 \leq \lambda \leq 1$ . We next consider the different cases for  $\mu$ .

Firstly, let  $\mathcal{W}_1 \leq \mu \leq \mathcal{W}_2$ , with the aid of Lemma 1 and Lemma 2, we obtain

$$\begin{aligned} & |a_3 - \mu a_2^2| \\ & \leq \frac{1}{3} \left| b_3 - \frac{3}{4}\mu b_2^2 \right| + \frac{\beta}{3^{n+1}(1+2\lambda)} \left| c_2 - \frac{1}{2}c_1^2 \right| \\ & \quad + \frac{\beta^2[(1+\lambda)^2 4^{n+1} - 2 \cdot 3^{n+1}(1+2\lambda)\mu]}{2 \cdot 12^{n+1}(1+\lambda)^2(1+2\lambda)} |c_1^2| + \beta \frac{4^{n+1}(1+\lambda)^2 - 2 \cdot 3^{n+1}(1+2\lambda)\mu}{6^{n+1}(1+\lambda)(1+2\lambda)} |c_1| \\ & \leq 1 - \mu + \frac{\beta}{3^{n+1}(1+2\lambda)} \left( 2 - \frac{1}{2}|c_1|^2 \right) + \frac{\beta^2[(1+\lambda)^2 4^{n+1} - 2 \cdot 3^{n+1}(1+2\lambda)\mu]}{2 \cdot 12^{n+1}(1+\lambda)^2(1+2\lambda)} |c_1|^2 \\ & \quad + \beta \frac{4^{n+1}(1+\lambda)^2 - 2 \cdot 3^{n+1}(1+2\lambda)\mu}{6^{n+1}(1+\lambda)(1+2\lambda)} |c_1| = \Lambda(x) \text{ say, with } x = |c_1|. \end{aligned} \tag{2.20}$$

Now, we take  $\Lambda'(x) = 0$ , it gives the stable point

$$x_0 = \frac{2^{n+1}(1 + \lambda)[4^{n+1}(1 + \lambda)^2 - 2 \cdot 3^{n+1}(1 + 2\lambda)\mu]}{4^{n+1}(1 + \lambda)^2(1 - \beta) + 2\beta \cdot 3^{n+1}(1 + 2\lambda)\mu},$$

moreover,

$$\Lambda''(x) = \frac{\beta^2[(1 + \lambda)^2 \cdot 4^{n+1} - 2 \cdot 3^{n+1}(1 + 2\lambda)\mu] - 4^{n+1}(1 + \lambda)^2\beta}{12^{n+1}(1 + 2\lambda)(1 + \lambda)^2} < 0,$$

it implies that  $\max\{\Lambda(x) : x = |c_1|\} = \Lambda(x_0)$ . So (2.20) gives the desired estimate on  $|a_3 - \mu a_2^2|$ .

In fact, since  $x = |c_1| \leq 2$ , it follows that  $\mu \geq \mathscr{W}_1$ . Furthermore, equality is attained for this case by choosing  $c_1 = x_0, c_2 = 2, b_2 = 2, b_3 = 3$  in (2.19).

Let, now  $\mu \leq \mathscr{W}_1$ , then

$$|a_3 - \mu a_2^2| \leq |a_3 - \mathscr{W}_1 a_2^2| + (\mathscr{W}_1 - \mu)|a_2|^2. \tag{2.21}$$

Using the result already proved in first case with  $\mu = \mathscr{W}_1$ , we have  $|a_3 - \mathscr{W}_1 a_2^2| \leq \Lambda(x_0)|_{\mu=\mathscr{W}_1}$ . Also, applying (2.17) we get  $|a_2| \leq \frac{\beta}{2^n(1+\lambda)} + 1$ . Thus (2.21) shows that

$$|a_3 - \mu a_2^2| \leq \Lambda(x_0)|_{\mu=\mathscr{W}_1} + (\mathscr{W}_1 - \mu) \left[ \frac{\beta}{2^n(1 + \lambda)} + 1 \right]^2. \tag{2.22}$$

The equality for (2.22) is attained when  $c_1 = x_0|_{\mu=\mathscr{W}_1}, c_2 = 2, b_2 = 2, b_3 = 3$  in (2.19).

Let, now  $\mathscr{W}_2 \leq \mu \leq \mathscr{W}_2 \frac{\beta+2}{\beta+1}$ . Then a computation shows that

$$a_3 - \mu a_2^2 = \left( \frac{\beta + 1}{\mathscr{W}_2} \mu - \beta - 1 \right) \left( a_3 - \mathscr{W}_2 \frac{\beta + 2}{\beta + 1} a_2^2 \right) + \frac{\beta + 1}{\mathscr{W}_2} \left( \mathscr{W}_2 \frac{\beta + 2}{\beta + 1} - \mu \right) \left( a_3 - \mathscr{W}_2 a_2^2 \right).$$

It yields

$$|a_3 - \mu a_2^2| = \left( \frac{\beta + 1}{\mathscr{W}_2} \mu - \beta - 1 \right) \left| a_3 - \mathscr{W}_2 \frac{\beta + 2}{\beta + 1} a_2^2 \right| + \frac{\beta + 1}{\mathscr{W}_2} \left( \mathscr{W}_2 \frac{\beta + 2}{\beta + 1} - \mu \right) |a_3 - \mathscr{W}_2 a_2^2|. \tag{2.23}$$

We deal first with the case  $\mu = \mathscr{W}_2 \frac{\beta+2}{\beta+1}$ . Since  $g \in S^*$ , so there is a function  $p(z) = 1 + p_1z + p_2z^2 + \dots \in \mathscr{P}$ , satisfying  $zg'(z) = g(z)p(z)$ , where  $b_2 = p_1, 2b_3 = p_1^2 + p_2$ . Thus we have

$$\begin{aligned} & a_3 - \mathscr{W}_2 \frac{\beta + 2}{\beta + 1} a_2^2 \\ &= \frac{1}{6} \left( p_2 - \frac{1}{2} p_1^2 \right) + \left[ \frac{1}{4} - \frac{1}{6} \left( \frac{4}{3} \right)^n \frac{(1 + \lambda)^2}{1 + 2\lambda} \frac{\beta + 2}{\beta + 1} \right] p_1^2 + \frac{\beta}{3^{n+1}(1 + 2\lambda)} \left( c_2 - \frac{1}{2} c_1^2 \right) \\ & \quad + \frac{-\beta^2}{2 \cdot 3^{n+1}(\beta + 1)(1 + 2\lambda)} c_1^2 + \beta \left[ \frac{2^n(1 + \lambda)}{3^{n+1}(1 + 2\lambda)} - \frac{1}{2} \left( \frac{2}{3} \right)^{n+1} \frac{1 + \lambda}{1 + 2\lambda} \frac{\beta + 2}{\beta + 1} \right] p_1 c_1, \end{aligned} \tag{2.24}$$

moreover,

$$\begin{aligned} & \left| a_3 - \mathscr{W}_2 \frac{\beta+2}{\beta+1} a_2^2 \right| \\ & \leq \frac{1}{6} \left| p_2 - \frac{1}{2} p_1^2 \right| + \left[ \frac{1}{6} \left( \frac{4}{3} \right)^n \frac{(1+\lambda)^2}{1+2\lambda} \frac{\beta+2}{\beta+1} - \frac{1}{4} \right] |p_1^2| + \frac{\beta}{3^{n+1}(1+2\lambda)} \left| c_2 - \frac{1}{2} c_1^2 \right| \\ & \quad + \frac{\beta^2}{2.3^{n+1}(\beta+1)(1+2\lambda)} |c_1^2| + \beta \left[ \frac{1}{2} \left( \frac{2}{3} \right)^{n+1} \frac{1+\lambda}{1+2\lambda} \frac{\beta+2}{\beta+1} - \frac{2^n(1+\lambda)}{3^{n+1}(1+2\lambda)} \right] |p_1 c_1| \\ & \leq \frac{1}{6} \left( 2 - \frac{1}{2} |p_1|^2 \right) + \left[ \frac{1}{6} \left( \frac{4}{3} \right)^n \frac{(1+\lambda)^2}{1+2\lambda} \frac{\beta+2}{\beta+1} - \frac{1}{4} \right] |p_1|^2 + \frac{\beta}{3^{n+1}(1+2\lambda)} \left( 2 - \frac{1}{2} |c_1|^2 \right) \\ & \quad + \frac{\beta^2}{2.3^{n+1}(\beta+1)(1+2\lambda)} |c_1|^2 + \beta \frac{2^{n+1}(1+\lambda)}{2.3^{n+1}(1+2\lambda)(\beta+1)} |p_1 c_1|, \\ & = \frac{1}{3} + \frac{2\beta}{3^{n+1}(1+2\lambda)} - \frac{2.3^n(1+2\lambda)(\beta+1) - 4^n(1+\lambda)^2(\beta+2)}{6.3^n(1+2\lambda)(\beta+1)} |p_1|^2 \\ & \quad - \frac{\beta}{2.3^{n+1}(\beta+1)(1+2\lambda)} |c_1|^2 + \beta \frac{2^{n+1}(1+\lambda)}{2.3^{n+1}(1+2\lambda)(\beta+1)} |p_1 c_1|. \end{aligned}$$

Letting  $p_1 = 2re^{i\theta}$ ,  $c_1 = 2Re^{i\varphi}$ , where  $0 \leq \theta \leq 2\pi$ ,  $0 \leq \varphi \leq 2\pi$ ,  $0 \leq r \leq 1$ , and  $0 \leq R \leq 1$ . Following the above inequality, we have

$$\begin{aligned} & \left| a_3 - \mathscr{W}_2 \frac{\beta+2}{\beta+1} a_2^2 \right| \leq \frac{1}{3} + \frac{2\beta}{3^{n+1}(1+2\lambda)} - 4r^2 \frac{2.3^n(1+2\lambda)(\beta+1) - 4^n(1+\lambda)^2(\beta+2)}{6.3^n(1+2\lambda)(\beta+1)} \\ & \quad - 4R^2 \frac{\beta}{2.3^{n+1}(\beta+1)(1+2\lambda)} + 4Rr\beta \frac{2^{n+1}(1+\lambda)}{2.3^{n+1}(1+2\lambda)(\beta+1)} = \Theta(R, r). \end{aligned}$$

Letting  $n$ ,  $\beta$  and  $\lambda$  fixed and differentiating  $\Theta(R, r)$  partially with  $n \geq 0$ ,  $0 \leq \beta \leq 1$ , and  $0 \leq \lambda \leq 1$ , we have

$$\Theta_{RR} \cdot \Theta_{rr} - \Theta_{Rr}^2 = \frac{128\beta \cdot 3^n(1+2\lambda)(\beta+1) - 128\beta \cdot 4^n(1+\lambda)^2(\beta+1)}{12.3^{2n+1}(1+2\lambda)^2(1+\beta)^2} < 0.$$

Therefore, the maximum of  $\Theta(R, r)$  occurs on the boundaries, which yields

$$\begin{aligned} & \left| a_3 - \mathscr{W}_2 \frac{\beta+2}{\beta+1} a_2^2 \right| \leq \Theta(R, r) \leq \Theta(1, 1) \\ & = \frac{1}{3} + \frac{2\beta}{3^{n+1}(1+2\lambda)} + \frac{2^{n+3}\beta(1+\lambda) - 4\beta - 8.3^n(1+2\lambda)(\beta+1) + 4^{n+1}(1+\lambda)^2(\beta+2)}{2.3^{n+1}(1+2\lambda)(\beta+1)}. \end{aligned} \tag{2.25}$$

Now, applying the first case with  $\mu = \mathscr{W}_2$ , we get  $|a_3 - \mathscr{W}_2 a_2^2| \leq \Lambda(x_0)|_{\mu=\mathscr{W}_2}$ . It follows from the (2.23) and (2.25) that

$$|a_3 - \mu a_2^2| \leq \left( \frac{\beta+1}{\mathscr{W}_2} \mu - \beta - 1 \right) \Theta(1, 1) + \frac{\beta+1}{\mathscr{W}_2} \left( \mathscr{W}_2 \frac{\beta+2}{\beta+1} - \mu \right) \cdot \Lambda(x_0)|_{\mu=\mathscr{W}_2}.$$

Equality is attained on choosing  $c_1 = b_2 = 2i$ ,  $c_2 = -2$ ,  $b_3 = -3$  in (2.19).

Finally, if  $\mu \geq \mathscr{W}_2 \frac{\beta+2}{\beta+1}$ , then with the aid of the result already proved for  $\mu = \mathscr{W}_2$  and  $a_2 \leq \frac{\beta}{2^n(1+\lambda)} + 1$ , we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \left| a_3 - \mathscr{W}_2 \frac{\beta+2}{\beta+1} a_2^2 \right| + \left| \mathscr{W}_2 \frac{\beta+2}{\beta+1} - \mu \right| |a_2|^2 \\ &\leq \Theta(1, 1) + \left( \mu - \mathscr{W}_2 \frac{\beta+2}{\beta+1} \right) \left[ \frac{\beta}{2^n(1+\lambda)} + 1 \right]^2. \end{aligned}$$

Equality is attained on choosing  $c_1 = b_2 = 2i$ ,  $c_2 = -2$ ,  $b_3 = -3$  in (2.19). Thus the proof is completed.  $\square$

REMARK.

(1) Setting  $n = 0$ ,  $b = 1$ ,  $\lambda = 0$  in Theorem 1, Theorem 2, Theorem 3 and Theorem 4, respectively, we obtain the corresponding results on the classes  $S^*(\varphi)$  defined by Ma-Minda [12].

(2) Setting  $n = 0$ ,  $b = 1$ ,  $\lambda = 1$  in Theorem 1, Theorem 2, Theorem 3 and Theorem 4, respectively, we obtain the corresponding results on the classes  $C(\varphi)$  defined by Ma-Minda [12].

(3) Setting  $n = 0$ ,  $b = 1$ ,  $\lambda = 0$ ,  $\varphi(z) = \frac{1+Az}{1+Bz}$  ( $-1 \leq A < B \leq 1$ ) in Theorem 1, Theorem 2, Theorem 3 and Theorem 4, respectively, we obtain the corresponding results on the classes  $S^*[A, B]$  defined by Janowski [7].

(4) Setting  $n = 0$ ,  $b = 1$ ,  $\lambda = 1$ ,  $\varphi(z) = \frac{1+Az}{1+Bz}$  ( $-1 \leq A < B \leq 1$ ) in Theorem 1, Theorem 2, Theorem 3 and Theorem 4, respectively, we obtain the corresponding results on the classes  $C[A, B]$  defined by Janowski [7].

(5) Setting  $n = 0$ ,  $b = 1$ ,  $\lambda = 0$  in Theorem 5, we obtain the results proved by Abdel-Gawad [1].

(6) Setting  $n = 0$ ,  $b = 1$ ,  $\lambda = 0$ ,  $\beta = 1$  in Theorem 5, we obtain the results proved by Keogh [9].

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