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ON THE RANGE OF THE PARAMETERS FOR THE GRAND FURUTA INEQUALITY TO BE VALID II

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Abstract. We will investigate the range of the parameters which make the grand Furuta inequality valid.

1. Introduction

A bounded linear operator T on a Hilbert space is said to be positive semidefinite (denoted by $0 \leqslant T$) if $0 \leqslant (Th,h)$ for all vectors h. We write $0 < T$ if T is positive semidefinite and invertible.

Furuta obtained an epochmaking extension of the Löwner-Heinz inequality.

THEOREM 1.1. [2] *Let* $0 \leqslant p$, $1 \leqslant q$ *and* $0 \leqslant r$ *with* $p+r \leqslant (1+r)q$. *If* $0 \leqslant B \leqslant A$ *holds, then*

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \leqslant A^{\frac{p+r}{q}}.$$

It is well-known that Theorem 1.1 is equivalent to the next theorem, which is often called the essential case of the Furuta inequality.

THEOREM 1.2. *Let* $1 \leqslant p$ *and* $0 \leqslant r$. *If* $0 \leqslant B \leqslant A$ *holds, then*

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} \leqslant A^{1+r}.$$

The following result by Tanahashi is a full description of the best possibility of the range

$$p+r \leqslant (1+r)q \quad \text{and} \quad 1 \leqslant q$$

as far as all parameters are positive. We would like to emphasize that the theorem can be divided into 2 cases.

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THEOREM 1.3. [7] *Let* p,q,r *be positive real numbers. If* $(1+r)q<p+r$ *or* $0<q<1$*, then there exist* 2×2 *matrices* A,B *with* $0<B\leqslant A$ *that do not satisfy the inequality*

$$\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}\leqslant A^{\frac{p+r}{q}}.$$

The next proposition is corresponding to the case $(1+r)q<p+r$ of the previous theorem by putting $q=\dfrac{p+r}{(1+r)\alpha}$.

PROPOSITION 1.4. *Let* $0<p,\ 0\leqslant r$*. If* $1<\alpha$*, then there exist* 2×2 *matrices* A,B *with* $0<B\leqslant A$ *that do not satisfy the inequality*

$$\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}\alpha}\leqslant A^{(1+r)\alpha}.$$

On the other hand, the following "α-free" proposition is corresponding to the case $0<q<1$ of Theorem 1.3 by putting $q=\dfrac{p+r}{1+r}$.

PROPOSITION 1.5. *Let* $0<p<1$ *and* $0<r$*. Then there exist* 2×2 *matrices* A,B *with* $0<B\leqslant A$ *that do not satisfy the inequality*

$$\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}}\leqslant A^{1+r}.$$

Since the condition $q=\dfrac{p+r}{1+r}$ is the essential case for the Furuta inequality, our interest about Proposition 1.5 is not at all inferior to Proposition 1.4.

By the way, Furuta gave a unifying extension of both Theorem 1.1 and the Ando-Hiai inequality [1], which is often called the grand Furuta inequality.

THEOREM 1.6. [3] *Let* $1\leqslant p,\ 1\leqslant s,\ 0\leqslant t\leqslant 1$ *and* $t\leqslant r$*. If* $0\leqslant B\leqslant A$ *with* $0<A$*, then the following inequality holds:*

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}}\right)^sA^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}}\leqslant A^{1-t+r}. \tag{1}$$

Again, Tanahashi showed that the outside powers in this theorem are best possible.

THEOREM 1.7. [8] *Let* $1\leqslant p,\ 1\leqslant s,\ 0\leqslant t\leqslant 1$ *and* $t\leqslant r$*. If* $1<\alpha$*, then there exist* 2×2 *matrices* A,B *with* $0<B\leqslant A$ *that do not satisfy the inequality*

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}}\right)^sA^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}\alpha}\leqslant A^{(1-t+r)\alpha}.$$

REMARK 1.8. In [8], Theorem 1.7 is originally stated as follows:

Let p,r,s,t be real numbers satisfying $1<s$, $0<t<1$, $t\leqslant r$, $1\leqslant p$. If

$$\frac{1-t+r}{(p-t)s+r}<\alpha,$$

then there exist invertible matrices A, B with $0 \leqslant B \leqslant A$ which do not satisfy

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}}\right)^{s}A^{\frac{r}{2}}\right\}^{\alpha} \leqslant A^{\{(p-t)s+r\}\alpha}.$$

These are just rephrasing, although their α differs each other. Theorem 1.7 can be naturally considered as an extension of Proposition 1.4. Indeed, if we put $s=1$, $t=0$ in Theorem 1.7, then we obtain Proposition 1.4 restricted as $1 \leqslant p$. On the other hand, being different from Theorem 1.3, even if all parameters are positive, Theorem 1.7 does not show that the range

$$1 \leqslant p,\ 1 \leqslant s,\ 0 \leqslant t \leqslant 1,\ t \leqslant r$$

can not be expanded anymore for the grand Furuta inequality to be valid. Thus the clarification of best possibility of the grand Furuta inequality is less satisfactory than that of the Furuta inequality. So our problem is to determine the range:

$$\left\{(p,s,t,r) \in \mathbb{R}_{+}^{4};\ \text{the inequality (1) holds whenever } 0 < B \leqslant A\right\}.$$

Although it will be a nice theorem if one could show the full solution only at once, it seems difficult to the author. Therefore, we should pile up several main cases of the problem.

It is quite natural to expect "α-free" version which can be regarded as corresponding to Proposition 1.5. The following result obtained by Koizumi and the author is such an attempt.

THEOREM 1.9. [5] *Let $0 < p,\ 0 < s,\ 0 < t \leqslant 1$ and $t \leqslant r$. Suppose that*

$$t < p \quad \textit{and} \quad \frac{1-t+r}{(p-t)s+r} \cdot sp < 1.$$

Then there exist 2×2 matrices A, B with $0 < B \leqslant A$ that do not satisfy the inequality (1).

REMARK 1.10. The quantity $\dfrac{1-t+r}{(p-t)s+r} \cdot sp$ in the above assumption has an essential meaning. It also appears in a certain functional inequality (cf. [9]).

(a) If $1 \leqslant p,\ 1 \leqslant s,\ 0 \leqslant t \leqslant 1$ and $t \leqslant r$, then $\dfrac{1-t+r}{(p-t)s+r} \leqslant 1 \leqslant \dfrac{1-t+r}{(p-t)s+r} \cdot sp$.

(b) If $0 < p,\ 0 < s < 1,\ sp < 1$ and $0 < t \leqslant r$, then $\dfrac{1-t+r}{(p-t)s+r} \cdot sp < 1$.

REMARK 1.11. The case (ii) of [5, Theorem 2.1] by Koizumi and the author treats the case $0 < p = t < 1$, $0 < s$, $t < r$. However, we have $A_1 < A_2$ by the notations in [5] and the proof for (i) is not applicable to (ii). It seems still open.

The author obtained the following theorem, which can be regarded as corresponding to Proposition 1.5. The advantage is that the assumption on parameters other than $0 < s < 1$ are very mild. It is not required to assume $sp < 1$ as in the condition (b) of Remark 1.10.

THEOREM 1.12. [10] *Let* $1 < p$, $0 < s < 1$, $0 < t < 1 + r$ *and* $ts < r$. *Then there exist* 2×2 *matrices* A, B *with* $0 < B \leqslant A$ *that do not satisfy the inequality (1).*

In [10], we used matrices

$$\begin{pmatrix} x^2+1 & x \\ x & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

The purpose of this article is to show another theorem which has the same conclusion as Theorem 1.12, for parameters of a different condition, by using other matrices.

2. Preliminaries

In this section, we will explain the outline of Tanahashi's argument in [7] and [8] without proofs.

Let A, B be 2×2 matrices with $0 < B \leqslant A$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$, and let U be a unitary which diagonalizes A as $U^*AU = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$. Assume A and B satisfy the grand Furuta inequality (1). Put $\alpha = 1 - t + r$ and $\psi = (p-t)s + r$. Then

$$\left\{U^*A^{\frac{r}{2}}U\left(U^*A^{-\frac{t}{2}}UU^*B^pUU^*A^{-\frac{t}{2}}U\right)^s U^*A^{\frac{r}{2}}U\right\}^{\frac{\alpha}{\psi}} \leqslant U^*A^{\alpha}U,$$

hence we have

$$\left\{\begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix}\left[\begin{pmatrix} d_1^{-\frac{t}{2}} & 0 \\ 0 & d_2^{-\frac{t}{2}} \end{pmatrix} U^* \begin{pmatrix} 1 & 0 \\ 0 & b^p \end{pmatrix} U \begin{pmatrix} d_1^{-\frac{t}{2}} & 0 \\ 0 & d_2^{-\frac{t}{2}} \end{pmatrix}\right]^s \begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix}\right\}^{\frac{\alpha}{\psi}}$$
$$\leqslant \begin{pmatrix} d_1^{\alpha} & 0 \\ 0 & d_2^{\alpha} \end{pmatrix}. \tag{2}$$

Denote

$$\begin{pmatrix} d_1^{-\frac{t}{2}} & 0 \\ 0 & d_2^{-\frac{t}{2}} \end{pmatrix} U^* \begin{pmatrix} 1 & 0 \\ 0 & b^p \end{pmatrix} U \begin{pmatrix} d_1^{-\frac{t}{2}} & 0 \\ 0 & d_2^{-\frac{t}{2}} \end{pmatrix} = k \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix},$$

where k is a positive scalar to be specified later.

LEMMA 2.1. *Suppose that* $A_1 < A_2$ *and* $A_3 < 0$. *Let*

$$V = \frac{1}{\sqrt{-A_1 + A_2 + 2\varepsilon_1}} \begin{pmatrix} \sqrt{\varepsilon_1} & -\sqrt{-A_1 + A_2 + \varepsilon_1} \\ -\sqrt{-A_1 + A_2 + \varepsilon_1} & -\sqrt{\varepsilon_1} \end{pmatrix}$$

where

$$2\varepsilon_1 = A_1 - A_2 + \sqrt{(A_1 - A_2)^2 + 4A_3^2}.$$

Then $A_3 = -\sqrt{(-A_1+A_2+\varepsilon_1)\varepsilon_1}$, V *is unitary and*

$$V^*\begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix} V = \begin{pmatrix} A_2+\varepsilon_1 & 0 \\ 0 & A_1-\varepsilon_1 \end{pmatrix}.$$

The formula (2) implies

$$\left\{\begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix} k^s V \begin{pmatrix} (A_2+\varepsilon_1)^s & 0 \\ 0 & (A_1-\varepsilon_1)^s \end{pmatrix} V^* \begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix}\right\}^{\frac{\alpha}{\psi}} \leqslant \begin{pmatrix} d_1^{\alpha} & 0 \\ 0 & d_2^{\alpha} \end{pmatrix}.$$

Write the left-hand matrix as

$$k^{s\frac{\alpha}{\psi}}(-A_1+A_2+2\varepsilon_1)^{-\frac{\alpha}{\psi}}\begin{pmatrix} B_1 & B_3 \\ B_3 & B_2 \end{pmatrix}^{\frac{\alpha}{\psi}},$$

where

$$B_1 = d_1^r\{\varepsilon_1(A_2+\varepsilon_1)^s + (-A_1+A_2+\varepsilon_1)(A_1-\varepsilon_1)^s\}$$
$$B_2 = d_2^r\{(-A_1+A_2+\varepsilon_1)(A_2+\varepsilon_1)^s + \varepsilon_1(A_1-\varepsilon_1)^s\}$$
$$B_3 = -d_1^{\frac{r}{2}}d_2^{\frac{r}{2}}\sqrt{(-A_1+A_2+\varepsilon_1)\varepsilon_1}\{(A_2+\varepsilon_1)^s-(A_1-\varepsilon_1)^s\}.$$

LEMMA 2.2. *Keep the situation as above. Assume that* $B_2 < B_1$. *Then the following inequality holds:*

$$\varepsilon_2\left\{\gamma d_1^{\alpha}-(B_2-\varepsilon_2)^{\frac{\alpha}{\psi}}\right\}\left\{(B_1+\varepsilon_2)^{\frac{\alpha}{\psi}}-\gamma d_2^{\alpha}\right\}$$
$$\leqslant (B_1-B_2+\varepsilon_2)\left\{\gamma d_1^{\alpha}-(B_1+\varepsilon_2)^{\frac{\alpha}{\psi}}\right\}\left\{\gamma d_2^{\alpha}-(B_2-\varepsilon_2)^{\frac{\alpha}{\psi}}\right\}, \tag{3}$$

where

$$\varepsilon_2 = \frac{1}{2}\left(-B_1+B_2+\sqrt{(B_1-B_2)^2+4B_3^2}\right),$$
$$\gamma = \left\{k^{-s}(-A_1+A_2+2\varepsilon_1)\right\}^{\frac{\alpha}{\psi}}.$$

3. Results

The method of our proof of the following theorem is the same as Tanahashi's argument, which is also used in [5] and [10] by the author. However, the details are different, and details are important for this kind of problems under consideration. Needless to say, different parameters may bring different conclusions (if we change $0 < p < 1$, $1 \leqslant t < 1+r$ to $1 \leqslant p$, $0 \leqslant t \leqslant 1$, $t \leqslant r$ and try to trace the argument, then any contradiction does not arise). We would like to emphasize there are several branching points such that the conditions about parameters in the assumption are to be reflected to powers or coefficients in calculations.

THEOREM 3.1. *Let $0 < p < 1$, $1 < s$, $1 \leqslant t < 1+r$ and $0 < (p-t)s+r$. Then there exist 2×2 matrices A, B with $0 < B \leqslant A$ that do not satisfy the inequality*

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}}\right)^{s}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}} \leqslant A^{1-t+r}.$$

Proof. We will consider matrices

$$A = \begin{pmatrix} 2 & x \\ x & 2x^2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix}.$$

Then we have $0 < B \leqslant A$. The eigenvalues of A are $x^2+1\pm\sqrt{x^4-x^2+1}$. Let

$$c = \frac{\sqrt{x^4-x^2+1}-x^2+1}{x} \qquad \text{and} \qquad U = \frac{1}{\sqrt{c^2+1}}\begin{pmatrix} c & 1 \\ 1 & -c \end{pmatrix}.$$

Then U is unitary and $U^*AU = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, where

$$d_1 = x^2+1+\sqrt{x^4-x^2+1}, \qquad d_2 = x^2+1-\sqrt{x^4-x^2+1}.$$

Assume A and B satisfy the grand Furuta inequality (1). Then we have

$$A_1 = d_1^{-t}(x^{2p}+c^2), \quad A_2 = d_2^{-t}(1+x^{2p}c^2), \quad A_3 = -d_1^{-\frac{t}{2}}d_2^{-\frac{t}{2}}(x^{2p}-1)c, \quad k = \frac{1}{c^2+1}$$

by the notation of the previous section.

In this article, it is sufficient to estimate only main terms. One can easily establish the following formulae:

$$\sqrt{x^4-x^2+1} = x^2 - \frac{1}{2} + o(1), \qquad c = \frac{1}{2x}(1+o(1)),$$
$$d_1 = 2x^2(1+o(1)), \qquad d_2 = \frac{3}{2}(1+o(1)),$$
$$A_1 = 2^{-t}x^{2(p-t)}(1+o(1)), \quad A_2 = \left(\frac{3}{2}\right)^{-t}(1+o(1)), \quad A_3^2 = \frac{3^{-t}}{4}x^{4p-2t-2}(1+o(1)),$$

where $f(x) = o(1)$ means that $f(x) \to 0$ $(x \to +\infty)$. □

By $0 < p < 1 \leqslant t$, we have $A_1 < A_2$ for sufficiently large x. It is elementary to see that

$$A_2 - A_1 = 3^{-t}2^t(1+o(1)), \qquad (A_2-A_1)\left(\frac{4A_3^2}{(A_1-A_2)^2}\right)^2 \sim x^{8p-4t-4} = x^{4p-2t-2}o(1)$$

and

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{2}(A_2 - A_1)\left(-1 + \sqrt{1 + \frac{4A_3^2}{(A_1 - A_2)^2}}\right) \\
&= \frac{1}{2}(A_2 - A_1)\left\{\binom{\frac{1}{2}}{1}\frac{4A_3^2}{(A_1 - A_2)^2} + \binom{\frac{1}{2}}{2}\left(\frac{4A_3^2}{(A_1 - A_2)^2}\right)^2 + \cdots\right\} \\
&= \frac{A_3^2}{A_2 - A_1} + x^{4p-2t-2}o(1) \\
&= \frac{3^{-t}}{4}x^{4p-2t-2}(1+o(1))\left\{3^{-t}2^t(1+o(1))\right\}^{-1} + x^{4p-2t-2}o(1) \\
&= 2^{-t-2}x^{4p-2t-2}(1+o(1)).
\end{aligned}$$

Obviously, the main terms of $-A_1 + A_2 + 2\varepsilon_1$, $-A_1 + A_2 + \varepsilon_1$ and $A_2 + \varepsilon_1$ (resp. $A_1 - \varepsilon_1$) are the same as A_2 (resp. A_1), so

$$\begin{aligned}
(-A_1 + A_2 + \varepsilon_1)(A_2 + \varepsilon_1)^s &= 2^{t+ts}3^{-t-ts}(1+o(1)), \\
\varepsilon_1(A_1 - \varepsilon_1)^s &= 2^{-t-2-ts}x^{4p-2t-2+2(p-t)s}(1+o(1)), \\
(-A_1 + A_2 + \varepsilon_1)(A_1 - \varepsilon_1)^s &= 2^{t-ts}3^{-t}x^{2(p-t)s}(1+o(1)), \\
\varepsilon_1(A_2 + \varepsilon_1)^s &= 2^{-t-2+ts}3^{-ts}x^{4p-2t-2}(1+o(1)).
\end{aligned}$$

Since $4p - 2t - 2 + 2(p-t)s < 0$, we have

$$\varepsilon_1(A_1 - \varepsilon_1)^s < (-A_1 + A_2 + \varepsilon_1)(A_2 + \varepsilon_1)^s$$

for sufficiently large x. Hence we can obtain

$$\begin{aligned}
B_2 &= 2^{-r+t+ts}3^{r-t-ts}(1+o(1)), \\
B_3^2 &= 2^{-2+2ts}3^{r-t-2ts}x^{2r+4p-2t-2}(1+o(1)).
\end{aligned}$$

On the other hand, the signature of $2(p-t)s - (4p-2t-2)$ is still undetermined; it is positive if $s \approx 1$, $p \approx 0$, and negative if s is large. Hence we have to divide our argument into cases in order to obtain the main term of

$$B_1 = 2^r x^{2r}\left\{2^{-t-2}x^{4p-2t-2}3^{-ts}2^{ts}(1+o(1)) + 3^{-t}2^t2^{-ts}x^{2(p-t)s}(1+o(1))\right\}.$$

3.1. $s < 1 + \dfrac{1-p}{t-p}$

Let $4p - 2t - 2 < 2(p-t)s$. In this case, we have $\varepsilon_1(A_2 + \varepsilon_1)^s < (-A_1 + A_2 + \varepsilon_1)(A_1 - \varepsilon_1)^s$ for sufficiently large x, hence

$$B_1 = 2^{r+t-ts}3^{-t}x^{2r+2(p-t)s}(1+o(1)).$$

Since $0<\psi$, we have $B_2<B_1$ for sufficiently large x, the main term of B_1-B_2 is the same as B_1, and hence

$$\begin{aligned}
&\frac{B_3^2}{(B_1-B_2)^2}\sim\frac{x^{2r+4p-2t-2}}{x^{4(p-t)s+4r}}=\frac{1}{x^{4(p-t)s+2r-4p+2t+2}},\\
&(B_1-B_2)\left(\frac{4B_3^2}{(B_1-B_2)^2}\right)^2\sim x^{2(p-t)s+2r}\cdot\left(\frac{1}{x^{4(p-t)s+2r-4p+2t+2}}\right)^2\\
&=x^{-2(p-t)s+4p-2t-2}\cdot\frac{1}{x^{4(p-t)s+2r-4p+2t+2}}=x^{-2(p-t)s+4p-2t-2}o(1),
\end{aligned}$$

where we use $\{2(p-t)s-4p+2t+2\}+2\{(p-t)s+r\}>0$. Therefore,

$$\begin{aligned}
\varepsilon_2&=\frac{1}{2}\left(-B_1+B_2+\sqrt{(B_1-B_2)^2+4B_3^2}\right)=\frac{1}{2}(B_1-B_2)\left(-1+\sqrt{1+\frac{4B_3^2}{(B_1-B_2)^2}}\right)\\
&=\frac{1}{2}(B_1-B_2)\left\{\binom{\frac{1}{2}}{1}\frac{4B_3^2}{(B_1-B_2)^2}+\binom{\frac{1}{2}}{2}\left(\frac{4B_3^2}{(B_1-B_2)^2}\right)^2+\cdots\right\}\\
&=\frac{B_3^2}{B_1-B_2}+x^{-2(p-t)s+4p-2t-2}o(1)\\
&=2^{-2+3ts-r-t}3^{r-2ts}x^{-2(p-t)s+4p-2t-2}(1+o(1)),
\end{aligned}$$

so we have

$$\begin{aligned}
B_2-\varepsilon_2&=2^{-r+t+ts}3^{r-t-ts}(1+o(1))-2^{-2+3ts-r-t}3^{r-2ts}x^{-2(p-t)s+4p-2t-2}(1+o(1))\\
&=2^{-r+t+ts}3^{r-t-ts}(1+o(1)),
\end{aligned}$$

where we use $0<2ps-2ts-4p+2t+2$.

The main term of $B_1+\varepsilon_2$ is the same as B_1, and so

$$(B_1+\varepsilon_2)^{\frac{\alpha}{\psi}}=\left(2^{r+t-ts}3^{-t}\right)^{\frac{\alpha}{\psi}}x^{2\alpha}(1+o(1)).$$

Now we should apply Lemma 2.2 to derive a contradiction.

Since $\left(c^2+1\right)^s(-A_1+A_2+2\varepsilon_1)=2^t3^{-t}(1+o(1))$, we have $\gamma=2^{t\frac{\alpha}{\psi}}3^{-t\frac{\alpha}{\psi}}(1+o(1))$.

One can easily see the estimations of 5 factors in the formula (3) as follows:

$$\begin{aligned}
\gamma d_1^\alpha-(B_2-\varepsilon_2)^{\frac{\alpha}{\psi}}&=2^{t\frac{\alpha}{\psi}+\alpha}3^{-t\frac{\alpha}{\psi}}x^{2\alpha}(1+o(1)),\\
(B_1+\varepsilon_2)^{\frac{\alpha}{\psi}}-\gamma d_2^\alpha&=\left(2^{r+t-ts}3^{-t}\right)^{\frac{\alpha}{\psi}}x^{2\alpha}(1+o(1)),\\
B_1-B_2+\varepsilon_2&=2^{r+t-ts}3^{-t}x^{2(p-t)s+2r}(1+o(1)),\\
\gamma d_1^\alpha-(B_1+\varepsilon_2)^{\frac{\alpha}{\psi}}&=2^{t\frac{\alpha}{\psi}}\left(2^\alpha-2^{(r-ts)\frac{\alpha}{\psi}}\right)3^{-t\frac{\alpha}{\psi}}x^{2\alpha}(1+o(1)),\\
\gamma d_2^\alpha-(B_2-\varepsilon_2)^{\frac{\alpha}{\psi}}&=2^{t\frac{\alpha}{\psi}}3^{-t\frac{\alpha}{\psi}}\left\{\left(\frac{3}{2}\right)^\alpha-\left(\frac{3}{2}\right)^{(r-ts)\frac{\alpha}{\psi}}\right\}(1+o(1)),
\end{aligned}$$

where we use $0 < \alpha$. Applying these estimations to the inequality (3), we obtain

$$\begin{aligned}
&2^{3ts-r-t-2}3^{r-2ts}2^{\alpha}\left(2^{r+t-ts}3^{-t}\right)^{\frac{\alpha}{\psi}}(1+o(1))\\
\leqslant &2^{r+t-ts}3^{-t}\left(2^{\alpha}-2^{(r-ts)\frac{\alpha}{\psi}}\right)2^{t\frac{\alpha}{\psi}}3^{-t\frac{\alpha}{\psi}}\left\{\left(\frac{3}{2}\right)^{\alpha}-\left(\frac{3}{2}\right)^{(r-ts)\frac{\alpha}{\psi}}\right\}\\
&\cdot x^{2(p-t)s+2r}x^{-\{-2(p-t)s+4p-2t-2\}}x^{-2\alpha}(1+o(1)).
\end{aligned}$$

For the power of x, we have

$$2(p-t)s+2r-\{-2(p-t)s+4p-2t-2\}-2\alpha = 4(p-t)(s-1) < 0,$$

where we use $p < t,\ 1 < s$. Letting $x \to \infty$, we have

$$0 < 2^{3ts-r-t-2}3^{r-2ts}2^{\alpha}\left(2^{r+t-ts}3^{-t}\right)^{\frac{\alpha}{\psi}} \leqslant 0.$$

This is a contradiction.

3.2. $s = 1+\dfrac{1-p}{t-p}$

Let $4p-2t-2 = 2(p-t)s$. It is not difficult to modify the argument in the previous subsection and to obtain

$$\begin{aligned}
\gamma d_1^{\alpha}-(B_2-\varepsilon_2)^{\frac{\alpha}{\psi}} &= c_1x^{2\alpha}(1+o(1)),\\
(B_1+\varepsilon_2)^{\frac{\alpha}{\psi}}-\gamma d_2^{\alpha} &= c_2x^{2\alpha}(1+o(1)),\\
B_1-B_2+\varepsilon_2 &= c_3x^{2(p-t)s+2r}(1+o(1)),\\
\gamma d_1^{\alpha}-(B_1+\varepsilon_2)^{\frac{\alpha}{\psi}} &= c_4x^{2\alpha}(1+o(1)),\\
\gamma d_2^{\alpha}-(B_2-\varepsilon_2)^{\frac{\alpha}{\psi}} &= c_5(1+o(1)),
\end{aligned}$$

where $c_j (j = 1,\cdots 5)$ are constants with $0 < c_1, c_2$.

These estimations can be applied to the inequality (3). By letting $x \to \infty$ and using $0 < p < 1$, we can obtain a contradiction $0 < c_1c_2 \leqslant 0$ of the same type as the subsection 3.1.

3.3. $1+\dfrac{1-p}{t-p} < s$

Let $2(p-t)s < 4p-2t-2$. In this case, we have $(-A_1+A_2+\varepsilon_1)(A_1-\varepsilon_1)^s < \varepsilon_1(A_2+\varepsilon_1)^s$ for sufficiently large x, hence

$$B_1 = 2^{r-t-2+ts}3^{-ts}x^{2r+4p-2t-2}(1+o(1)).$$

Since $2ps-2ts-4p+2t+2 < 0$, it is obvious that $-4p+2t+2 < 2(t-p)s < 2r$, so we have $B_2 < B_1$ for sufficiently large x, and the main term of B_1-B_2 is the same as

B_1. We can still apply Lemma 2.2, however, the estimations are changed, so we should check the calculation.

$$\frac{B_3^2}{(B_1-B_2)^2} \sim \frac{x^{2r+4p-2t-2}}{x^{2(2r+4p-2t-2)}} = \frac{1}{x^{2r+4p-2t-2}},$$
$$(B_1-B_2)\left(\frac{4B_3^2}{(B_1-B_2)^2}\right)^2 \sim x^{2r+4p-2t-2}\cdot\left(\frac{1}{x^{2r+4p-2t-2}}\right)^2 = \frac{1}{x^{2r+4p-2t-2}} = o(1).$$

Therefore,

$$\begin{aligned}\varepsilon_2 &= \frac{B_3^2}{B_1-B_2} + o(1)\\ &= 2^{-2+2ts}3^{r-t-2ts}x^{2r+4p-2t-2}(1+o(1))\cdot\left(2^{r-t-2+ts}3^{-ts}x^{2r+4p-2t-2}(1+o(1))\right)^{-1} + o(1)\\ &= 2^{ts-r+t}3^{r-t-ts}(1+o(1)),\end{aligned}$$

so the main terms of B_2 and ε_2 are cancelled by subtraction,

$$B_2 - \varepsilon_2 = 2^{-r+t+ts}3^{r-t-ts}(1+o(1)) - 2^{ts-r+t}3^{r-t-ts}(1+o(1)) = o(1).$$

The main term of $B_1+\varepsilon_2$ is the same as B_1,

$$(B_1+\varepsilon_2)^{\frac{\alpha}{\psi}} = \left(2^{r-t-2+ts}3^{-ts}x^{2r+4p-2t-2}\right)^{\frac{\alpha}{\psi}}(1+o(1)).$$

One can easily see the estimations of the factors in the formula (3).

$$\begin{aligned}\gamma d_1^{\alpha} - (B_2-\varepsilon_2)^{\frac{\alpha}{\psi}} &= 2^{t\frac{\alpha}{\psi}+\alpha}3^{-t\frac{\alpha}{\psi}}x^{2\alpha}(1+o(1)),\\ (B_1+\varepsilon_2)^{\frac{\alpha}{\psi}} - \gamma d_2^{\alpha} &= \left(2^{r-t-2+ts}3^{-ts}x^{2r+4p-2t-2}\right)^{\frac{\alpha}{\psi}}(1+o(1)),\\ B_1 - B_2 + \varepsilon_2 &= 2^{r-t-2+ts}3^{-ts}x^{2r+4p-2t-2}(1+o(1)),\\ \gamma d_1^{\alpha} - (B_1+\varepsilon_2)^{\frac{\alpha}{\psi}} &= 2^{t\frac{\alpha}{\psi}}3^{-t\frac{\alpha}{\psi}}(1+o(1))\cdot 2^{\alpha}x^{2\alpha}(1+o(1))\\ &\qquad - \left(2^{r-t-2+ts}3^{-ts}x^{2r+4p-2t-2}\right)^{\frac{\alpha}{\psi}}(1+o(1))\\ &= -\left(2^{r-t-2+ts}3^{-ts}x^{2r+4p-2t-2}\right)^{\frac{\alpha}{\psi}}(1+o(1)),\\ \gamma d_2^{\alpha} - (B_2-\varepsilon_2)^{\frac{\alpha}{\psi}} &= 2^{t\frac{\alpha}{\psi}-\alpha}3^{-t\frac{\alpha}{\psi}+\alpha}(1+o(1)),\end{aligned}$$

where we used

$$2\alpha - (2r+4p-2t-2)\frac{\alpha}{\psi} = \frac{\alpha}{\psi}\{2(p-t)s-4p+2t+2\} < 0.$$

Applying these estimations to the inequality (3), we obtain

$$\begin{aligned}&2^{ts-r+t}3^{r-t-ts}2^{t\frac{\alpha}{\psi}+\alpha}3^{-t\frac{\alpha}{\psi}}\left(2^{r-t-2+ts}3^{-ts}\right)^{\frac{\alpha}{\psi}}(1+o(1))\\ &\leqslant -2^{r-t-2+ts}3^{-ts}\left(2^{r-t-2+ts}3^{-ts}\right)^{\frac{\alpha}{\psi}}2^{t\frac{\alpha}{\psi}-\alpha}3^{-t\frac{\alpha}{\psi}+\alpha}x^{2r+4p-2t-2}x^{-2\alpha}(1+o(1)).\end{aligned}$$

For the power of x, we have

$$(2r+4p-2t-2)-2\alpha = 4p-4 < 0.$$

Letting $x \to \infty$, we have

$$0 < 2^{ts-r+t}3^{r-t-ts}2^{t\frac{\alpha}{\psi}+\alpha}3^{-t\frac{\alpha}{\psi}}\left(2^{r-t-2+ts}3^{-ts}\right)^{\frac{\alpha}{\psi}} \leqslant 0.$$

This is a contradiction and completes the proof. □

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