REFINEMENTS OF SOME INEQUALITIES RELATED TO JENSEN’S INEQUALITY

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Dedicated to Professor Sin-Ei Takahasi on the occasion of his 70th birthday

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Abstract. A finite form of Jensen’s inequality for a continuous convex function from a topological abelian semigroup to another topological ordered abelian semigroup is given by the author and S.-E. Takahasi. As an application of this abstract Jensen’s inequality, two inequalities with respect to geometric mean and arithmetic mean are obtained. The first gives a new refinement of the geometric-arithmetic mean inequality. The second gives a refinement between the arithmetic mean and a certain mean.

1. Introduction

The finite form of Jensen’s inequality proved by Jensen [1] in 1906 asserts that if \( t_1, \ldots, t_n \) are positive numbers with \( \sum_{i=1}^{n} t_i = 1 \) and \( f \) is a continuous convex (resp. concave) function on a real interval \( I \), then

\[
f \left( \sum_{i=1}^{n} t_i x_i \right) \leq \sum_{i=1}^{n} t_i f(x_i) \quad \text{resp.} \quad f \left( \sum_{i=1}^{n} t_i x_i \right) \geq \sum_{i=1}^{n} t_i f(x_i)
\]

holds for all \( x_1, \ldots, x_n \in I \).

In [3], the author and Takahasi have introduced a concept called \( (\ast, \circ) \)-convex (or concave) for a continuous function from a topological abelian semigroup \( (I, \ast) \) to another topological ordered abelian semigroup \( (J, \circ) \), and give an abstract Jensen’s inequality for such a function. Applying this abstract Jensen’s inequality, we give two interesting inequalities related to the geometric mean and the arithmetic mean. The first (Theorem 1) is a new refinement of the geometric-arithmetic mean inequality and the second (Theorem 2) is a refinement between the arithmetic mean and a certain mean related to the geometric mean.


Keywords and phrases: Jensen’s inequality, mean, geometric-arithmetic mean inequality, topological abelian semigroup operation.
2. Terminology and main results

Let $I$ be a topological space and $\ast$ a topological abelian semigroup operation on $I$. For any $x \in I$ and $n \in \mathbb{N}$, define the $n$-th power $x^{(n)_\ast}$ of $x$ recursively by $x^{(1)_\ast} = x$ and $x^{(n+1)_\ast} = x^{(n)_\ast} \ast x$ for $n \geq 1$.

We assume that

$(\sharp_1)$ any $n$-th power function: $x \mapsto x^{(n)_\ast}$ is a bijection of $I$ onto itself.

By the assumption $(\sharp_1)$, for each $x \in I$ and $n \in \mathbb{N}$, there exists a unique element $a$ of $I$ such that $a^{(n)_\ast} = x$. Denote by $x^{(1/n)_\ast}$ such an element $a$. Moreover, we define

$$x^{(m/n)_\ast} = \left(x^{(1/n)_\ast}\right)^{(m)_\ast}$$

for each $m,n \in \mathbb{N}$. Then we can easily see that this definition is well-defined. In this case, we can easily show that the following power laws:

$$x^{(p+q)_\ast} = x^{(p)_\ast} \ast y^{(q)_\ast}, x^{(pq)_\ast} = \left(x^{(p)_\ast}\right)^{(q)_\ast} \quad \text{and} \quad (x \ast y)^{(p)_\ast} = x^{(p)_\ast} \ast y^{(p)_\ast}$$

(1)

for all $p,q \in \mathbb{Q}_+$ and $x,y \in I$. Here $\mathbb{Q}_+$ denotes the set of all positive rational numbers. Furthermore, we assume that

$(\sharp_2)$ for each $x \in I$, the function $p \mapsto x^{(p)_\ast}$ is continuous on $\mathbb{Q}_+$ and it has a continuous extension to $\mathbb{R}_+$, say $t \mapsto x^{(t)_\ast}$.

Here $\mathbb{R}_+$ denotes the set of all positive real numbers. Therefore power laws (1) hold for all $p,q \in \mathbb{R}_+$. Denote by $\mathcal{A}_+ (I)$ the set of all topological abelian semigroup operations on $I$ satisfying both $(\sharp_1)$ and $(\sharp_2)$. Our assumption $(\sharp_1)$ leads to the following important concept called mean. For each $x,y \in I$, put

$$M_\ast(x,y) = (x \ast y)^{(1/2)_\ast}.$$  

We call $M_\ast(x,y)$ the mean of $x$ and $y$ with respect to the operation $\ast$.

Now let $J$ be a topological ordered space with relation $\leq$, and denote by $\mathcal{A}_+^0 (J) = \mathcal{A}_+^0 (J, \leq)$ the set of all operations $o \in \mathcal{A}_+ (J)$ satisfying the following two conditions:

$(\flat_1)$ $a \leq b \iff a \circ c \leq b \circ c$ for all $a,b,c \in J$

and

$(\flat_2)$ $a \leq b \Rightarrow a^{(t)_\circ} \leq b^{(t)_\circ}$ for all $a,b \in J$ and $t \in \mathbb{R}_+$.

Let $C(I,J)$ be the set of all continuous functions from $I$ to $J$. Take $\ast \in \mathcal{A}_+(I)$, $o \in \mathcal{A}_+^0 (J, \leq)$ and $f \in C(I,J)$ arbitrarily. If $f$ satisfies

$$f(M_\ast(x,y)) \leq M_\circ(f(x),f(y)) \quad \text{(resp. } f(M_\ast(x,y)) \geq M_\circ(f(x),f(y))\text{)}$$

for all $x,y \in I$, then $f$ is said to be $(\ast, o)$-convex (resp. concave).

In [3], the author and Takahasi have shown the following theorem which states a finite form of Jensen’s inequality for a $(\ast, o)$-convex (or concave) function.
THEOREM A. Let \( * \in \mathcal{A}_{+}(I) \) and \( \circ \in \mathcal{A}_{+}^{0}(J, \leq) \). If \( f \in C(I, J) \) is \((*, \circ)\)-convex, then
\[
f(x_1^{(t_1)} * \cdots * x_n^{(t_n)}) \leq f(x_1)^{(t_1)} \circ \cdots \circ f(x_n)^{(t_n)}
\]
holds for all \( n \in \mathbb{N} \), \( x_1, \ldots, x_n \in I \) and \( t_1, \ldots, t_n \in \mathbb{R}_{+} \) with \( t_1 + \cdots + t_n = 1 \). If \( f \) is \((*, \circ)\)-concave, then the inequality above is reversed.

REMARK 1. The above theorem is the inheritance of the idea of [2, Theorem 1] which gives a new interpretation of Jensen’s inequality by \( \varphi \)-mean.

As an application of Theorem A, we have the following

THEOREM 1. Let \( a_1, \ldots, a_n, t_1, \ldots, t_n > 0 \) with \( t_1 + \cdots + t_n = 1 \) and put
\[
(GA)_t = \prod_{i=1}^{n} (a_i + t)^{t_i} - t
\]
for each \( t \geq 0 \). Then \( \{(GA)_t : t \geq 0\} \) is strictly monotone increasing and
\[
\lim_{t \to \infty} (GA)_t = \sum_{i=1}^{n} t_i a_i
\]
holds.

REMARK 2. The above theorem gives a strict refinement of the geometric-arithmetic mean inequality:
\[
\prod_{i=1}^{n} a_i^{t_i} < (GA)_t < \sum_{i=1}^{n} t_i a_i \quad (t > 0).
\]
Furthermore, as another application of Theorem A, we have the following

THEOREM 2. Let \( 0 < a_1, \ldots, a_n, t_1, \ldots, t_n < 1 \) with \( t_1 + \cdots + t_n = 1 \) and put
\[
(AP)_t = \frac{1}{t} \frac{\prod_{i=1}^{n} (1 + t a_i)^{t_i} - \prod_{i=1}^{n} (1 - t a_i)^{t_i}}{\prod_{i=1}^{n} (1 + a_i)^{t_i} + \prod_{i=1}^{n} (1 - a_i)^{t_i}}
\]
for each \( t \) with \( 0 < t \leq 1 \). Then \( \{(AP)_t : 0 < t \leq 1\} \) is strictly monotone increasing and
\[
\lim_{t \downarrow 0} (AP)_t = \sum_{i=1}^{n} t_i a_i
\]
holds.

REMARK 3. The above theorem gives a strict refinement between \( \sum_{i=1}^{n} t_i a_i \) and \( (AP)_1 \):
\[
\sum_{i=1}^{n} t_i a_i < (AP)_t < \frac{\prod_{i=1}^{n} (1 + a_i)^{t_i} - \prod_{i=1}^{n} (1 - a_i)^{t_i}}{\prod_{i=1}^{n} (1 + a_i)^{t_i} + \prod_{i=1}^{n} (1 - a_i)^{t_i}} \quad (0 < t < 1).
\]
3. Proofs of main results

In order to show our main results, we must prepare several lemmas.

**LEMMA 1.** Let $I$ and $J$ be two ordered topological spaces. Suppose that there exists a homeomorphism $\varphi$ of $I$ onto $J$ such that both $\varphi$ and $\varphi^{-1}$ are monotone increasing or monotone decreasing. Let $\circ \in \mathcal{A}_+^0(J)$ and put $a \circ_{\varphi} b = \varphi^{-1}((\varphi(a) \circ \varphi(b)))$ for each $a, b \in I$. Then

(i) $\circ_{\varphi} \in \mathcal{A}_+^0(I)$.

(ii) $a_1^{(t_1)_{\circ_{\varphi}}} \circ_{\varphi} \cdots \circ_{\varphi} a_n^{(t_n)_{\circ_{\varphi}}} = \varphi^{-1}\left(\varphi\left(a_1^{(t_1)_{\circ}}\right) \circ \cdots \circ \varphi\left(a_n^{(t_n)_{\circ}}\right)\right)$ holds for all $a_1, \cdots, a_n \in I$ and $t_1, \cdots, t_n > 0$.

**Proof.** (i) It is obvious that $\circ_{\varphi}$ is a topological abelian semigroup operation on $I$. Note that $a^{(n)_{\circ_{\varphi}}} = \varphi^{-1}((\varphi(a)^{(n)_{\circ}}))$ for each $a \in I$ and $n \in \mathbb{N}$. Then $\circ_{\varphi}$ satisfies the condition $(\sharp_1)$. Moreover we have that $a^{(p)_{\circ_{\varphi}}} = \varphi^{-1}((\varphi(a)^{(p)_{\circ}}))$ for each $a \in I$ and $p \in \mathbb{Q}_+$. Then $\circ_{\varphi}$ satisfies the condition $(\sharp_2)$. Suppose that both $\varphi$ and $\varphi^{-1}$ are monotone increasing. Then

$$a \leq b \Leftrightarrow \varphi(a) \leq \varphi(b)$$

$$\Leftrightarrow \varphi(a) \circ \varphi(c) \leq \varphi(b) \circ \varphi(c)$$

$$\Leftrightarrow \varphi^{-1}(\varphi(a) \circ \varphi(c)) \leq \varphi^{-1}(\varphi(b) \circ \varphi(c))$$

$$\Leftrightarrow a \circ_{\varphi} c \leq b \circ_{\varphi} c$$

for all $a, b, c \in I$. Then $\circ_{\varphi}$ satisfies the condition $(\flat_1)$. Furthermore we have

$$a \leq b \Rightarrow \varphi(a) \leq \varphi(b)$$

$$\Rightarrow \varphi(a)^{(t)_{\circ}} \leq \varphi(b)^{(t)_{\circ}}$$

$$\Rightarrow \varphi^{-1}(\varphi(a)^{(t)_{\circ}}) \leq \varphi^{-1}(\varphi(b)^{(t)_{\circ}})$$

$$\Rightarrow a^{(t)_{\circ_{\varphi}}} \leq b^{(t)_{\circ_{\varphi}}}$$

for all $a, b \in I$ and $t \in \mathbb{R}_+$. Then $\circ_{\varphi}$ satisfies the condition $(\flat_2)$. Consequently we obtain that $\circ_{\varphi} \in \mathcal{A}_+^0(J)$. If both $\varphi$ and $\varphi^{-1}$ are monotone decreasing, then we obtain the same result by using the same method above.

(ii) Let $a_1, \cdots, a_n, t_1, \cdots, t_n > 0$. Then we have

$$a_1^{(t_1)_{\circ_{\varphi}}} \circ_{\varphi} \cdots \circ_{\varphi} a_n^{(t_n)_{\circ_{\varphi}}}$$

$$= \varphi^{-1}\left(\varphi\left(a_1^{(t_1)_{\circ}}\right) \circ_{\varphi} \cdots \circ_{\varphi} \varphi^{-1}\left(\varphi\left(a_n^{(t_n)_{\circ}}\right)\right)\right)$$

$$= \varphi^{-1}\left(\varphi\left(a_1^{(t_1)_{\circ}} \circ_{\varphi} a_2^{(t_2)_{\circ}}\right) \circ_{\varphi} \cdots \circ_{\varphi} \varphi^{-1}\left(\varphi\left(a_n^{(t_n)_{\circ}}\right)\right)\right)$$

$$\vdots$$

$$= \varphi^{-1}\left(\varphi\left(a_1^{(t_1)_{\circ}} \circ \cdots \circ \varphi\left(a_n^{(t_n)_{\circ}}\right)\right)\right),$$
and hence the desired equality holds. □

**Lemma 2.** Let \( f_1, \ldots, f_n \) be differentiable positive-valued functions on a real interval \( I \) and \( t_1, \ldots, t_n > 0 \). Then

\[
\frac{d}{dx} \prod_{i=1}^{n} f_i(x)^{t_i} = \prod_{i=1}^{n} f_i(x)^{t_i} \sum_{i=1}^{n} t_i f_i'(x) \quad \frac{1}{f_i(x)}
\]

holds for all \( x \in I \).

**Proof.** Straightforward. □

Now note that \( \mathbb{R} \) is an ordered topological space with the ordinary order and the ordinary topology. Let \( + \) be the ordinary additive operation on \( \mathbb{R} \). Then it is obvious that \( + \in \mathcal{A}_0^0(\mathbb{R}) \).

**Proof of Theorem 1.** Take \( t \geq 0 \) arbitrarily and put

\[
I_t = \{ x \in \mathbb{R} : x > -t \}.
\]

Then \( I_t \) is an ordered topological spaces with the ordinary order and the ordinary topology. Consider the following function

\[
\phi_t(x) = \log(x + t) \quad (x \in I_t).
\]

Then \( \phi_t \) is a homeomorphism of \( I_t \) onto \( \mathbb{R} \). Moreover both \( \phi_t \) and \( \phi_t^{-1} \) are strictly monotone increasing, and hence \( (+)_{\phi_t} \in \mathcal{A}_+^0(I_t) \) by Lemma 1-(i). For the sake of simplicity, let \( \circ_t = (+)_{\phi_t} \). Since \( \phi_t^{-1}(y) = e^y - t \) for all \( y \in \mathbb{R} \), we have from simple computation that

\[
a \circ_t b = (a + t)(b + t) - t
\]

for all \( a, b \in I_t \).

Now let \( a_1, \ldots, a_n \in I_t \) and \( t_1, \ldots, t_n > 0 \). Then we have from Lemma 1-(ii) that

\[
a_1^{(t_1)_{\phi_t}} \circ \cdots \circ a_n^{(t_n)_{\phi_t}} = \phi_t^{-1}(t_1 \phi_t(a_1) + \cdots + t_n \phi_t(a_n))
\]

\[
= \prod_{i=1}^{n} (a_i + t)^{t_i} - t = (GA)_t.
\]

Assume that \( 0 \leq t < s \), and so \( I_t \subset I_s \). Let \( t : I_t \to I_s \) be the identity mapping. Then \( t \) is \((\circ_t, \circ_s)\)-convex. Indeed,

\[
tM_{\circ_t}(a, b) = M_{\circ_t}(a, b) = (a \circ_t b)^{(1/2)}_{\circ_t}
\]

\[
= \sqrt{(a + t)(b + t) - t}
\]

\[
\leq \sqrt{(a + s)(b + s) - s}
\]

\[
= M_{\circ_s}(a, b) = M_{\circ_s}(ta, tb)
\]

holds for all \( a, b \in I_t \). Therefore if \( t_1 + \cdots + t_n = 1 \), then we have from Theorem A that \((GA)_t \leq (GA)_s \). However, \((GA)_t < (GA)_s \) holds. Indeed, if \((GA)_t = (GA)_s \), then
\[ \prod_{i=1}^{n} (a_i + x)^{t_i} = x + c \] must hold for all \( x \in \mathbb{R} \) with \( t < x < s \) and a constant \( c \in \mathbb{R} \). This is a contradiction.

We next show that \( \lim_{t \to -\infty} (GA)_t = \sum_{i=1}^{n} t_i a_i \). To do this, put
\[
p(t) = \prod_{i=1}^{n-1} \left( \frac{a_i + t}{a_n + t} \right)^{t_i}
\]
for each \( t > 0 \). Then we have that \( (GA)_t = a_n p(t) + (p(t) - 1)t \) \( (t > 0) \). Obviously, \( \lim_{t \to -\infty} p(t) = 1 \). Also we have from Lemma 2 that
\[
\lim_{t \to -\infty} (p(t) - 1)t = \lim_{x \to 0} \frac{p(1/x) - 1}{x}
\]
\[
= \lim_{x \to 0} \frac{d}{dx} \prod_{i=1}^{n-1} \left( \frac{1 + a_i x}{1 + a_n x} \right)^{t_i}
\]
\[
= \lim_{x \to 0} \prod_{i=1}^{n-1} \left( \frac{1 + a_i x}{1 + a_n x} \right)^{t_i} \sum_{i=1}^{n-1} \frac{t_i (a_i - a_n)}{(1 + a_i x)(1 + a_n x)}
\]
\[
= t_1 a_1 + \cdots + t_{n-1} a_{n-1} - (t_1 + \cdots + t_{n-1}) a_n.
\]

Therefore
\[
\lim_{t \to -\infty} (GA)_t = a_n + t_1 a_1 + \cdots + t_{n-1} a_{n-1} - (t_1 + \cdots + t_{n-1}) a_n = \sum_{i=1}^{n} t_i a_i,
\]
as required. Thus we obtain the desired result. \( \square \)

**Proof of Theorem 2.** Let \( 0 < t \leq 1 \) and put \( I_t = \{ x \in \mathbb{R} : 0 < x < 1/t \} \). Then \( I_t \) is an ordered topological spaces with the ordinary order and the ordinary topology. Consider the following function
\[
\varphi_t(x) = \tanh^{-1}(tx) \quad (x \in I_t)
\]
Then \( \varphi_t \) is a homeomorphism of \( I_t \) onto \( \mathbb{R}_+ \). Moreover both \( \varphi_t \) and \( \varphi_t^{-1} \) are strictly monotone increasing, and hence \( (+) \varphi_t \in \mathcal{G}^0_+(I_t) \) by Lemma 1-(i). For the sake of simplicity, let \( \varphi_t = (+) \varphi_t \).

Now let \( 0 < a_1, \ldots, a_n, t_1, \ldots, t_n < 1 \) with \( t_1 + \cdots + t_n = 1 \). Then \( a_1, \ldots, a_n \in I_t \), and hence we have from Lemma 1-(ii) that
\[
a_1^{(t_1)_{xy}} \circ_t \cdots \circ_t a_n^{(t_n)_{xy}} = \frac{1}{t} \tanh\left( t_1 \tanh^{-1}(t a_1) + \cdots + t_n \tanh^{-1}(t a_n) \right).
\]
Since \( \tanh^{-1}(x) = \log \sqrt{\frac{1+x}{1-x}} \quad (0 < x < 1) \), it follows that
\[
t_1 \tanh^{-1}(t a_1) + \cdots + t_n \tanh^{-1}(t a_n) = \log \prod_{i=1}^{n} \left( \frac{1 + t a_i}{1 - t a_i} \right)^{t_i/2}.
\]
Therefore we have
\[
a_{(t_1)\gamma} \circ_t \cdots \circ_t a_{(t_n)\gamma} = \frac{1}{t} \text{tanh} \log \prod_{i=1}^{n} \left( \frac{1+ta_i}{1-ta_i} \right)^{t_i/2} = \frac{1}{t} \Pi_{i=1}^{n} \left( \frac{1+ta_i}{1-ta_i} \right)^{t_i/2} - \Pi_{i=1}^{n} \left( \frac{1+ta_i}{1-ta_i} \right)^{-t_i/2}
\]

On the other hand, we have \((AP)_t\).

Let \(a, b \in I_t\). Then the solution of the equation \(x \circ_t x = a \circ_t b, x \in I_t\) is given by
\[
M_{\circ_t}(a, b) = \frac{1 + t^2ab - \sqrt{(1 - t^2a^2)(1 - t^2b^2)}}{t^2(a+b)}.
\]

Put \(f(x) = M_{\circ_t}(a, b)\) for each \(x \in \mathbb{R}\) with \(0 < x < t\). By simple computation, we have
\[
f'(x) = \frac{2x^2 - a^2 - b^2 - 2\sqrt{(x^2 - a^2)(x^2 - b^2)}}{(a+b)x\sqrt{(x^2 - a^2)(x^2 - b^2)}} \geq 0 \quad (0 < x < t).
\]

Then \(M_{\circ_t}(a, b) \geq M_{\circ_s}(a, b)\) holds when \(0 < s < t\). This means that the identity mapping from \(I_t\) to \(I_s\) is \((\circ_t, \circ_s)\)-concave when \(0 < s < t\). Therefore it follows from Theorem A that
\[
(AP)_t = a_{(t_1)\gamma} \circ_t \cdots \circ_t a_{(t_n)\gamma} \geq a_{(t_1)\gamma} \circ_s \cdots \circ_s a_{(t_n)\gamma} = (AP)_s \quad (0 < s < t)
\]
holds. However, \((AP)_t < (AP)_s\) \((0 < s < t)\) holds. Indeed, if \((AP)_t = (AP)_s\) and \(0 < s < t\), then
\[
\frac{\prod_{i=1}^{n}(1 + xai)_{ti} - \prod_{i=1}^{n}(1 - xai)_{ti}}{\prod_{i=1}^{n}(1 + xai)_{ti} + \prod_{i=1}^{n}(1 - xai)_{ti}} = cx
\]
must hold for all \(x \in \mathbb{R}\) with \(s < x < t\) and a constant \(c \in \mathbb{R}\). This is a contraction. Also we have from Lemma 2 that
\[
\lim_{t \downarrow 0} (AP)_t = \frac{1}{2} \lim_{t \downarrow 0} \frac{d}{dt} \left( \prod_{i=1}^{n}(1 + a_{it})_{ti} - \prod_{i=1}^{n}(1 - a_{it})_{ti} \right)
\]
\[
= \frac{1}{2} \lim_{t \downarrow 0} \left( \prod_{i=1}^{n}(1 + a_{it})_{ti} \sum_{i=1}^{n} \frac{t_i a_i}{1 + a_{it}} + \prod_{i=1}^{n}(1 - a_{it})_{ti} \sum_{i=1}^{n} \frac{t_i a_i}{1 - a_{it}} \right)
\]
\[
= \sum_{i=1}^{n} t_i a_i.
\]
Thus we obtain the desired result.  

REMARK 4. We can show that \(\sum_{i=1}^{n} t_i a_i \leq (AP)_t\) \((0 < t \leq 1)\) in other ways. Indeed,
\[
M_\sigma(a, b) - M_+(a, b) = \frac{2 - t^2 a^2 - t^2 b^2 - 2\sqrt{(1 - t^2 a^2)(1 - t^2 b^2)}}{2t^2(a + b)} \geq 0
\]
holds for all \(a, b \in I_t\). Then the identity mapping from \(I_t\) to \(\mathbb{R}_+\) is \((\sigma_t, +)-\)concave, and hence
\[
(AP)_t = a_1^{(t_1)} \sigma_t \cdots \sigma_t a_n^{(t_n)} \geq a_1^{(t_1)} + \cdots + a_n^{(t_n)} = \sum_{i=1}^{n} t_i a_i
\]
holds by Theorem A.

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