

ON THE NORMS OF r -CIRCULANT MATRICES WITH THE HYPER-FIBONACCI AND LUCAS NUMBERS

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Abstract. In this paper, we study norms of circulant matrices $F = \text{Circ}(F_0^{(k)}, F_1^{(k)}, \dots, F_{n-1}^{(k)})$, $L = \text{Circ}(L_0^{(k)}, L_1^{(k)}, \dots, L_{n-1}^{(k)})$ and r -circulant matrices $F_r = \text{Circ}_r(F_0^{(k)}, F_1^{(k)}, \dots, F_{n-1}^{(k)})$, $L_r = \text{Circ}_r(L_0^{(k)}, L_1^{(k)}, \dots, L_{n-1}^{(k)})$, where $F_n^{(k)}$ and $L_n^{(k)}$ denote the hyper-Fibonacci and hyper-Lucas numbers, respectively.

1. Introduction

The circulant matrices and r -circulant matrices play important role in signal processing, coding theory, image processing, linear forecast and so on. An $n \times n$ matrix C_r is called an r -circulant matrix if it is of the form

$$C_r = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ r c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ r c_{n-2} & r c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r c_1 & r c_2 & r c_3 & \cdots & r c_{n-1} & c_0 \end{bmatrix}.$$

The matrix C_r is determined by its first row elements and r , thus we denote $C_r = \text{Circ}_r(c_0, c_1, \dots, c_{n-1})$. When we take $r = 1$, the matrix $C_1 = C$ is called a circulant matrix. We denote $C_1 = C = \text{Circ}(c_0, c_1, \dots, c_{n-1})$. Circulant matrices are especially tractable class of matrices since their inverses, conjugate transposes, sums and products are also circulant. Moreover, circulant matrices are normal matrices [4]. Also, by means of [4, 9], it is well known that the eigenvalues of C are

$$\lambda_m = \sum_{k=0}^{n-1} c_k w^{-mk} \quad (1)$$

where $w = e^{\frac{2\pi i}{n}}$ and $i = \sqrt{-1}$, and the corresponding eigenvectors are

$$x_m = \left(1, w^m, w^{2m}, \dots, w^{(n-1)m} \right)^T \quad (2)$$

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Recently, there have been many papers on the norms of special matrices with special elements such as Fibonacci and Lucas numbers [1, 2, 8, 11, 14–18, 20]. Solak [16, 17] has computed the spectral and Euclidean norms of circulant matrices with the Fibonacci and Lucas numbers. Shen and Cen [15] have given upper and lower bounds for the spectral norms of r -circulant matrices in the forms $A = C_r(F_0, F_1, \dots, F_{n-1})$ and $B = C_r(L_0, L_1, \dots, L_{n-1})$. Yazlik and Taskara [20] have presented upper and lower bounds for the spectral norm of an r -circulant matrix with the generalized k -Horadam numbers. As for us, in this paper, we compute the spectral norms of circulant and r -circulant matrices with the hyper-Fibonacci and hyper-Lucas numbers.

The main contents of this paper are organized as follows: In Section 2, we give some preliminaries, definitions and lemmas related to our study. In Section 3, we derive some bounds for the spectral norms of r -circulant matrices with the hyper-Fibonacci and hyper-Lucas numbers of the forms $F_r = \text{Circr}(F_0^{(k)}, F_1^{(k)}, \dots, F_{n-1}^{(k)})$, $L_r = \text{Circr}(L_0^{(k)}, L_1^{(k)}, \dots, L_{n-1}^{(k)})$ and their Hadamard and Kronecker products. For this, we firstly compute the spectral and Euclidean norms of circulant matrices of the forms $F = \text{Circ}(F_0^{(k)}, F_1^{(k)}, \dots, F_{n-1}^{(k)})$ and $L = \text{Circ}(L_0^{(k)}, L_1^{(k)}, \dots, L_{n-1}^{(k)})$. Moreover, we give some examples related to special cases of our results.

2. Preliminaries

The sequence of the Fibonacci numbers is one of the most well-known sequences, and it has many applications to different fields such as mathematics, statistics and physics. The Fibonacci numbers are defined by the second order linear recurrence relation: $F_{n+1} = F_n + F_{n-1}$ ($n \geq 1$), $F_0 = 0$ and $F_1 = 1$. Similarly, the Lucas numbers are defined by $L_{n+1} = L_n + L_{n-1}$ ($n \geq 1$), $L_0 = 2$ and $L_1 = 1$. Fibonacci and Lucas numbers have generating functions and many generalizations [3, 5, 10, 12, 13, 19]. In [5], Dil and Mezö introduced new concepts as hyper-Fibonacci numbers and hyper-Lucas numbers. These concepts are defined as

$$F_n^{(k)} = \sum_{s=0}^n F_s^{(k-1)}, \text{ with } F_n^{(0)} = F_n, F_0^{(k)} = 0 \text{ and } F_1^{(k)} = 1 \tag{3}$$

and

$$L_n^{(k)} = \sum_{s=0}^n L_s^{(k-1)}, \text{ with } L_n^{(0)} = L_n, L_0^{(k)} = 2, L_1^{(k)} = 2k + 1. \tag{4}$$

The hyper-Fibonacci and the hyper-Lucas numbers have the recurrence relations $F_n^{(k)} = F_{n-1}^{(k)} + F_n^{(k-1)}$ and $L_n^{(k)} = L_{n-1}^{(k)} + L_n^{(k-1)}$, respectively. Also, $F_n^{(k)}$ and $L_n^{(k)}$ have the following more explicit forms when $k = 1, 2, 3$.

$$F_n^{(1)} = F_{n+2} - 1, F_n^{(2)} = F_{n+4} - n - 3 \text{ and } F_n^{(3)} = F_{n+6} - \frac{n^2 + 7n + 16}{2}, \tag{5}$$

$$L_n^{(1)} = L_{n+2} - 1, L_n^{(2)} = L_{n+4} - n - 5 \text{ and } L_n^{(3)} = L_{n+6} - \frac{n^2 + 11n + 32}{2}. \tag{6}$$

Now we give some definitions and lemmas related to our study.

DEFINITION 1. Let $A = (a_{ij})$ be any $m \times n$ matrix. The *Euclidean norm* of A is

$$\|A\|_E = \sqrt{\left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2\right)}.$$

DEFINITION 2. Let $A = (a_{ij})$ be any $m \times n$ matrix. The *spectral norm* of A is

$$\|A\|_2 = \sqrt{\max_i \lambda_i(A^H A)},$$

where $\lambda_i(A^H A)$ are eigenvalues of $A^H A$ and A^H is conjugate transpose of A .

There are two well known relations between Euclidean norm and spectral norm as the following:

$$\frac{1}{\sqrt{n}} \|A\|_E \leq \|A\|_2 \leq \|A\|_E \tag{7}$$

$$\|A\|_2 \leq \|A\|_E \leq \sqrt{n} \|A\|_2. \tag{8}$$

DEFINITION 3. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. Then their *Hadamard product* $A \circ B$ is defined

$$A \circ B = [a_{ij} b_{ij}].$$

DEFINITION 4. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ and $p \times r$ matrices, respectively. Then their *Kronecker product* $A \otimes B$ is defined

$$A \otimes B = [a_{ij} B].$$

LEMMA 1. [7] *Let A and B be two $m \times n$ matrices. Then we have*

$$\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2.$$

LEMMA 2. [7] *Let A and B be two $m \times n$ matrices. Then we have*

$$\|A \circ B\|_2 \leq r_1(A) c_1(B)$$

where $r_1(A) = \max_{1 \leq i \leq m} \sqrt{\sum_{j=1}^n |a_{ij}|^2}$ and $c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^m |b_{ij}|^2}$.

LEMMA 3. [7] *Let A and B be two $m \times n$ matrices. Then we have*

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2.$$

LEMMA 4. [6] *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then, A is a normal matrix if and only if the eigenvalues of $A^H A$ are $|\lambda_1|^2, |\lambda_2|^2, \dots, |\lambda_n|^2$.*

3. Main results

THEOREM 1. *The spectral norm of the matrix $F = \text{Circ}(F_0^{(k)}, F_1^{(k)}, \dots, F_{n-1}^{(k)})$ is*

$$\|F\|_2 = F_{n-1}^{(k+1)}.$$

Proof. Since the circulant matrix F is normal, its spectral norm is equal to its spectral radius. Furthermore, by considering F is irreducible and its entries are non-negative, we have that the spectral radius (or spectral norm) of the matrix F is equal to its Perron root. We select an n -dimensional column vector $v = (1, 1, \dots, 1)^T$, then

$$Fv = \left(\sum_{s=0}^{n-1} F_s^{(k)} \right) v.$$

Obviously, $\sum_{s=0}^{n-1} F_s^{(k)}$ is an eigenvalue of F associated with v and it is the Perron root of F . Hence, by (3) we have

$$\|F\|_2 = \sum_{s=0}^{n-1} F_s^{(k)} = F_{n-1}^{(k+1)}.$$

This completes the proof. \square

EXAMPLE 1. By using Theorem 1 and the equations in (5), we have

$$\|F\|_2 = \begin{cases} F_{n+1} - 1, & \text{if } k = 0, \\ F_{n+3} - n - 2, & \text{if } k = 1, \\ F_{n+5} - \frac{n^2 + 5n + 10}{2}, & \text{if } k = 2. \end{cases}$$

COROLLARY 1. *For the Euclidean norm of the matrix $F = \text{Circ}(F_0^{(k)}, F_1^{(k)}, \dots, F_{n-1}^{(k)})$, we have*

$$F_{n-1}^{(k+1)} \leq \|F\|_E \leq \sqrt{n} F_{n-1}^{(k+1)}.$$

Proof. The proof is trivial from Theorem 1 and the relation between spectral norm and Euclidean norm in (8). \square

COROLLARY 2. *For the sum of squares of hyper-Fibonacci numbers, we have*

$$\frac{1}{\sqrt{n}} F_{n-1}^{(k+1)} \leq \sqrt{\sum_{s=0}^{n-1} (F_s^{(k)})^2} \leq F_{n-1}^{(k+1)}. \tag{9}$$

Proof. This follows from the definition of Euclidean norm and Corollary 1. \square

THEOREM 2. *The spectral norm of the matrix $L = \text{Circ}(L_0^{(k)}, L_1^{(k)}, \dots, L_{n-1}^{(k)})$ is*

$$\|L\|_2 = L_{n-1}^{(k+1)}.$$

Proof. This theorem can be proved by using a similar method to method of the proof of Theorem 1. But, we will use another method. Since L is a circulant matrix, from (1) its eigenvalues are of the form

$$\lambda_m = \sum_{s=0}^{n-1} L_s^{(k)} e^{\frac{-2\pi i m s}{n}}.$$

Then for $m = 0$, by using (4) we have

$$\lambda_0 = \sum_{s=0}^{n-1} L_s^{(k)} = L_{n-1}^{(k+1)}. \tag{10}$$

Also, we have

$$|\lambda_m| = \left| \sum_{s=0}^{n-1} L_s^{(k)} e^{\frac{-2\pi i m s}{n}} \right| \leq \sum_{s=0}^{n-1} |L_s^{(k)}| \left| e^{\frac{-2\pi i m s}{n}} \right| \leq \sum_{s=0}^{n-1} |L_s^{(k)}| = \sum_{s=0}^{n-1} L_s^{(k)}. \tag{11}$$

By using Lemma 4 and the fact that the matrix L is a normal matrix, we have

$$\|L\|_2 = \max_{0 \leq m \leq n-1} |\lambda_m| = \max \left(|\lambda_0|, \max_{1 \leq m \leq n-1} |\lambda_m| \right). \tag{12}$$

From (10), (11) and (12), we have

$$\|L\|_2 = L_{n-1}^{(k+1)}.$$

Thus the proof is completed. \square

EXAMPLE 2. By using Theorem 2 and the equations in (6), we have

$$\|L\|_2 = \begin{cases} L_{n+1} - 1, & \text{if } k = 0, \\ L_{n+3} - n - 4, & \text{if } k = 1, \\ L_{n+5} - \frac{n^2 + 9n + 22}{2}, & \text{if } k = 2. \end{cases}$$

COROLLARY 3. *For the Euclidean norm of the matrix $L = \text{Circ}(L_0^{(k)}, L_1^{(k)}, \dots, L_{n-1}^{(k)})$, we have*

$$L_{n-1}^{(k+1)} \leq \|L\|_E \leq \sqrt{n} L_{n-1}^{(k+1)}.$$

Proof. The proof is trivial from Theorem 2 and the relation between spectral norm and Euclidean norm in (8). \square

COROLLARY 4. *For the sum of squares of hyper-Lucas numbers, we have*

$$\frac{1}{\sqrt{n}}L_{n-1}^{(k+1)} \leq \sqrt{\sum_{s=0}^{n-1} \left(L_s^{(k)}\right)^2} \leq L_{n-1}^{(k+1)}. \tag{13}$$

Proof. This follows from the definition of Euclidean norm and Corollary 3. \square

COROLLARY 5. *The spectral norm of the Hadamard product of $F = \text{Circ}(F_0^{(k)}, F_1^{(k)}, \dots, F_{n-1}^{(k)})$ and $L = \text{Circ}(L_0^{(k)}, L_1^{(k)}, \dots, L_{n-1}^{(k)})$ holds*

$$\|F \circ L\|_2 \leq F_{n-1}^{(k+1)} L_{n-1}^{(k+1)}.$$

Proof. The proof is trivial since $\|F \circ L\|_2 \leq \|F\|_2 \|L\|_2$. \square

COROLLARY 6. *The spectral norm of the Kronecker product of $F = \text{Circ}(F_0^{(k)}, F_1^{(k)}, \dots, F_{n-1}^{(k)})$ and $L = \text{Circ}(L_0^{(k)}, L_1^{(k)}, \dots, L_{n-1}^{(k)})$ holds*

$$\|F \otimes L\|_2 = F_{n-1}^{(k+1)} L_{n-1}^{(k+1)}.$$

Proof. The proof is trivial since $\|F \otimes L\|_2 = \|F\|_2 \|L\|_2$. \square

THEOREM 3. *Let $F_r = \text{Circ}_r(F_0^{(k)}, F_1^{(k)}, \dots, F_{n-1}^{(k)})$ be an r -circulant matrix.*

i) *If $|r| \geq 1$, then*

$$\frac{1}{\sqrt{n}}F_{n-1}^{(k+1)} \leq \|F_r\|_2 \leq |r| \left(F_{n-1}^{(k+1)}\right)^2$$

ii) *If $|r| < 1$, then*

$$\frac{|r|}{\sqrt{n}}F_{n-1}^{(k+1)} \leq \|F_r\|_2 \leq \sqrt{n-1}F_{n-1}^{(k+1)}.$$

Proof. Since the matrix F_r is of the form

$$F_r = \begin{bmatrix} F_0^{(k)} & F_1^{(k)} & F_2^{(k)} & \dots & F_{n-2}^{(k)} & F_{n-1}^{(k)} \\ rF_{n-1}^{(k)} & F_0^{(k)} & F_1^{(k)} & \dots & F_{n-3}^{(k)} & F_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rF_2^{(k)} & rF_3^{(k)} & rF_4^{(k)} & \dots & F_0^{(k)} & F_1^{(k)} \\ rF_1^{(k)} & rF_2^{(k)} & rF_3^{(k)} & \dots & rF_{n-1}^{(k)} & F_0^{(k)} \end{bmatrix}$$

and from the definition of Euclidean norm, we have

$$\|F_r\|_E = \sqrt{\sum_{s=0}^{n-1} (n-s) \left(F_s^{(k)}\right)^2 + \sum_{s=0}^{n-1} s|r|^2 \left(F_s^{(k)}\right)^2}.$$

i) Since $|r| \geq 1$, by (9) we have

$$\|F_r\|_E \geq \sqrt{\sum_{s=0}^{n-1} (n-s) (F_s^{(k)})^2 + \sum_{s=0}^{n-1} s (F_s^{(k)})^2} = \sqrt{n \sum_{s=0}^{n-1} (F_s^{(k)})^2} \geq F_{n-1}^{(k+1)}.$$

From (7)

$$\|F_r\|_2 \geq \frac{1}{\sqrt{n}} F_{n-1}^{(k+1)}.$$

Now, let the matrices B and C be as

$$B = \begin{bmatrix} rF_0^{(k)} & 1 & 1 & \cdots & 1 & 1 \\ rF_{n-1}^{(k)} & rF_0^{(k)} & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rF_2^{(k)} & rF_3^{(k)} & rF_4^{(k)} & \cdots & rF_0^{(k)} & 1 \\ rF_1^{(k)} & rF_2^{(k)} & rF_3^{(k)} & \cdots & rF_{n-1}^{(k)} & rF_0^{(k)} \end{bmatrix}$$

and

$$C = \begin{bmatrix} F_0^{(k)} & F_1^{(k)} & F_2^{(k)} & \cdots & F_{n-2}^{(k)} & F_{n-1}^{(k)} \\ 1 & F_0^{(k)} & F_1^{(k)} & \cdots & F_{n-3}^{(k)} & F_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & F_0^{(k)} & F_1^{(k)} \\ 1 & 1 & 1 & \cdots & 1 & F_0^{(k)} \end{bmatrix}.$$

That is, $F_r = B \circ C$. Then we obtain

$$r_1(B) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^n |b_{ij}|^2} = \sqrt{\sum_{j=1}^n |b_{nj}|^2} = \sqrt{|r|^2 \sum_{s=0}^{n-1} (F_s^{(k)})^2}$$

and

$$c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^n |c_{ij}|^2} = \sqrt{\sum_{i=1}^n |c_{in}|^2} = \sqrt{\sum_{s=0}^{n-1} (F_s^{(k)})^2}.$$

Hence, from (9) and Lemma 2, we have

$$\|F_r\|_2 \leq r_1(B) c_1(B) \leq |r| (F_{n-1}^{(k+1)})^2.$$

Thus, we write

$$\frac{1}{\sqrt{n}} F_{n-1}^{(k+1)} \leq \|F_r\|_2 \leq |r| (F_{n-1}^{(k+1)})^2.$$

ii) Since $|r| < 1$, by (9) we have

$$\begin{aligned} \|F_r\|_E &= \sqrt{\sum_{s=0}^{n-1} (n-s) \left(F_s^{(k)}\right)^2 + \sum_{s=0}^{n-1} s|r|^2 \left(F_s^{(k)}\right)^2} \\ &\geq \sqrt{\sum_{s=0}^{n-1} (n-s)|r|^2 \left(F_s^{(k)}\right)^2 + \sum_{s=0}^{n-1} s|r|^2 \left(F_s^{(k)}\right)^2} \\ &= |r| \sqrt{n \sum_{s=0}^{n-1} \left(F_s^{(k)}\right)^2} \geq |r| F_{n-1}^{(k+1)}. \end{aligned}$$

From (7)

$$\|F_r\|_2 \geq \frac{|r|}{\sqrt{n}} F_{n-1}^{(k+1)}.$$

Now, let the matrices B and C be as

$$B = \begin{bmatrix} F_0^{(k)} & 1 & 1 & \cdots & 1 & 1 \\ r & F_0^{(k)} & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r & r & r & \cdots & F_0^{(k)} & 1 \\ r & r & r & \cdots & r & F_0^{(k)} \end{bmatrix}$$

and

$$C = \begin{bmatrix} F_0^{(k)} & F_1^{(k)} & F_2^{(k)} & \cdots & F_{n-2}^{(k)} & F_{n-1}^{(k)} \\ F_{n-1}^{(k)} & F_0^{(k)} & F_1^{(k)} & \cdots & F_{n-3}^{(k)} & F_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ F_2^{(k)} & F_3^{(k)} & F_4^{(k)} & \cdots & F_0^{(k)} & F_1^{(k)} \\ F_1^{(k)} & F_2^{(k)} & F_3^{(k)} & \cdots & F_{n-1}^{(k)} & F_0^{(k)} \end{bmatrix}.$$

That is, $F_r = B \circ C$. Then we obtain

$$r_1(B) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^n |b_{ij}|^2} = \sqrt{F_0^{(k)} + n - 1} = \sqrt{n - 1}$$

and

$$c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^n |c_{ij}|^2} = \sqrt{\sum_{s=0}^{n-1} \left(F_s^{(k)}\right)^2}.$$

Hence, from (9) and Lemma 2, we have

$$\|F_r\|_2 \leq r_1(B) c_1(B) \leq \sqrt{n-1} F_{n-1}^{(k+1)}.$$

Thus, we write

$$\frac{|r|}{\sqrt{n}} F_{n-1}^{(k+1)} \leq \|F_r\|_2 \leq \sqrt{n-1} F_{n-1}^{(k+1)}.$$

Thus, the proof is completed. \square

EXAMPLE 3. By using Theorem 3 and the equations in (5), if $|r| \geq 1$, we have

$$\frac{1}{\sqrt{n}}(F_{n+1} - 1) \leq \|F_r\|_2 \leq |r|(F_{n+1} - 1)^2, \text{ if } k = 0,$$

$$\frac{1}{\sqrt{n}}(F_{n+3} - n - 2) \leq \|F_r\|_2 \leq |r|(F_{n+3} - n - 2)^2, \text{ if } k = 1,$$

$$\frac{1}{\sqrt{n}}\left(F_{n+5} - \frac{n^2 + 5n + 10}{2}\right) \leq \|F_r\|_2 \leq |r|\left(F_{n+5} - \frac{n^2 + 5n + 10}{2}\right)^2, \text{ if } k = 2,$$

and if $|r| < 1$, we have

$$\frac{|r|}{\sqrt{n}}(F_{n+1} - 1) \leq \|F_r\|_2 \leq \sqrt{n-1}(F_{n+1} - 1), \text{ if } k = 0,$$

$$\frac{|r|}{\sqrt{n}}(F_{n+3} - n - 2) \leq \|F_r\|_2 \leq \sqrt{n-1}(F_{n+3} - n - 2), \text{ if } k = 1,$$

$$\frac{|r|}{\sqrt{n}}\left(F_{n+5} - \frac{n^2 + 5n + 10}{2}\right) \leq \|F_r\|_2 \leq \sqrt{n-1}\left(F_{n+5} - \frac{n^2 + 5n + 10}{2}\right), \text{ if } k = 2.$$

THEOREM 4. Let $L_r = \text{Circr}(L_0^{(k)}, L_1^{(k)}, \dots, L_{n-1}^{(k)})$ be an r -circulant matrix.

i) If $|r| \geq 1$, then

$$\frac{1}{\sqrt{n}}L_{n-1}^{(k+1)} \leq \|L_r\|_2 \leq |r|(L_{n-1}^{(k+1)})^2.$$

ii) If $|r| < 1$, then

$$\frac{|r|}{\sqrt{n}}L_{n-1}^{(k+1)} \leq \|L_r\|_2 \leq \sqrt{n}L_{n-1}^{(k+1)}.$$

Proof. Since the matrix L_r is of the form

$$L_r = \begin{bmatrix} L_0^{(k)} & L_1^{(k)} & L_2^{(k)} & \cdots & L_{n-2}^{(k)} & L_{n-1}^{(k)} \\ rL_{n-1}^{(k)} & L_0^{(k)} & L_1^{(k)} & \cdots & L_{n-3}^{(k)} & L_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rL_2^{(k)} & rL_3^{(k)} & rL_4^{(k)} & \cdots & L_0^{(k)} & L_1^{(k)} \\ rL_1^{(k)} & rL_2^{(k)} & rL_3^{(k)} & \cdots & rL_{n-1}^{(k)} & L_0^{(k)} \end{bmatrix}$$

and from the definition of Euclidean norm, we have

$$\|L_r\|_E = \sqrt{\sum_{s=0}^{n-1} (n-s) (L_s^{(k)})^2 + \sum_{s=0}^{n-1} s |r|^2 (L_s^{(k)})^2}.$$

i) Since $|r| \geq 1$, by (13) and we have

$$\|L_r\|_E \geq \sqrt{\sum_{s=0}^{n-1} (n-s) \left(L_s^{(k)}\right)^2 + \sum_{s=0}^{n-1} s \left(L_s^{(k)}\right)^2} = \sqrt{n \sum_{s=0}^{n-1} \left(L_s^{(k)}\right)^2} \geq L_{n-1}^{(k+1)}.$$

From (7)

$$\|L_r\|_2 \geq \frac{1}{\sqrt{n}} L_{n-1}^{(k+1)}.$$

Now, let the matrices B and C be as

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ rL_{n-1}^{(k)} & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rL_2^{(k)} & rL_3^{(k)} & rL_4^{(k)} & \cdots & 1 & 1 \\ rL_1^{(k)} & rL_2^{(k)} & rL_3^{(k)} & \cdots & rL_{n-1}^{(k)} & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} L_0^{(k)} & L_1^{(k)} & L_2^{(k)} & \cdots & L_{n-2}^{(k)} & L_{n-1}^{(k)} \\ 1 & L_0^{(k)} & L_1^{(k)} & \cdots & L_{n-3}^{(k)} & L_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & L_0^{(k)} & L_1^{(k)} \\ 1 & 1 & 1 & \cdots & 1 & L_0^{(k)} \end{bmatrix}.$$

That is, $L_r = B \circ C$. Then we obtain

$$\begin{aligned} r_1(B) &= \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^n |b_{ij}|^2} = \sqrt{\sum_{j=1}^n |b_{nj}|^2} = \sqrt{1 + \sum_{s=1}^{n-1} |r|^2 \left(L_s^{(k)}\right)^2} \\ &\leq |r| \sqrt{\sum_{s=0}^{n-1} \left(L_s^{(k)}\right)^2} \end{aligned}$$

and

$$c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^n |c_{ij}|^2} = \sqrt{\sum_{i=1}^n |c_{in}|^2} = \sqrt{\sum_{s=0}^{n-1} \left(L_s^{(k)}\right)^2}.$$

Hence, from (13) and Lemma 2, we have

$$\|L_r\|_2 \leq r_1(B) c_1(B) \leq |r| \left(L_{n-1}^{(k+1)}\right)^2.$$

Thus, we write

$$\frac{1}{\sqrt{n}} L_{n-1}^{(k+1)} \leq \|L_r\|_2 \leq |r| \left(L_{n-1}^{(k+1)}\right)^2.$$

ii) Since $|r| < 1$, by (13) and we have

$$\begin{aligned} \|L_r\|_E &= \sqrt{\sum_{s=0}^{n-1} (n-s) \left(L_s^{(k)}\right)^2 + \sum_{s=0}^{n-1} s |r|^2 \left(L_s^{(k)}\right)^2} \\ &\geq \sqrt{\sum_{s=0}^{n-1} (n-s) |r|^2 \left(L_s^{(k)}\right)^2 + \sum_{s=0}^{n-1} s |r|^2 \left(L_s^{(k)}\right)^2} \\ &= \sqrt{n |r|^2 \sum_{s=0}^{n-1} \left(L_s^{(k)}\right)^2} \geq |r| \left(L_{n-1}^{(k+1)}\right). \end{aligned}$$

From (7)

$$\|L_r\|_2 \geq \frac{|r|}{\sqrt{n}} L_{n-1}^{(k+1)}.$$

Now, let the matrices B and C be as

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ r & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r & r & r & \cdots & 1 & 1 \\ r & r & r & \cdots & r & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} L_0^{(k)} & L_1^{(k)} & L_2^{(k)} & \cdots & L_{n-2}^{(k)} & L_{n-1}^{(k)} \\ L_{n-1}^{(k)} & L_0^{(k)} & L_1^{(k)} & \cdots & L_{n-3}^{(k)} & L_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ L_2^{(k)} & L_3^{(k)} & L_4^{(k)} & \cdots & L_0^{(k)} & L_1^{(k)} \\ L_1^{(k)} & L_2^{(k)} & L_3^{(k)} & \cdots & L_{n-1}^{(k)} & L_0^{(k)} \end{bmatrix}.$$

That is, $L_r = B \circ C$. Then we obtain

$$r_1(B) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^n |b_{ij}|^2} = \sqrt{n}$$

and

$$c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^n |c_{ij}|^2} = \sqrt{\sum_{s=0}^{n-1} \left(L_s^{(k)}\right)^2}.$$

Hence, from (13) and Lemma 2, we have

$$\|L_r\|_2 \leq r_1(B) c_1(B) \leq \sqrt{n} L_{n-1}^{(k+1)}.$$

Thus, we write

$$\frac{|r|}{\sqrt{n}} L_{n-1}^{(k+1)} \leq \|L_r\|_2 \leq \sqrt{n} L_{n-1}^{(k+1)}.$$

This completes the proof. \square

EXAMPLE 4. By using Theorem 4 and the equations in (6), if $|r| \geq 1$, we have

$$\begin{aligned} \frac{1}{\sqrt{n}}(L_{n+1} - 1) &\leq \|L_r\|_2 \leq |r|(L_{n+1} - 1)^2, \text{ if } k = 0, \\ \frac{1}{\sqrt{n}}(L_{n+3} - n - 4) &\leq \|L_r\|_2 \leq |r|(L_{n+3} - n - 4)^2, \text{ if } k = 1, \\ \frac{1}{\sqrt{n}}\left(L_{n+5} - \frac{n^2 + 9n + 22}{2}\right) &\leq \|L_r\|_2 \leq |r|\left(L_{n+5} - \frac{n^2 + 9n + 22}{2}\right)^2, \text{ if } k = 2, \end{aligned}$$

and if $|r| < 1$, we have

$$\begin{aligned} \frac{|r|}{\sqrt{n}}(L_{n+1} - 1) &\leq \|L_r\|_2 \leq \sqrt{n}(L_{n+1} - 1), \text{ if } k = 0, \\ \frac{|r|}{\sqrt{n}}(L_{n+3} - n - 4) &\leq \|L_r\|_2 \leq \sqrt{n}(L_{n+3} - n - 4), \text{ if } k = 1, \\ \frac{|r|}{\sqrt{n}}\left(L_{n+5} - \frac{n^2 + 9n + 22}{2}\right) &\leq \|L_r\|_2 \leq \sqrt{n}\left(L_{n+5} - \frac{n^2 + 9n + 22}{2}\right), \text{ if } k = 2. \end{aligned}$$

COROLLARY 7. *The spectral norm of the Hadamard product of $F_r = \text{Circr}(F_0^{(k)}, F_1^{(k)}, \dots, F_{n-1}^{(k)})$ and $L_r = \text{Circr}(L_0^{(k)}, L_1^{(k)}, \dots, L_{n-1}^{(k)})$ holds*

i) *If $|r| \geq 1$, then*

$$\|F_r \circ L_r\|_2 \leq |r|^2 \left(F_{n-1}^{(k+1)}\right)^2 \left(L_{n-1}^{(k+1)}\right)^2.$$

ii) *If $|r| < 1$, then*

$$\|F_r \circ L_r\|_2 \leq \sqrt{n(n-1)} F_{n-1}^{(k+1)} L_{n-1}^{(k+1)}.$$

Proof. The proof is trivial since $\|F_r \circ L_r\|_2 \leq \|F_r\|_2 \|L_r\|_2$. \square

COROLLARY 8. *The spectral norm of the Kronecker product of $F_r = \text{Circr}(F_0^{(k)}, F_1^{(k)}, \dots, F_{n-1}^{(k)})$ and $L_r = \text{Circr}(L_0^{(k)}, L_1^{(k)}, \dots, L_{n-1}^{(k)})$ holds*

i) *If $|r| \geq 1$, then*

$$\frac{1}{n} F_{n-1}^{(k+1)} L_{n-1}^{(k+1)} \leq \|F_r \otimes L_r\|_2 \leq |r|^2 \left(F_{n-1}^{(k+1)}\right)^2 \left(L_{n-1}^{(k+1)}\right)^2.$$

ii) *If $|r| < 1$, then*

$$\frac{|r|^2}{n} L_{n-1}^{(k+1)} F_{n-1}^{(k+1)} \leq \|F_r \otimes L_r\|_2 \leq \sqrt{n(n-1)} F_{n-1}^{(k+1)} L_{n-1}^{(k+1)}.$$

Proof. The proof is trivial since $\|F_r \otimes L_r\|_2 = \|F_r\|_2 \|L_r\|_2$. \square

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