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ON THE NORMS OF r–CIRCULANT MATRICES WITH THE HYPER–FIBONACCI AND LUCAS NUMBERS

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Abstract. In this paper, we study norms of circulant matrices $F = \mathrm{Circ}(F_0^{(k)},\ F_1^{(k)},\ldots,F_{n-1}^{(k)})$, $L = \mathrm{Circ}(L_0^{(k)},\ L_1^{(k)},\ldots,L_{n-1}^{(k)})$ and r-circulant matrices $F_r = \mathrm{Circr}(F_0^{(k)},\ F_1^{(k)},\ldots,F_{n-1}^{(k)})$, $L_r = \mathrm{Circr}(L_0^{(k)},\ L_1^{(k)},\ldots,L_{n-1}^{(k)})$, where $F_n^{(k)}$ and $L_n^{(k)}$ denote the hyper-Fibonacci and hyper-Lucas numbers, respectively.

1. Introduction

The circulant matrices and r-circulant matrices play important role in signal processing, coding theory, image processing, linear forecast and so on. An $n \times n$ matrix C_r is called an r-circulant matrix if it is of the form

$$C_r = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \ldots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{bmatrix}.$$

The matrix C_r is determined by its first row elements and r, thus we denote $C_r = \mathrm{Circr}\left(c_0, c_1, \ldots, c_{n-1}\right)$. When we take $r = 1$, the matrix $C_1 = C$ is called a circulant matrix. We denote $C_1 = C$ $=\mathrm{Circ}\left(c_0, c_1, \ldots, c_{n-1}\right)$. Circulant matrices are especially tractable class of matrices since their inverses, conjugate transposes, sums and products are also circulant. Moreover, circulant matrices are normal matrices [4]. Also, by means of [4, 9], it is well known that the eigenvalues of C are

$$\underset{0 \leqslant m \leqslant n-1}{\lambda_m} = \sum_{k=0}^{n-1} c_k w^{-mk} \tag{1}$$

where $w = e^{\frac{2\pi i}{n}}$ and $i = \sqrt{-1}$, and the corresponding eigenvectors are

$$\underset{0 \leqslant m \leqslant n-1}{x_m} = \left(1, w^m, w^{2m}, \ldots, w^{(n-1)m}\right)^T. \tag{2}$$

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Recently, there have been many papers on the norms of special matrices with special elements such as Fibonacci and Lucas numbers [1, 2, 8, 11, 14–18, 20]. Solak [16, 17] has computed the spectral and Euclidean norms of circulant matrices with the Fibonacci and Lucas numbers. Shen and Cen [15] have given upper and lower bounds for the spectral norms of r-circulant matrices in the forms $A = C_r(F_0, F_1, \ldots, F_{n-1})$ and $B = C_r(L_0, L_1, \ldots, L_{n-1})$. Yazlik and Taskara [20] have presented upper and lower bounds for the spectral norm of an r-circulant matrix with the generalized k-Horadam numbers. As for us, in this paper, we compute the spectral norms of circulant and r-circulant matrices with the hyper-Fibonacci and hyper-Lucas numbers.

The main contents of this paper are organized as follows: In Section 2, we give some preliminaries, definitions and lemmas related to our study. In Section 3, we derive some bounds for the spectral norms of r-circulant matrices with the hyper-Fibonacci and hyper-Lucas numbers of the forms $F_r = \mathrm{Circr}(F_0^{(k)},\ F_1^{(k)}, \ldots, F_{n-1}^{(k)})$, $L_r = \mathrm{Circr}(L_0^{(k)}, L_1^{(k)}, \ldots, L_{n-1}^{(k)})$ and their Hadamard and Kronecker products. For this, we firstly compute the spectral and Euclidean norms of circulant matrices of the forms $F = \mathrm{Circ}(F_0^{(k)},\ F_1^{(k)}, \ldots, F_{n-1}^{(k)})$ and $L = \mathrm{Circ}(L_0^{(k)},\ L_1^{(k)}, \ldots, L_{n-1}^{(k)})$. Moreover, we give some examples related to special cases of our results.

2. Preliminaries

The sequence of the Fibonacci numbers is one of the most well-known sequences, and it has many applications to different fields such as mathematics, statistics and physics. The Fibonacci numbers are defined by the second order linear recurrence relation: $F_{n+1} = F_n + F_{n-1}\ (n \geqslant 1)$, $F_0 = 0$ and $F_1 = 1$. Similarly, the Lucas numbers are defined by $L_{n+1} = L_n + L_{n-1}\ \ (n \geqslant 1)$, $L_0 = 2$ and $L_1 = 1$. Fibonacci and Lucas numbers have generating functions and many generalizations [3, 5, 10, 12, 13, 19]. In [5], Dil and Mezö introduced new concepts as hyper-Fibonacci numbers and hyper-Lucas numbers. These concepts are defined as

$$F_n^{(k)} = \sum_{s=0}^{n} F_s^{(k-1)}, \ \text{ with } F_n^{(0)} = F_n,\ F_0^{(k)} = 0 \text{ and } F_1^{(k)} = 1 \tag{3}$$

and

$$L_n^{(k)} = \sum_{s=0}^{n} L_s^{(k-1)}, \ \text{ with } L_n^{(0)} = L_n,\ L_0^{(k)} = 2,\ L_1^{(k)} = 2k+1. \tag{4}$$

The hyper-Fibonacci and the hyper-Lucas numbers have the recurrence relations $F_n^{(k)} = F_{n-1}^{(k)} + F_n^{(k-1)}$ and $L_n^{(k)} = L_{n-1}^{(k)} + L_n^{(k-1)}$, respectively. Also, $F_n^{(k)}$ and $L_n^{(k)}$ have the following more explicit forms when $k = 1, 2, 3$.

$$F_n^{(1)} = F_{n+2} - 1,\ \ F_n^{(2)} = F_{n+4} - n - 3 \ \text{ and } \ F_n^{(3)} = F_{n+6} - \frac{n^2 + 7n + 16}{2}, \tag{5}$$

$$L_n^{(1)} = L_{n+2} - 1,\ \ \ L_n^{(2)} = L_{n+4} - n - 5 \ \text{ and } \ L_n^{(3)} = L_{n+6} - \frac{n^2 + 11n + 32}{2}. \tag{6}$$

Now we give some definitions and lemmas related to our study.

DEFINITION 1. Let $A=(a_{ij})$ be any $m\times n$ matrix. The *Euclidean norm* of A is

$$\|A\|_E=\sqrt{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{ij}\right|^2\right)}.$$

DEFINITION 2. Let $A=(a_{ij})$ be any $m\times n$ matrix. The *spectral norm* of A is

$$\|A\|_2=\sqrt{\max_i \lambda_i\left(A^HA\right)},$$

where $\lambda_i\left(A^HA\right)$ are eigenvalues of A^HA and A^H is conjugate transpose of A.

There are two well known relations between Euclidean norm and spectral norm as the following:

$$\frac{1}{\sqrt{n}}\|A\|_E\leqslant\|A\|_2\leqslant\|A\|_E \tag{7}$$

$$\|A\|_2\leqslant\|A\|_E\leqslant\sqrt{n}\,\|A\|_2. \tag{8}$$

DEFINITION 3. Let $A=(a_{ij})$ and $B=(b_{ij})$ be $m\times n$ matrices. Then their Hadamard product A $\circ$ B is defined

$$A\circ B=[a_{ij}b_{ij}].$$

DEFINITION 4. Let $A=(a_{ij})$ and $B=(b_{ij})$ be $m\times n$ and $p\times r$ matrices, respectively. Then their Kronecker product A $\otimes$ B is defined

$$A\otimes B=[a_{ij}B].$$

LEMMA 1. [7] *Let A and B be two $m\times n$ matrices. Then we have*

$$\|A\circ B\|_2\leqslant\|A\|_2\,\|B\|_2.$$

LEMMA 2. [7] *Let A and B be two $m\times n$ matrices. Then we have*

$$\|A\circ B\|_2\leqslant r_1(A)\,c_1(B)$$

where $r_1(A)=\max\limits_{1\leqslant i\leqslant m}\sqrt{\sum\limits_{j=1}^{n}\left|a_{ij}\right|^2}$ *and* $c_1(B)=\max\limits_{1\leqslant j\leqslant n}\sqrt{\sum\limits_{i=1}^{m}\left|b_{ij}\right|^2}$.

LEMMA 3. [7] *Let A and B be two $m\times n$ matrices. Then we have*

$$\|A\otimes B\|_2=\|A\|_2\,\|B\|_2.$$

LEMMA 4. [6] *Let A be an $n\times n$ matrix with eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$. Then, A is a normal matrix if and only if the eigenvalues of A^HA are $|\lambda_1|^2,|\lambda_2|^2,\ldots,|\lambda_n|^2$.*

3. Main results

THEOREM 1. *The spectral norm of the matrix* $F = \mathrm{Circ}(F_0^{(k)},\ F_1^{(k)}, \dots, F_{n-1}^{(k)})$ *is*

$$\|F\|_2 = F_{n-1}^{(k+1)}.$$

Proof. Since the circulant matrix F is normal, its spectral norm is equal to its spectral radius. Furthermore, by considering F is irreducible and its entries are non-negative, we have that the spectral radius (or spectral norm) of the matrix F is equal to its Perron root. We select an n-dimensional column vector $v = (1,1,\dots,1)^T$, then

$$Fv = \left(\sum_{s=0}^{n-1} F_s^{(k)}\right) v.$$

Obviously, $\sum_{s=0}^{n-1} F_s^{(k)}$ is an eigenvalue of F associated with v and it is the Perron root of F. Hence, by (3) we have

$$\|F\|_2 = \sum_{s=0}^{n-1} F_s^{(k)} = F_{n-1}^{(k+1)}.$$

This completes the proof. □

EXAMPLE 1. By using Theorem 1 and the equations in (5), we have

$$\|F\|_2 = \begin{cases} F_{n+1} - 1, & \text{if } k = 0, \\ F_{n+3} - n - 2, & \text{if } k = 1, \\ F_{n+5} - \frac{n^2+5n+10}{2}, & \text{if } k = 2. \end{cases}$$

COROLLARY 1. *For the Euclidean norm of the matrix* $F = \mathrm{Circ}(F_0^{(k)},\ F_1^{(k)}, \dots, F_{n-1}^{(k)})$, *we have*

$$F_{n-1}^{(k+1)} \leqslant \|F\|_E \leqslant \sqrt{n} F_{n-1}^{(k+1)}.$$

Proof. The proof is trivial from Theorem 1 and the relation between spectral norm and Euclidean norm in (8). □

COROLLARY 2. *For the sum of squares of hyper-Fibonacci numbers, we have*

$$\frac{1}{\sqrt{n}} F_{n-1}^{(k+1)} \leqslant \sqrt{\sum_{s=0}^{n-1} \left(F_s^{(k)}\right)^2} \leqslant F_{n-1}^{(k+1)}. \tag{9}$$

Proof. This follows from the definition of Euclidean norm and Corollary 1. □

THEOREM 2. *The spectral norm of the matrix* $L = \mathrm{Circ}(L_0^{(k)},\ L_1^{(k)},\dots,L_{n-1}^{(k)})$ *is*

$$\|L\|_2 = L_{n-1}^{(k+1)}.$$

Proof. This theorem can be proved by using a similar method to method of the proof of Theorem 1. But, we will use another method. Since L is a circulant matrix, from (1) its eigenvalues are of the form

$$\lambda_m_{0\leqslant m\leqslant n-1} = \sum_{s=0}^{n-1} L_s^{(k)} e^{\frac{-2\pi ims}{n}}.$$

Then for $m=0$, by using (4) we have

$$\lambda_0 = \sum_{s=0}^{n-1} L_s^{(k)} = L_{n-1}^{(k+1)}. \tag{10}$$

Also, we have

$$\underset{1\leqslant m\leqslant n-1}{|\lambda_m|} = \left|\sum_{s=0}^{n-1} L_s^{(k)} e^{\frac{-2\pi ims}{n}}\right| \leqslant \sum_{s=0}^{n-1} \left|L_s^{(k)}\right| \left|e^{\frac{-2\pi ims}{n}}\right| \leqslant \sum_{s=0}^{n-1} \left|L_s^{(k)}\right| = \sum_{s=0}^{n-1} L_s^{(k)}. \tag{11}$$

By using Lemma 4 and the fact that the matrix L is a normal matrix, we have

$$\|L\|_2 = \max_{0\leqslant m\leqslant n-1} |\lambda_m| = \max\left(|\lambda_0|,\ \max_{1\leqslant m\leqslant n-1} |\lambda_m|\right). \tag{12}$$

From (10), (11) and (12), we have

$$\|L\|_2 = L_{n-1}^{(k+1)}.$$

Thus the proof is completed. $\square$

EXAMPLE 2. By using Theorem 2 and the equations in (6), we have

$$\|L\|_2 = \begin{cases} L_{n+1} - 1, & \text{if } k=0, \\ L_{n+3} - n - 4, & \text{if } k=1, \\ L_{n+5} - \frac{n^2+9n+22}{2}, & \text{if } k=2. \end{cases}$$

COROLLARY 3. *For the Euclidean norm of the matrix* $L = \mathrm{Circ}(L_0^{(k)},\ L_1^{(k)},\dots, L_{n-1}^{(k)})$, *we have*

$$L_{n-1}^{(k+1)} \leqslant \|L\|_E \leqslant \sqrt{n} L_{n-1}^{(k+1)}.$$

Proof. The proof is trivial from Theorem 2 and the relation between spectral norm and Euclidean norm in (8). $\square$

COROLLARY 4. *For the sum of squares of hyper-Lucas numbers, we have*

$$\frac{1}{\sqrt{n}} L_{n-1}^{(k+1)} \leqslant \sqrt{\sum_{s=0}^{n-1} \left(L_s^{(k)}\right)^2} \leqslant L_{n-1}^{(k+1)}. \tag{13}$$

Proof. This follows from the definition of Euclidean norm and Corollary 3. $\square$

COROLLARY 5. *The spectral norm of the Hadamard product of* $F = \mathrm{Circ}(F_0^{(k)}, F_1^{(k)}, \ldots, F_{n-1}^{(k)})$ *and* $L = \mathrm{Circ}(L_0^{(k)}, L_1^{(k)}, \ldots, L_{n-1}^{(k)})$ *holds*

$$\|F \circ L\|_2 \leqslant F_{n-1}^{(k+1)} L_{n-1}^{(k+1)}.$$

Proof. The proof is trivial since $\|F \circ L\|_2 \leqslant \|F\|_2 \|L\|_2$. $\square$

COROLLARY 6. *The spectral norm of the Kronecker product of* $F = \mathrm{Circ}(F_0^{(k)}, F_1^{(k)}, \ldots, F_{n-1}^{(k)})$ *and* $L = \mathrm{Circ}(L_0^{(k)}, L_1^{(k)}, \ldots, L_{n-1}^{(k)})$ *holds*

$$\|F \otimes L\|_2 = F_{n-1}^{(k+1)} L_{n-1}^{(k+1)}.$$

Proof. The proof is trivial since $\|F \otimes L\|_2 = \|F\|_2 \|L\|_2$. $\square$

THEOREM 3. *Let* $F_r = \mathrm{Circr}(F_0^{(k)}, F_1^{(k)}, \ldots, F_{n-1}^{(k)})$ *be an* r*-circulant matrix.*
i) If $|r| \geqslant 1$, *then*

$$\frac{1}{\sqrt{n}} F_{n-1}^{(k+1)} \leqslant \|F_r\|_2 \leqslant |r| \left(F_{n-1}^{(k+1)}\right)^2$$

ii) If $|r| < 1$, *then*

$$\frac{|r|}{\sqrt{n}} F_{n-1}^{(k+1)} \leqslant \|F_r\|_2 \leqslant \sqrt{n-1} F_{n-1}^{(k+1)}.$$

Proof. Since the matrix F_r is of the form

$$F_r = \begin{bmatrix} F_0^{(k)} & F_1^{(k)} & F_2^{(k)} & \cdots & F_{n-2}^{(k)} & F_{n-1}^{(k)} \\ rF_{n-1}^{(k)} & F_0^{(k)} & F_1^{(k)} & \cdots & F_{n-3}^{(k)} & F_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rF_2^{(k)} & rF_3^{(k)} & rF_4^{(k)} & \cdots & F_0^{(k)} & F_1^{(k)} \\ rF_1^{(k)} & rF_2^{(k)} & rF_3^{(k)} & \cdots & rF_{n-1}^{(k)} & F_0^{(k)} \end{bmatrix}$$

and from the definition of Euclidean norm, we have

$$\|F_r\|_E = \sqrt{\sum_{s=0}^{n-1} (n-s) \left(F_s^{(k)}\right)^2 + \sum_{s=0}^{n-1} s |r|^2 \left(F_s^{(k)}\right)^2}.$$

i) Since $|r| \geqslant 1,$ by (9) we have

$$\|F_r\|_E \geqslant \sqrt{\sum_{s=0}^{n-1}(n-s)\left(F_s^{(k)}\right)^2+\sum_{s=0}^{n-1}s\left(F_s^{(k)}\right)^2}=\sqrt{n\sum_{s=0}^{n-1}\left(F_s^{(k)}\right)^2}\geqslant F_{n-1}^{(k+1)}.$$

From (7)

$$\|F_r\|_2 \geqslant \frac{1}{\sqrt{n}}F_{n-1}^{(k+1)}.$$

Now, let the matrices B and C be as

$$B=\begin{bmatrix} rF_0^{(k)} & 1 & 1 & \cdots & 1 & 1 \\ rF_{n-1}^{(k)} & rF_0^{(k)} & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rF_2^{(k)} & rF_3^{(k)} & rF_4^{(k)} & \cdots & rF_0^{(k)} & 1 \\ rF_1^{(k)} & rF_2^{(k)} & rF_3^{(k)} & \cdots & rF_{n-1}^{(k)} & rF_0^{(k)} \end{bmatrix}$$

and

$$C=\begin{bmatrix} F_0^{(k)} & F_1^{(k)} & F_2^{(k)} & \cdots & F_{n-2}^{(k)} & F_{n-1}^{(k)} \\ 1 & F_0^{(k)} & F_1^{(k)} & \cdots & F_{n-3}^{(k)} & F_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & F_0^{(k)} & F_1^{(k)} \\ 1 & 1 & 1 & \cdots & 1 & F_0^{(k)} \end{bmatrix}.$$

That is, $F_r = B\circ C.$ Then we obtain

$$r_1(B)=\max_{1\leqslant i\leqslant n}\sqrt{\sum_{j=1}^{n}\left|b_{ij}\right|^2}=\sqrt{\sum_{j=1}^{n}\left|b_{nj}\right|^2}=\sqrt{|r|^2\sum_{s=0}^{n-1}\left(F_s^{(k)}\right)^2}$$

and

$$c_1(B)=\max_{1\leqslant j\leqslant n}\sqrt{\sum_{i=1}^{n}\left|c_{ij}\right|^2}=\sqrt{\sum_{i=1}^{n}\left|c_{in}\right|^2}=\sqrt{\sum_{s=0}^{n-1}\left(F_s^{(k)}\right)^2}.$$

Hence, from (9) and Lemma 2, we have

$$\|F_r\|_2 \leqslant r_1(B)\,c_1(B)\leqslant |r|\left(F_{n-1}^{(k+1)}\right)^2.$$

Thus, we write

$$\frac{1}{\sqrt{n}}F_{n-1}^{(k+1)}\leqslant \|F_r\|_2\leqslant |r|\left(F_{n-1}^{(k+1)}\right)^2.$$

ii) Since $|r|<1,$ by (9) we have

$$\begin{aligned}\|F_r\|_E &= \sqrt{\sum_{s=0}^{n-1}(n-s)\left(F_s^{(k)}\right)^2+\sum_{s=0}^{n-1}s\,|r|^2\left(F_s^{(k)}\right)^2}\\ &\geqslant \sqrt{\sum_{s=0}^{n-1}(n-s)\,|r|^2\left(F_s^{(k)}\right)^2+\sum_{s=0}^{n-1}s\,|r|^2\left(F_s^{(k)}\right)^2}\\ &= |r|\sqrt{n\sum_{s=0}^{n-1}\left(F_s^{(k)}\right)^2}\geqslant |r|\,F_{n-1}^{(k+1)}.\end{aligned}$$

From (7)

$$\|F_r\|_2 \geqslant \frac{|r|}{\sqrt{n}}F_{n-1}^{(k+1)}.$$

Now, let the matrices B and C be as

$$B=\begin{bmatrix} F_0^{(k)} & 1 & 1\cdots & 1 & 1\\ r & F_0^{(k)} & 1\cdots & 1 & 1\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ r & r & r\cdots & F_0^{(k)} & 1\\ r & r & r\cdots & r & F_0^{(k)}\end{bmatrix}$$

and

$$C=\begin{bmatrix} F_0^{(k)} & F_1^{(k)} & F_2^{(k)} & \cdots & F_{n-2}^{(k)} & F_{n-1}^{(k)}\\ F_{n-1}^{(k)} & F_0^{(k)} & F_1^{(k)} & \cdots & F_{n-3}^{(k)} & F_{n-2}^{(k)}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ F_2^{(k)} & F_3^{(k)} & F_4^{(k)} & \cdots & F_0^{(k)} & F_1^{(k)}\\ F_1^{(k)} & F_2^{(k)} & F_3^{(k)} & \cdots & F_{n-1}^{(k)} & F_0^{(k)}\end{bmatrix}.$$

That is, $F_r=B\circ C.$ Then we obtain

$$r_1(B)=\max_{1\leqslant i\leqslant n}\sqrt{\sum_{j=1}^{n}\left|b_{ij}\right|^2}=\sqrt{F_0^{(k)}+n-1}=\sqrt{n-1}$$

and

$$c_1(B)=\max_{1\leqslant j\leqslant n}\sqrt{\sum_{i=1}^{n}\left|c_{ij}\right|^2}=\sqrt{\sum_{s=0}^{n-1}\left(F_s^{(k)}\right)^2}.$$

Hence, from (9) and Lemma 2, we have

$$\|F_r\|_2\leqslant r_1(B)\,c_1(B)\leqslant\sqrt{n-1}F_{n-1}^{(k+1)}.$$

Thus, we write

$$\frac{|r|}{\sqrt{n}}F_{n-1}^{(k+1)}\leqslant\|F_r\|_2\leqslant\sqrt{n-1}F_{n-1}^{(k+1)}.$$

Thus, the proof is completed. $\square$

EXAMPLE 3. By using Theorem 3 and the equations in (5), if $|r| \geqslant 1$, we have

$$\frac{1}{\sqrt{n}}(F_{n+1}-1) \leqslant \|F_r\|_2 \leqslant |r|(F_{n+1}-1)^2, \text{ if } k=0,$$

$$\frac{1}{\sqrt{n}}(F_{n+3}-n-2) \leqslant \|F_r\|_2 \leqslant |r|(F_{n+3}-n-2)^2, \text{ if } k=1,$$

$$\frac{1}{\sqrt{n}}\left(F_{n+5}-\frac{n^2+5n+10}{2}\right) \leqslant \|F_r\|_2 \leqslant |r|\left(F_{n+5}-\frac{n^2+5n+10}{2}\right)^2, \text{ if } k=2,$$

and if $|r|<1$, we have

$$\frac{|r|}{\sqrt{n}}(F_{n+1}-1) \leqslant \|F_r\|_2 \leqslant \sqrt{n-1}(F_{n+1}-1), \text{ if } k=0,$$

$$\frac{|r|}{\sqrt{n}}(F_{n+3}-n-2) \leqslant \|F_r\|_2 \leqslant \sqrt{n-1}(F_{n+3}-n-2), \text{ if } k=1,$$

$$\frac{|r|}{\sqrt{n}}\left(F_{n+5}-\frac{n^2+5n+10}{2}\right) \leqslant \|F_r\|_2 \leqslant \sqrt{n-1}\left(F_{n+5}-\frac{n^2+5n+10}{2}\right), \text{ if } k=2.$$

THEOREM 4. *Let* $L_r = \operatorname{Circr}(L_0^{(k)}, L_1^{(k)}, \ldots, L_{n-1}^{(k)})$ *be an* r*-circulant matrix.*
i) If $|r| \geqslant 1$*, then*

$$\frac{1}{\sqrt{n}} L_{n-1}^{(k+1)} \leqslant \|L_r\|_2 \leqslant |r|\left(L_{n-1}^{(k+1)}\right)^2.$$

ii) If $|r|<1$*, then*

$$\frac{|r|}{\sqrt{n}} L_{n-1}^{(k+1)} \leqslant \|L_r\|_2 \leqslant \sqrt{n} L_{n-1}^{(k+1)}.$$

Proof. Since the matrix L_r is of the form

$$L_r = \begin{bmatrix} L_0^{(k)} & L_1^{(k)} & L_2^{(k)} & \cdots & L_{n-2}^{(k)} & L_{n-1}^{(k)} \\ rL_{n-1}^{(k)} & L_0^{(k)} & L_1^{(k)} & \cdots & L_{n-3}^{(k)} & L_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rL_2^{(k)} & rL_3^{(k)} & rL_4^{(k)} & \cdots & L_0^{(k)} & L_1^{(k)} \\ rL_1^{(k)} & rL_2^{(k)} & rL_3^{(k)} & \cdots & rL_{n-1}^{(k)} & L_0^{(k)} \end{bmatrix}$$

and from the definition of Euclidean norm, we have

$$\|L_r\|_E = \sqrt{\sum_{s=0}^{n-1}(n-s)\left(L_s^{(k)}\right)^2 + \sum_{s=0}^{n-1} s|r|^2\left(L_s^{(k)}\right)^2}.$$

i) Since $|r| \geqslant 1$, by (13) and we have

$$\|L_r\|_E \geqslant \sqrt{\sum_{s=0}^{n-1}(n-s)\left(L_s^{(k)}\right)^2 + \sum_{s=0}^{n-1} s\left(L_s^{(k)}\right)^2} = \sqrt{n\sum_{s=0}^{n-1}\left(L_s^{(k)}\right)^2} \geqslant L_{n-1}^{(k+1)}.$$

From (7)

$$\|L_r\|_2 \geqslant \frac{1}{\sqrt{n}} L_{n-1}^{(k+1)}.$$

Now, let the matrices B and C be as

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ rL_{n-1}^{(k)} & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rL_2^{(k)} & rL_3^{(k)} & rL_4^{(k)} & \cdots & 1 & 1 \\ rL_1^{(k)} & rL_2^{(k)} & rL_3^{(k)} & \cdots & rL_{n-1}^{(k)} & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} L_0^{(k)} & L_1^{(k)} & L_2^{(k)} & \cdots & L_{n-2}^{(k)} & L_{n-1}^{(k)} \\ 1 & L_0^{(k)} & L_1^{(k)} & \cdots & L_{n-3}^{(k)} & L_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & L_0^{(k)} & L_1^{(k)} \\ 1 & 1 & 1 & \cdots & 1 & L_0^{(k)} \end{bmatrix}.$$

That is, $L_r = B \circ C$. Then we obtain

$$\begin{aligned} r_1(B) &= \max_{1\leqslant i\leqslant n}\sqrt{\sum_{j=1}^{n}\left|b_{ij}\right|^2} = \sqrt{\sum_{j=1}^{n}\left|b_{nj}\right|^2} = \sqrt{1+\sum_{s=1}^{n-1}|r|^2\left(L_s^{(k)}\right)^2} \\ &\leqslant |r|\sqrt{\sum_{s=0}^{n-1}\left(L_s^{(k)}\right)^2} \end{aligned}$$

and

$$c_1(B) = \max_{1\leqslant j\leqslant n}\sqrt{\sum_{i=1}^{n}\left|c_{ij}\right|^2} = \sqrt{\sum_{i=1}^{n}|c_{in}|^2} = \sqrt{\sum_{s=0}^{n-1}\left(L_s^{(k)}\right)^2}.$$

Hence, from (13) and Lemma 2, we have

$$\|L_r\|_2 \leqslant r_1(B)\,c_1(B) \leqslant |r|\left(L_{n-1}^{(k+1)}\right)^2.$$

Thus, we write

$$\frac{1}{\sqrt{n}} L_{n-1}^{(k+1)} \leqslant \|L_r\|_2 \leqslant |r|\left(L_{n-1}^{(k+1)}\right)^2.$$

ii) Since $|r|<1,$ by (13) and we have

$$\begin{aligned}
\|L_r\|_E &= \sqrt{\sum_{s=0}^{n-1}(n-s)\left(L_s^{(k)}\right)^2+\sum_{s=0}^{n-1}s\,|r|^2\left(L_s^{(k)}\right)^2} \\
&\geqslant \sqrt{\sum_{s=0}^{n-1}(n-s)\,|r|^2\left(L_s^{(k)}\right)^2+\sum_{s=0}^{n-1}s\,|r|^2\left(L_s^{(k)}\right)^2} \\
&= \sqrt{n\,|r|^2\sum_{s=0}^{n-1}\left(L_s^{(k)}\right)^2} \geqslant |r|\left(L_{n-1}^{(k+1)}\right).
\end{aligned}$$

From (7)

$$\|L_r\|_2 \geqslant \frac{|r|}{\sqrt{n}}L_{n-1}^{(k+1)}.$$

Now, let the matrices B and C be as

$$B=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1\\ r & 1 & 1 & \cdots & 1 & 1\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ r & r & r & \cdots & 1 & 1\\ r & r & r & \cdots & r & 1\end{bmatrix}$$

and

$$C=\begin{bmatrix} L_0^{(k)} & L_1^{(k)} & L_2^{(k)} & \cdots & L_{n-2}^{(k)} & L_{n-1}^{(k)}\\ L_{n-1}^{(k)} & L_0^{(k)} & L_1^{(k)} & \cdots & L_{n-3}^{(k)} & L_{n-2}^{(k)}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ L_2^{(k)} & L_3^{(k)} & L_4^{(k)} & \cdots & L_0^{(k)} & L_1^{(k)}\\ L_1^{(k)} & L_2^{(k)} & L_3^{(k)} & \cdots & L_{n-1}^{(k)} & L_0^{(k)}\end{bmatrix}.$$

That is, $L_r=B\circ C.$ Then we obtain

$$r_1(B)=\max_{1\leqslant i\leqslant n}\sqrt{\sum_{j=1}^{n}\left|b_{ij}\right|^2}=\sqrt{n}$$

and

$$c_1(B)=\max_{1\leqslant j\leqslant n}\sqrt{\sum_{i=1}^{n}\left|c_{ij}\right|^2}=\sqrt{\sum_{s=0}^{n-1}\left(L_s^{(k)}\right)^2}.$$

Hence, from (13) and Lemma 2, we have

$$\|L_r\|_2\leqslant r_1(B)\,c_1(B)\leqslant\sqrt{n}L_{n-1}^{(k+1)}.$$

Thus, we write

$$\frac{|r|}{\sqrt{n}}L_{n-1}^{(k+1)}\leqslant\|L_r\|_2\leqslant\sqrt{n}L_{n-1}^{(k+1)}.$$

This completes the proof. □

EXAMPLE 4. By using Theorem 4 and the equations in (6), if $|r| \geqslant 1$, we have

$$\frac{1}{\sqrt{n}}\left(L_{n+1}-1\right) \leqslant \|L_r\|_2 \leqslant |r|\left(L_{n+1}-1\right)^2, \text{ if } k=0,$$

$$\frac{1}{\sqrt{n}}\left(L_{n+3}-n-4\right) \leqslant \|L_r\|_2 \leqslant |r|\left(L_{n+3}-n-4\right)^2, \text{ if } k=1,$$

$$\frac{1}{\sqrt{n}}\left(L_{n+5}-\frac{n^2+9n+22}{2}\right) \leqslant \|L_r\|_2 \leqslant |r|\left(L_{n+5}-\frac{n^2+9n+22}{2}\right)^2, \text{ if } k=2,$$

and if $|r| < 1$, we have

$$\frac{|r|}{\sqrt{n}}\left(L_{n+1}-1\right) \leqslant \|L_r\|_2 \leqslant \sqrt{n}\left(L_{n+1}-1\right), \text{ if } k=0,$$

$$\frac{|r|}{\sqrt{n}}\left(L_{n+3}-n-4\right) \leqslant \|L_r\|_2 \leqslant \sqrt{n}\left(L_{n+3}-n-4\right), \text{ if } k=1,$$

$$\frac{|r|}{\sqrt{n}}\left(L_{n+5}-\frac{n^2+9n+22}{2}\right) \leqslant \|L_r\|_2 \leqslant \sqrt{n}\left(L_{n+5}-\frac{n^2+9n+22}{2}\right), \text{ if } k=2.$$

COROLLARY 7. *The spectral norm of the Hadamard product of* $F_r = \mathrm{Circr}(F_0^{(k)}, F_1^{(k)}, \ldots, F_{n-1}^{(k)})$ *and* $L_r = \mathrm{Circr}(L_0^{(k)}, L_1^{(k)}, \ldots, L_{n-1}^{(k)})$ *holds*

i) If $|r| \geqslant 1$*, then*

$$\|F_r \circ L_r\|_2 \leqslant |r|^2 \left(F_{n-1}^{(k+1)}\right)^2 \left(L_{n-1}^{(k+1)}\right)^2.$$

ii) If $|r| < 1$*, then*

$$\|F_r \circ L_r\|_2 \leqslant \sqrt{n(n-1)}F_{n-1}^{(k+1)}L_{n-1}^{(k+1)}.$$

Proof. The proof is trivial since $\|F_r \circ L_r\|_2 \leqslant \|F_r\|_2 \|L_r\|_2$. □

COROLLARY 8. *The spectral norm of the Kronecker product of* $F_r = \mathrm{Circr}(F_0^{(k)}, F_1^{(k)}, \ldots, F_{n-1}^{(k)})$ *and* $L_r = \mathrm{Circr}(L_0^{(k)}, L_1^{(k)}, \ldots, L_{n-1}^{(k)})$ *holds*

i) If $|r| \geqslant 1$*, then*

$$\frac{1}{n}F_{n-1}^{(k+1)}L_{n-1}^{(k+1)} \leqslant \|F_r \otimes L_r\|_2 \leqslant |r|^2 \left(F_{n-1}^{(k+1)}\right)^2 \left(L_{n-1}^{(k+1)}\right)^2.$$

ii) If $|r| < 1$*, then*

$$\frac{|r|^2}{n}L_{n-1}^{(k+1)}F_{n-1}^{(k+1)} \leqslant \|F_r \otimes L_r\|_2 \leqslant \sqrt{n(n-1)}F_{n-1}^{(k+1)}L_{n-1}^{(k+1)}.$$

Proof. The proof is trivial since $\|F_r \otimes L_r\|_2 = \|F_r\|_2 \|L_r\|_2$. □

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