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ASYMPTOTIC EXPANSIONS OF BIVARIATE CLASSICAL MEANS AND RELATED INEQUALITIES

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Abstract. The subject of this paper are classical bivariate means and theirs behaviour for translated values of the arguments. The asymptotic expansions for eleven means are derived, with respect to the shift variable. This approach enables better understanding of various relations between these means.

1. Introduction

Classical means are subject of intensive study over decades. There are many papers and monographs on this subject, let us mention only the well known textbooks [4, 5, 15]. In recent years a dozens of papers are concerned to the question of comparison of various means. In this paper we shall consider various classical bivariate means and study theirs behaviour for large values of arguments. Our intention is to develop the theory of asymptotic inequalities between means, as a very efficient tool for establishing inequalities between these means.

To be more precise, we will first fix notation for considered means. Let us denote

$$
\begin{aligned}
N(s,t) &= \frac{s^2+t^2}{s+t}, & L(s,t) &= \frac{t-s}{\log t - \log s},\\
Q(s,t) &= \sqrt{\frac{s^2+t^2}{2}}, & G(s,t) &= \sqrt{st},\\
C(s,t) &= \frac{2}{3}\cdot\frac{s^2+st+t^2}{s+t}, & cL(s,t) &= \frac{\log t - \log s}{\frac{1}{s}-\frac{1}{t}},\\
A(s,t) &= \frac{s+t}{2}, & cI(s,t) &= e\left(\frac{s^t}{t^s}\right)^{\frac{1}{t-s}},\\
I(s,t) &= \frac{1}{e}\left(\frac{t^t}{s^s}\right)^{\frac{1}{t-s}}, & H(s,t) &= \frac{2st}{s+t},
\end{aligned}
$$

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$$He(s,t) = \frac{s+\sqrt{st}+t}{3},$$

The names of these means are: contraharmonic mean (N), quadratic mean (Q), centroidal mean (C), arithmetic mean (A), identric mean (I), Heronian mean (He), logarithmic mean (L), geometric mean (G), cologarithmic mean (cL), coidentric mean (cI), harmonic mean (H).

These means are written down in falling order with respect to their values, it is well known that it holds

$$N \geqslant Q \geqslant C \geqslant A \geqslant I \geqslant He \geqslant L \geqslant G \geqslant cL \geqslant cI \geqslant H.$$

If $M(s,t)$ is any of the means above, our intention is to find asymptotic expansion for the function $x \mapsto M(x+s,x+t)$ as $x \to \infty$.

For all x it holds

$$A(x+s,x+t) = x + A(s,t),$$

the asymptotic expansion for this mean has only these two terms. Since arguments $x+s$ and $x+t$ become close as $x \to \infty$, and

$$H(x+s,x+t) - x = \frac{(s+t)x+2st}{2x+s+t} \to A(s,t),$$

$$N(x+s,x+t) - x = \frac{(s+t)x+s^2+t^2}{2x+s+t} \to A(s,t)$$

all means in the scale above become close one to another and for any of them we have

$$M(x+s,x+t) \sim x + \frac{s+t}{2} + O(x^{-1}), \qquad \text{as } x \to \infty. \tag{1.1}$$

Such problems are studied in papers [2, 3, 14] where this property is proved for some n-variable means. Let us note that this asymptotic behaviour need not to be satisfied for every mean, since $\min(t,s)$ and $\max(t,s)$ are also means. Another example is the mean

$$F(s,t) = \log \frac{e^t - e^s}{t-s}$$

(diferential or integral mean of the exponential function) since for all x we have

$$F(x+s,x+t) = x + F(s,t).$$

Therefore, we expect to obtain asymptotic expansions of the form

$$M(x+s,x+t) = x + \frac{s+t}{2} + \sum_{n=2}^{\infty} c_n(t,s) x^{-n+1}$$

for any of the mean from the list above. It turns out that $c_n(t,s)$ are homogeneous polynomials of degree n. According to obtained expansions, the theory of asymptotic expansions ensures that the following inequality is valid

$$M(x+s,x+t) \lesseqgtr x + \frac{s+t}{2} + \frac{c_2(t,s)}{x}$$

for large values of argument x. We shall show that these inequalities are in fact satisfied for all positive values of x, s and t, for each mean from selected list given above.

The main advantage of such expansions is however to establish an efficient algorithm for the comparison of means.

The better notation will be obtained if we introduce the following variables instead of s and t:

$$\alpha = \frac{t+s}{2}, \qquad \beta = \frac{t-s}{2}.$$

Then $t = \alpha + \beta$ and $s = \alpha - \beta$. Let us denote also

$$\gamma = st = \alpha^2 - \beta^2, \qquad \delta = \frac{s^2+t^2}{2} = \alpha^2 + \beta^2.$$

In all examples, the asymptotic expansions and related inequalities will be stated in terms of α and β.

All means considered here are symmetric, therefore, one may suppose that $t > s$. In the whole paper, except in the Section 7, we shall assume $s > 0$. This is equivalent to $\alpha > \beta$.

We shall derive an asymptotic expansion for each of the means mentioned in the introduction, and prove related inequalities for the well known ones: geometric, quadratic, harmonic, logarithmic and identric mean.

2. Lemmas

In this section the general result concerning composition of asymptotic series is given.

The following lemmas about functional transformations of asymptotic series goes back to Euler's work. See e.g. [13] for the explanation in the case of Taylor series, and [12]:

LEMMA 2.1. *Let $g(x)$ be a function with asymptotic expansion*

$$g(x) \sim \sum_{n=0}^{\infty} a_n x^{-n},$$

where $a_0 = 1$. Then for all real p it holds

$$g(x)^p \sim \sum_{n=0}^{\infty} P_n(p) x^{-n},$$

where

$$\begin{aligned} &P_0(p) = 1, \\ &P_n(p) = \frac{1}{n}\sum_{k=1}^{n}[k(1+p)-n]a_k P_{n-k}(p). \end{aligned} \tag{2.1}$$

LEMMA 2.2. *Let*

$$g(x)=\sum_{n=1}^{\infty}a_nx^{-n}$$

be a given asymptotic expansion. Then the composition $\exp(g(x))$ *has asymptotic expansion of the following form*

$$\exp(g(x))=\sum_{n=0}^{\infty}b_nx^{-n}$$

where $b_0=1$ *and*

$$b_n=\frac{1}{n}\sum_{k=1}^{n}ka_kb_{n-k},\qquad n\geqslant 1. \tag{2.2}$$

3. Geometric, quadratic and Heronian means

Let us consider first geometric and quadratic mean. They both have similar asymptotic expansion and can be treated together. Since

$$G(x+s,x+t)=\sqrt{(x+s)(x+t)}=x\left(1+\frac{2\alpha}{x}+\frac{\gamma}{x^2}\right)^{1/2}, \tag{3.1}$$

using algorithm in Lemma 2.1 we get the following result.

THEOREM 3.1. *Geometric mean has the following asymptotic expansion*

$$G(x+s,x+t)=x\cdot\sum_{n=0}^{\infty}c_nx^{-n}$$

where $c_0=1$, $c_1=\alpha$ *and*

$$c_n=\left(\frac{3}{n}-2\right)\alpha c_{n-1}+\left(\frac{3}{n}-1\right)(\alpha^2-\beta^2)c_{n-2},\qquad n\geqslant 2$$

The first few coefficients are

$$\begin{aligned}
c_0&=1,\\
c_1&=\alpha,\\
c_2&=-\tfrac{1}{2}\beta^2,\\
c_3&=\tfrac{1}{2}\alpha\beta^2,\\
c_4&=-\tfrac{1}{8}\beta^2(4\alpha^2+\beta^2),\\
c_5&=\tfrac{1}{8}\alpha\beta^2(4\alpha^2+3\beta^2),\\
c_6&=-\tfrac{1}{16}\beta^2(8\alpha^4+12\alpha^2\beta^2+\beta^4).
\end{aligned}$$

THEOREM 3.2. *The following inequality is satisfied for all* $x>0$*:*

$$G(x+s,x+t) > x+\alpha-\frac{\beta^2}{2x}. \tag{3.2}$$

Proof. If the right side of the inequality (3.2) is positive, then this inequality is equivalent to

$$(x+s)(x+t) > \left(x+\alpha-\frac{\beta^2}{2x}\right)^2$$

which is equivalent to

$$4\alpha x > \beta^2. \tag{3.3}$$

It is enough to prove that the right side of (3.2) is negative if (3.3) is not satisfied. First, $x>0$ and

$$x+\alpha-\frac{\beta^2}{2x} > 0$$

is equivalent to

$$x > \frac{\alpha}{2}\left[-1+\sqrt{1+\frac{2\beta^2}{\alpha^2}}\right].$$

Since $\alpha>\beta$ this implies

$$\sqrt{1+\frac{2\beta^2}{\alpha^2}} = \sqrt{\left(1+\frac{\beta^2}{2\alpha^2}\right)^2+\frac{\beta^2}{\alpha^2}\left(1-\frac{\beta^2}{4\alpha^2}\right)} > 1+\frac{\beta^2}{2\alpha^2}.$$

Therefore,

$$x > \frac{\alpha}{2}\left[-1+\sqrt{1+\frac{2\beta^2}{\alpha^2}}\right] > \frac{\beta^2}{4\alpha}$$

which proves the theorem.

In the case of quadratic mean, the function can be written in the form

$$Q(x+s,x+t) = \sqrt{\frac{(x+s)^2+(x+t)^2}{2}} = x\left(1+\frac{2\alpha}{x}+\frac{\delta}{x^2}\right)^{1/2} \tag{3.4}$$

and the same reasoning as before leads to the following results.

THEOREM 3.3. *Quadratic mean has the following asymptotic expansion*

$$Q(x+s,x+t) = x\cdot\sum_{n=0}^{\infty} c_n x^{-n}$$

where $c_0=1$, $c_1=\alpha$ *and*

$$c_n = \left(\frac{3}{n}-2\right)\alpha c_{n-1} + \left(\frac{3}{n}-1\right)(\alpha^2+\beta^2)c_{n-2}, \qquad n\geqslant 2.$$

The first few coefficients are

$$\begin{aligned}
c_0 &= 1,\\
c_1 &= \alpha,\\
c_2 &= \tfrac{1}{2}\beta^2,\\
c_3 &= -\tfrac{1}{2}\alpha\beta^2,\\
c_4 &= \tfrac{1}{8}\beta^2(4\alpha^2-\beta^2),\\
c_5 &= -\tfrac{1}{8}\alpha\beta^2(4\alpha^2-3\beta^2),\\
c_6 &= \tfrac{1}{16}\beta^2(8\alpha^4-12\alpha^2\beta^2+\beta^4).
\end{aligned}$$

The proof of the following theorem is obvious.

THEOREM 3.4. *The following inequality is satisfied for all* $x>0$,

$$Q(x+s,x+t) < x+\alpha+\frac{\beta^2}{2x}. \tag{3.5}$$

Heronian mean can be written as

$$He(s,t) = \tfrac{2}{3}A(s,t)+\tfrac{1}{3}G(s,t),$$

hence, its coefficients for $n \geqslant 2$ equal one third of the coefficients of geometrical mean.

4. Harmonic, centroidal and contraharmonic means

For the mentioned means, an explicit formula for coefficients of asymptotic expansions can be derived.

THEOREM 4.1. *Harmonic, contraharmonic and centroidal means have the following expansions*

$$H(x+s,x+t) = x+\alpha-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\beta^2\alpha^{n-1}}{x^n}, \tag{4.1}$$

$$N(x+s,x+t) = x+\alpha+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\beta^2\alpha^{n-1}}{x^n}, \tag{4.2}$$

$$C(x+s,x+t) = x+\alpha+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\beta^2\alpha^{n-1}}{3x^n} \tag{4.3}$$

Proof. We can write

$$\begin{aligned}
H(x+s,x+t) &= \frac{x^2+2\alpha x+\gamma}{x+\alpha}\\
&= x+\alpha+\frac{\gamma-\alpha^2}{x+\alpha}
\end{aligned}$$

$$= x + \alpha - \frac{\beta^2}{x+\alpha}$$
$$= x + \alpha - \frac{\beta^2}{x}\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^n}{x^n}$$

and (4.1) follows.

For the counterharmonic mean, it holds

$$N(x+s,x+t) = \frac{x^2+2\alpha x+\delta}{x+\alpha}$$
$$= x+\alpha+\frac{\delta-\alpha^2}{x+\alpha} = x+\alpha+\frac{\beta^2}{x+\alpha}$$

and the rest follows as before. The centroidal mean can be written as

$$C(s,t) = \tfrac{2}{3}N(s,t) + \tfrac{1}{3}H(s,t)$$

and the assertion follows from previous two expansions.

THEOREM 4.2. *The following inequality is fulfilled for all x:*

$$H(x+s,x+t) > x+\alpha-\frac{\beta^2}{x}. \tag{4.4}$$

The proof is easy.

5. Logarithmic and cologarithmic mean

Let us denote

$$S_n = t^n - s^n = (\alpha+\beta)^n - (\alpha-\beta)^n. \tag{5.1}$$

The coefficients of asymptotic expansion of the logarithmic mean is given by recursive formula:

THEOREM 5.1. *Logarithmic mean has the following asymptotic expansion*

$$L(x+s,x+t) = x\cdot\sum_{n=0}^{\infty} c_n x^{-n}$$

where $c_0 = 1$ and

$$c_n = \sum_{k=1}^{n}(-1)^{k-1}\frac{S_{k+1}}{2(k+1)\beta}c_{n-k}, \qquad n \geqslant 1.$$

Proof. We have

$$L(x+t,x+s) = \frac{t-s}{\log(x+t)-\log(x+s)}$$

$$= x\left(\sum_{n=0}^{\infty}(-1)^n \frac{(\alpha+\beta)^{n+1}-(\alpha-\beta)^{n+1}}{2(n+1)\beta x^n}\right)^{-1}.$$

The assertion follows using algorithm in Lemma 2.1, with $p=-1$.

The first few coefficients are

$$\begin{aligned}
c_0 &= 1,\\
c_1 &= \alpha,\\
c_2 &= -\tfrac{1}{3}\beta^2,\\
c_3 &= \tfrac{1}{3}\alpha\beta^2,\\
c_4 &= -\tfrac{1}{45}\beta^2(15\alpha^2+4\beta^2),\\
c_5 &= \tfrac{1}{15}\alpha\beta^2(5\alpha^2+4\beta^2),\\
c_6 &= -\tfrac{1}{945}\beta^2(315\alpha^4+504\alpha^2\beta^2+44\beta^4).
\end{aligned}$$

THEOREM 5.2. *The following inequality is fulfilled for all $x>0$:*

$$L(x+s,x+t) > x+\alpha-\frac{\beta^2}{3x}. \tag{5.2}$$

Proof. Let us suppose that the right side of (5.2) is positive. (If this is not the case, then the inequality is fullfiled.) Then (5.2) can be written in equivalent form

$$f(x) = \frac{\log(x+\alpha+\beta)-\log(x+\alpha-\beta)}{2\beta} - \frac{1}{x+\alpha-\dfrac{\beta^2}{3x}} < 0.$$

Let us prove that the function f is increasing. Since $\lim_{x\to\infty} f(x) = 0$, we conclude that f must be negative for all x. Since

$$f'(x) = \frac{3\alpha(4x+\alpha)\beta^2-4\beta^4}{[(x+\alpha)^2-\beta^2][-3x(x+\alpha)+\beta^2]^2},$$

this function is positive if and only if

$$x > \frac{4\beta^2-3\alpha^2}{12\alpha}. \tag{5.3}$$

To finish the proof, we must show that the right side of (5.2) is negative if (5.3) is not satisfied. For $x>0$

$$x+\alpha-\frac{\beta^2}{3x} > 0$$

is equivalent to

$$x > -\frac{\alpha}{2}\left[1-\sqrt{1+\frac{4\beta^2}{3\alpha^2}}\right].$$

Now, we have for $\alpha > \beta$

$$\sqrt{1+\frac{4\beta^2}{3\alpha^2}} = \sqrt{\left(1+\frac{\beta^2}{6\alpha^2}\right)^2 + \frac{\beta^2}{\alpha^2}\left(1-\frac{\beta^2}{36\alpha^2}\right)} > 1-\frac{\beta^2}{6\alpha^2}.$$

Therefore, if the right side of (5.2) is positive, then the following must be satisfied:

$$x > \frac{\beta^2}{12\alpha} > \frac{4\beta^2-3\alpha^2}{12\alpha},$$

and the theorem is proved.

The coefficients for the cologarithmic mean can be obtained by explicit formula:

THEOREM 5.3. *Cologarithmic mean has the following asymptotic expansion*

$$cL(x+s,x+t) = x+\alpha+\sum_{n=1}^{\infty}\frac{(-1)^n}{\beta(n+2)}\left[\frac{\alpha S_{n+1}}{n+1}-\frac{(\alpha^2-\beta^2)S_n}{n}\right]x^{-n}, \qquad n \geqslant 1. \quad (5.4)$$

Proof. We have

$$\begin{aligned}
cL(x+s,x+t) &= \frac{\log(x+t)-\log(x+s)}{t-s}(x+t)(x+s)\\
&= \left(x^2+2\alpha x+\gamma\right)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{S_n}{2\beta n}x^{-n}\\
&= x+\alpha+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2\beta}\left[\frac{S_{n+2}}{n+2}-2\alpha\frac{S_{n+1}}{n+1}+\gamma\frac{S_n}{n}\right]x^{-n}.
\end{aligned}$$

Now we can use

$$S_{n+2} = 2\alpha S_{n+1} - \gamma S_n$$

and after reducing, we obtain (5.4).

The first few coeficients are

$$\begin{aligned}
c_0 &= 1,\\
c_1 &= \alpha,\\
c_2 &= -\tfrac{2}{3}\beta^2,\\
c_3 &= \tfrac{2}{3}\alpha\beta^2,\\
c_4 &= -\tfrac{2}{15}\beta^2(5\alpha^2+\beta^2),\\
c_5 &= \tfrac{2}{15}\alpha\beta^2(5\alpha^2+3\beta^2),\\
c_6 &= -\tfrac{2}{105}\beta^2(35\alpha^4+42\alpha^2\beta^2+3\beta^4).
\end{aligned}$$

6. Identric and coidentric mean

The last two of these classical means from our list are identric and coidentric mean.

THEOREM 6.1. *The identric mean has the following asymptotic expansion*

$$I(x+s,x+t) = x\left(\sum_{n=0}^{\infty} c_n x^{-n}\right),$$

where coefficients (c_n) *are given by* $c_0 = 1$ *and*

$$c_n = \frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}\frac{S_{k+1}}{2(k+1)\beta}c_{n-k}. \tag{6.1}$$

Proof. From the definition of identric mean one can write

$$I(x+s,x+t) = \frac{1}{e}\left(\frac{(x+t)^{x+t}}{(x+s)^{x+s}}\right)^{\frac{1}{t-s}}.$$

In order to expand this function into asymptotic series, the logarithm of the mean will be used:

$$\begin{aligned} g(x) &= \log I(x+s,x+t) \\ &= \frac{(x+t)\log(x+t)-(x+s)\log(x+s)}{t-s} - 1 \\ &= \log x + \frac{1}{t-s}\left[(x+t)\log\left(1+\frac{t}{x}\right) - (x+s)\log\left(1+\frac{s}{x}\right)\right] - 1 \\ &= \log x + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^{n+1}-s^{n+1}}{n(n+1)(t-s)}x^{-n}. \end{aligned}$$

Now we can apply algorithm (2.2) from Lemma 2.2.

The first few coefficients are

$$\begin{aligned} c_0 &= 1, \\ c_1 &= \alpha, \\ c_2 &= -\tfrac{1}{6}\beta^2, \\ c_3 &= \tfrac{1}{6}\alpha\beta^2, \\ c_4 &= -\tfrac{1}{360}\beta^2(60\alpha^2+13\beta^2), \\ c_5 &= \tfrac{1}{120}\alpha\beta^2(20\alpha^2+13\beta^2), \\ c_6 &= -\tfrac{1}{45360}\beta^2(7560\alpha^4+9828\alpha^2\beta^2+737\beta^4). \end{aligned}$$

THEOREM 6.2. *The following inequality is fulfilled for all* $x > 0$*:*

$$I(x+s,x+t) > x+\alpha-\frac{\beta^2}{6x}. \tag{6.2}$$

Proof. Suppose that the right side of (6.2) is positive. Let us define

$$f(x)=\log(I(x+s,x+t))-\log\left(x+\alpha-\frac{\beta^2}{6x}\right).$$

Then (6.2) is equivalent to $f(x)>0$. Now,

$$f'(x)=\frac{\log(x+t)-\log(x+s)}{t-s}-\frac{6x^2+\beta^2}{x(6x^2+6x\alpha-\beta^2)}.$$

and

$$f''(x)=\frac{\beta^2(72x^3\alpha-\alpha^2\beta^2+\beta^4+x^2(48\alpha^2-26\beta^2)+2x(6\alpha^3-7\alpha\beta^2))}{x^2(x+\alpha-\beta)(x+\alpha+\beta)(6x^2+6x\alpha-\beta^2)^2}$$

The nominator of this fraction is positive because the expression into parenthesis is equal to:

$$47\alpha x^3+\alpha x(5x-2\alpha)^2+(26x^2+2\alpha x)(\alpha^2-\beta^2)+(6x\alpha-\beta^2)^2+\alpha^2(6x^2+6\alpha x-\beta^2)>0.$$

It is easy to see that

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}f'(x)=0.$$

Therefore, $f''(x)>0$ for all x implies that f' is increasing with limit 0, so, $f'(x)<0$ for all x. Now, the same argument implies that $f(x)>0$ for all x, which has to be proved.

THEOREM 6.3. *The coidentric mean has the following asymptotic expansion*

$$cI(x+s,x+t)=x\left(\sum_{n=0}^{\infty}c_nx^{-n}\right),$$

where coefficients (c_n) *are given by* $c_0=1$ *and*

$$c_n=\frac{1}{n}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2(k+1)\beta}[2k\alpha S_k-(2k+1)(\alpha^2-\beta^2)S_{k-1}]c_{n-k}. \tag{6.3}$$

Proof. Using the same technique as in the proof of previous theorem, one get

$$\log cI(x+s,x+t)=\log x+\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{t^{n+1}-s^{n+1}}{(n+1)(t-s)}-ts\frac{t^{n-1}-s^{n-1}}{n(t-s)}\right]\frac{1}{x^n}$$

Now, from (5.1) this can be written as

$$\log cI(x+s,x+t)=\log x+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n(n+1)\beta}[2n\alpha S_n-(2n+1)(\alpha^2-\beta^2)S_{n-1}]\frac{1}{x^n}.$$

The rest follows from Lema 2.2.

The first few coefficients are

$$
\begin{aligned}
c_0 &= 1,\\
c_1 &= \alpha,\\
c_2 &= -\tfrac{5}{6}\beta^2,\\
c_3 &= \tfrac{5}{6}\alpha\beta^2,\\
c_4 &= -\tfrac{1}{360}\beta^2(300\alpha^2+37\beta^2),\\
c_5 &= \tfrac{1}{120}\alpha\beta^2(100\alpha^2+37\beta^2),\\
c_6 &= -\tfrac{1}{45360}\beta^2(37800\alpha^4+27972\alpha^2\beta^2+1405\beta^4).
\end{aligned}
$$

7. The case $\alpha \leqslant 0$

The condition $s > 0$ is the natural one when the mean $M(s,t)$ is considered. But, in the expression $M(x+s,x+t)$ this condition is not necessary. In fact, allowing negative values for the first argument will simplify many calculations and comparison between means. Let us see why.

The main expansion has the form

$$M(x+s,x+t) \sim x+\alpha+\sum_{n=0}^{\infty} c_{n+2}(\alpha,\beta)x^{-n-1} \tag{7.1}$$

and this relation is valid for all values of s and t, not only for the positive ones. This can be written as

$$M(x+\alpha-\beta,x+\alpha+\beta) \sim x+\alpha+\sum_{n=0}^{\infty} c_{n+2}(\alpha,\beta)x^{-n-1}$$

Now, choose in (7.1) $s=-\beta$, $t=\beta$. Then corresponding α is equal to zero, so

$$M(x-\beta,x+\beta) \sim x+\sum_{n=0}^{\infty} c_{n+2}(0,\beta)x^{-n-1},$$

and this holds for any sufficiently large x, hence, also for $x+\alpha$:

$$M(x+\alpha-\beta,x+\alpha+\beta) \sim x+\alpha+\sum_{n=0}^{\infty} c_{n+2}(0,\beta)(x+\alpha)^{-n-1}.$$

THEOREM 7.1. *The coefficients (c_n) of the asymptotic expansion of mean M satisfy*

$$c_{n+2}(\alpha,\beta) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\alpha^{n-k}c_{k+2}(0,\beta), \qquad n \geqslant 0. \tag{7.2}$$

Proof. From the analysis above, one can write

$$\begin{aligned}
\sum_{n=0}^{\infty} c_{n+2}(\alpha,\beta)x^{-n-1} &= \sum_{n=0}^{\infty} c_{n+2}(0,\beta)(x+\alpha)^{-n-1} \\
&= \sum_{n=0}^{\infty} c_{n+2}(0,\beta)\sum_{k=0}^{\infty}(-1)^k\binom{n+k}{k}\alpha^k x^{-n-1-k} \\
&= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\alpha^{n-k}c_{k+2}(0,\beta)\Big)x^{-n-1},
\end{aligned}$$

which has to be proved.

Therefore, asymptotic expansion of the general case can be restored from the asymptotic expansion in the case $\alpha = 0$. These expansions are much simpler, and reveals another properties of means. Let us write some of them:

1. Geometric mean. We can write now explicit formulae for the coefficients,

$$\begin{aligned}
G(x-\beta,x+\beta) &= x - \sum_{n=1}^{\infty}\frac{(2n-3)!!}{2^n n!}\beta^{2n}x^{-2n+1} \\
&= x - \frac{\beta^2}{2}x^{-1} - \frac{\beta^4}{8}x^{-3} - \frac{\beta^6}{16}x^{-5} - \frac{5\beta^8}{128}x^{-7} - \cdots
\end{aligned}$$

We see that all coeficients c_n are negative, for $n \geqslant 2$. From this, and (7.2) we conclude that these coefficients remain negative if $\alpha < 0$.

2. Quadratic mean. The expansion is similar,

$$\begin{aligned}
Q(x-\beta,x+\beta) &= x + \sum_{n=1}^{\infty}\frac{(2n-3)!!}{2^n n!}(-1)^{n-1}\beta^{2n}x^{-2n+1} \\
&= x + \frac{\beta^2}{2}x^{-1} - \frac{\beta^4}{8}x^{-3} + \frac{\beta^6}{16}x^{-5} - \frac{\beta^8}{128}x^{-7} + \cdots
\end{aligned}$$

Now, the coefficients are alternating, and remains such for $\alpha > 0$.

3. Harmonic mean. The expansion collapses:

$$H(x-\beta,x+\beta) = x - \frac{\beta^2}{x}.$$

4. Logarithmic mean.

$$L(x-\beta,x+\beta) = x\sum_{n=0}^{\infty}c_n\beta^n x^{-n}$$

where $c_0 = 1$, $c_{2n+1} = 0$, for $n \geqslant 0$ and

$$c_{2n} = -\sum_{k=1}^{n}\frac{1}{2k+1}c_{2n-2k}$$

Hence,

$$L(x-\beta,x+\beta) = x - \frac{\beta^2}{3}x^{-1} - \frac{4\beta^4}{45}x^{-3} - \frac{44\beta^6}{945}x^{-5} - \frac{428\beta^8}{14175}x^{-7} - \ldots$$

The non-nul coefficients are negative. Using (7.2) we conclude that they remain negative if $\alpha < 0$ and become alternating if $\alpha > 0$.

5. Identric mean. The expansion has similar structure to expansion of logarithmic mean. We have

$$\log(I(x-\beta, x+\beta)) = \ln x + \sum_{n=1}^{\infty} \frac{-1}{2n(2n+1)} \left(\frac{\beta}{x}\right)^{2n},$$

therefore

$$I(x-\beta, x+\beta) = x \sum_{n=0}^{\infty} c_n \beta^n x^{-n}$$

where $c_0 = 1$, $c_{2n+1} = 0$, for $n \geqslant 0$ and

$$c_{2n} = \frac{1}{2n} \sum_{k=1}^{n} \frac{-1}{2k+1} c_{2n-2k}.$$

Hence

$$I(x-\beta, x+\beta) = x - \frac{\beta^2}{6} x^{-1} - \frac{13\beta^4}{360} x^{-3} - \frac{737\beta^6}{45360} x^{-5} - \dots.$$

We shall now discuss the inequalities of the type

$$M(x+s, x+t) \lesseqgtr x + \alpha + \frac{c_2(\beta)}{x} \tag{7.3}$$

in the case $\alpha \leqslant 0$.

THEOREM 7.2. *Let $\alpha \leqslant 0$. Then the following inequalities are valid for all $x > -s$:*

$$G(x+s, x+t) < x + \alpha - \frac{\beta^2}{2x},$$
$$H(x+s, x+t) < x + \alpha - \frac{\beta^2}{x},$$
$$L(x+s, x+t) < x + \alpha - \frac{\beta^2}{3x},$$
$$I(x+s, x+t) < x + \alpha - \frac{\beta^2}{6x}.$$

Note that the sign in these inequalities is opposite to the one in theorems proved before.

These inequalities look at first sight paradoxical, one has for example

$$G(x+s, x+t) > x + \alpha - \frac{\beta^2}{2x}$$

if $\alpha > 0$, and with opposite sign if $\alpha < 0$!

Proof. Asymptotic expansion (7.1) is convergent at least for all $x > \max\{|s|,|t|\}$. This bound can be lovered, for example it is equal to $\sqrt{(s^2+t^2)/2}$ in the case of quadratic mean. But, in the case $\alpha < 0$ we have $s < 0$ and $|s| > |t|$, therefore, the series is convergent for $x > -s$. On the other hand, this condition is also a necessary one for the arguments of the mean $M(x+s,x+t)$ to be positive.

From the discussion above, we know that for $\alpha < 0$ the series

$$\sum_{n=0}^{\infty} c_{n+2}(\alpha,\beta)x^{-n-1}$$

is with alternating coefficients in the case of logarithmic and identric mean. The first neglected coefficient in the inequalities above is negative, therefore the mean is less than the indicated sum. The same is true for harmonic mean. In the case of geometric mean, all coeficients are negative.

This is enough for the proof of the theorem.

We have restricted ourselves to the inequalities of the type (7.3). It is clear that similar result one can obtain by similar reasoning for more than three term of asymptotic expansion included.

8. Some remarks and applications

The results obtained above show some properties of the coefficients (c_n) which have to be studied in the future.

Asymptotic expansions derived in this paper will be used as very efficient method in comparison of various means. We shall explain this idea in our forthcoming papers [9, **?**]. The following remark is a begining clue.

REMARK 8.1. The value of the coefficient c_2 in these expansions behaves as one would expect, having in mind the order between these means. Here is a list for the first five coefficients (note that we have $c_0 = 1$, $c_1 = \alpha$, $c_3 = \alpha c_2$ for all means):

mean	c_2	c_4
N	β^2	$\alpha^2\beta^2$
Q	$\frac{1}{2}\beta^2$	$\frac{1}{2}\alpha^2\beta^2 - \frac{1}{8}\beta^4$
C	$\frac{1}{3}\beta^2$	$\frac{1}{3}\alpha^2\beta^2$
A	0	0
I	$-\frac{1}{6}\beta^2$	$-\frac{1}{6}\alpha^2\beta^2 - \frac{13}{360}\beta^4$
He	$-\frac{1}{6}\beta^2$	$-\frac{1}{6}\alpha^2\beta^2 - \frac{1}{24}\beta^4$

mean	c_2	c_4
L	$-\frac{1}{3}\beta^2$	$-\frac{1}{3}\alpha^2\beta^2 - \frac{4}{45}\beta^4$
G	$-\frac{1}{2}\beta^2$	$-\frac{1}{2}\alpha^2\beta^2 - \frac{1}{8}\beta^4$
cL	$-\frac{2}{3}\beta^2$	$-\frac{2}{3}\alpha^2\beta^2 - \frac{2}{15}\beta^4$
cI	$-\frac{5}{6}\beta^2$	$-\frac{5}{6}\alpha^2\beta^2 - \frac{37}{360}\beta^4$
H	$-\beta^2$	$-\alpha^2\beta^2$

The results obtained in this paper can be extended to the case of more general means, like parameter means or integral means, see [10, 11]. Also, the same idea can be extended to n-variable means, like arithmetic, geometric, harmonic and power mean. Such problems were studied in [1]. Furthermore, some other types of inequalities can be established betwwen means as it was done in [6, 7, 8, 16].

Let us describe here only the immediate consequence of obtained results.

From the expansions derived in this paper, we can write

$$\begin{aligned}
G &= A - \frac{\beta^2}{2x} + \frac{\alpha\beta^2}{2x^2} - \frac{\alpha^2\beta^2}{2x^3} - \frac{\beta^4}{8x^3} + O(x^{-4}), \\
H &= A - \frac{\beta^2}{x} + \frac{\alpha\beta^2}{x^2} - \frac{\alpha^2\beta^2}{x^3} + O(x^{-4}), \\
Q &= A + \frac{\beta^2}{2x} - \frac{\alpha\beta^2}{2x^2} + \frac{\alpha^2\beta^2}{2x^3} - \frac{\beta^4}{8x^3} + O(x^{-4}),
\end{aligned}$$

From this list one can deduce in the case $\alpha = 0$

$$G = A - \frac{\beta^2}{2x} + O(x^{-3}),$$

which suggest the inequality $G \leqslant A$, and this is not a surprise. But

$$\begin{aligned}
2G &= H + A - \frac{\beta^4}{4x^3} + O(x^{-5}), \\
2A &= G + Q + \frac{\beta^4}{4x^3} + O(x^{-5}), \\
3A &= 2Q + H + \frac{\beta^4}{4x^3} + O(x^{-5}),
\end{aligned}$$

gives better understanding to the quality of approximations. Also, these relations are strong indications for the validity of inequalities

$$\begin{aligned}
2G &\leqslant H + A, \\
2A &\geqslant G + Q, \\
3A &\geqslant 2Q + H,
\end{aligned}$$

which are in fact the true ones, for all values of arguments of these means. Finally, we note the remarkable approximation

$$G + A - H - Q = -\frac{\beta^6}{8x^5} + O(x^{-7})$$

wherefrom one can derive the valid inequality

$$G + A \leqslant H + Q.$$

One may ask does these properties hold also for a general n-variable mean? Here is a table with numerical results.

Let us take an interval of the length 10 and let x takes value 100. It means that $a_1, a_2, \dots, a_n$ should be taken as random numbers from segment $[100, 110]$. We calculate all means and other quantities, and repeat this procedure for $N = 1000$ times. Finaly, the average of obtained results are calculated. In the first table, the result for $n = 2$ is written, it is covered by theory developed in this paper:

Q	105.028
A	105.008
G	104.988
H	104.967
$A-G$	$-.0202713$
$2G-H-A$	-9.0857×10^{-6}
$2A-G-Q$	9.08037×10^{-6}
$3A-2Q-H$	9.07504×10^{-6}
$G+A-H-Q$	-5.32721×10^{-9}

The results perfectly fit to derived formulas. But, if we take for example $n = 20$, i.e. generate 20 numbers from this interval and calculate their means (and repeat the procedure $N = 1000$ times) something like the following table will occur:

Q	105.030
A	104.993
G	104.955
H	104.917
$A-G$	0.0379678
$2G-H-A$	5.73459×10^{-6}
$2A-G-Q$	2.26065×10^{-5}
$3A-2Q-H$	5.09476×10^{-5}
$G+A-H-Q$	2.83411×10^{-5}

This different behaviour will be explained in our forthcoming papers.

Finally, one may ask a very natural question. Why average of geometrical mean is 104.955 and of harmonic one is 104.917? The deflection from arithmetic mean is two times bigger for harmonic mean. This behaviour can be explained using technique of asymptotic expansions.

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