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INEQUALITIES INVOLVING GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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Abstract. The Huygens-Wilker type inequalities involving generalized trigonometric functions and generalized hyperbolic functions are established. The first and the second inequalities of Huygens and Wilker, for classes of functions under discussion, are also investigated.

1. Introduction

During the past two decades after the publication of the paper [9], by P. Lindqvist, several researchers have studied the so-called p-trigonometric and the p-hyperbolic functions, which in the particular case $p=2$ reduce to classical functions. A list of published papers which deal with these two families of functions is long. For more details the interested reader is referred to [3, 8, 9, 2, 4, 1, 7, 13, 18, 19, 23] and to the references therein. In what follows we will call these families of functions the generalized trigonometric functions (*gtf*) and the generalized hyperbolic functions (*ghf*). The interest in investigations of these functions is justified by the fact they play an important in certain problems that arise in theory of differential equations. It is known that the *gtf* are eigenfunctions of the Dirichlet problem for the one-dimensional p-Laplacian. For more details, see [9, 3].

This paper deals with inequalities involving members of two families of functions mentioned earlier. In particular, when $p=2$, some of obtained inequalities simplify to four known inequalities for the trigonometric functions. The first two read as follows

$$1 < \frac{1}{3}\Big(2\frac{\sin x}{x} + \frac{\tan x}{x}\Big) \tag{1}$$

and

$$1 < \frac{1}{2}\Big[\Big(\frac{\sin x}{x}\Big)^2 + \frac{\tan x}{x}\Big] \tag{2}$$

$(0<|x|<\pi/2)$. Inequalities (1) and (2) have been obtained, respectively, by C. Huygens [6] and J.B. Wilker [20]. Several proofs of these results can be found in mathematical literature (see, e.g., [5, 10, 11, 12, 17, 22, 21, 24, 25, 26] and the references

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therein). In [17] the authors called inequalities (1) and (2) the first Huygens and the first Wilker inequalities, respectively, for the trigonometric functions.

The second Huygens and the second Wilker inequalities for the trigonometric functions

$$1 < \frac{1}{3}\Big(2\frac{x}{\sin x} + \frac{x}{\tan x}\Big) \tag{3}$$

and

$$1 < \frac{1}{2}\Big[\Big(\frac{x}{\sin x}\Big)^2 + \frac{x}{\tan x}\Big] \tag{4}$$

$(0 < |x| < \pi/2)$ have been also of interest for several researchers. For the proofs of the last two results the interested reader is referred to [17] and [21]. It is worth mentioning that inequalities which bear strong resemblance of inequalities (1)–(4) have been obtained recently for the Jacobian elliptic functions, Gauss lemniscate functions and other classes of functions as well (see, e.g., [11, 13, 14, 15, 16]).

Counterparts of inequalities (1)–(4) for the hyperbolic functions have also been proven. They have the same structure as inequalities (1)–(4) with the following modifications $\sin \to \sinh$ and $\tan \to \tanh$. For more details and additional references see, e.g., [17]. The goal of this paper is to obtain some inequalities which provide generalizations of the inequalities (1)–(4) for the *gtf* and the *ghf* functions. Definitions of these families of functions are given in Section 2. The Huygens-Wilker type inequalities for the *gtf* and the *ghf* are established in Section 3. The first and the second Huygens and Wilker inequalities for functions under discussion are obtained in Section 4. Therein, among other things, we demonstrate that the right sides of the second Huygens and the second Wilker inequalities for the *gtf* are comparable. A similar result for the *ghf* is also established.

2. Definitions and preliminaries

For the reader's convenience we recall first definition of the celebrated Gauss hypergeometric function $F(a,b;c;z)$:

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!}, \qquad |z| < 1,$$

where $(a,n) = a(a+1)\dots(a+n-1)$ $(n \neq 0)$ is the shifted factorial or Appell symbol, with $(a,0) = 1$ if $a \neq 0$, and $c \neq 0, -1, -2, \dots$.

In what follows we will always assume that the letter p represents a real number which is strictly greater 1, unless otherwise stated. Also, we will adopt notation and definitions used in [3].

Let

$$\pi_p = 2\frac{\pi/p}{\sin(\pi/p)}.$$

Further, let

$$a_p = \frac{\pi_p}{2}, \qquad b_p = 2^{-1/p} F\Big(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; \frac{1}{2}\Big), \qquad c_p = 2^{-1/p} F\Big(1, \frac{1}{p}; 1 + \frac{1}{p}; \frac{1}{2}\Big).$$

Also, let $I=(0,1)$ be the unit interval. The *gtf* and the *ghf* used in this paper are the following homeomorphisms

$$\sin_p : (0,a_p) \to I, \qquad \tan_p : (0,b_p) \to I$$

and

$$\sinh_p : (0,c_p) \to I, \qquad \tanh_p : (0,\infty) \to I.$$

For $x \in I$, their inverse functions are defined as follows

$$\sin_p^{-1} x = \int_0^x (1-t^p)^{-1/p} dt = xF\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};x^p\right), \tag{5}$$

$$\tan_p^{-1} x = \int_0^x (1+t^p)^{-1} dt = xF\left(1,\frac{1}{p};1+\frac{1}{p};-x^p\right), \tag{6}$$

$$\sinh_p^{-1} x = \int_0^x (1+t^p)^{-1/p} dt = xF\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};-x^p\right), \tag{7}$$

and

$$\tanh_p^{-1} x = \int_0^x (1-t^p)^{-1} dt = xF\left(1,\frac{1}{p};1+\frac{1}{p};x^p\right). \tag{8}$$

For the latter use we recall some formulas which involve functions defined above. They all appear in [8]. The generalized cosine function $\cos_p$ is defined as

$$\cos_p x = (1-(\sin_p x)^p)^{1/p}, \qquad x \in [0,a_p]$$

and the generalized tangent function satisfies the following equation

$$\tan_p x = \frac{\sin_p x}{\cos_p x}, \qquad x \in [0,a_p).$$

The counterparts of the last two formulas for the *ghf* are defined below

$$\cosh_p x = (1+(\sinh_p x)^p)^{1/p}, \qquad x>0,$$

and

$$\tanh_p x = \frac{\sinh_p x}{\cosh_p x}.$$

Some results of this paper are established with the aid of the following.

PROPOSITION 1. Let u,v,λ,μ be positive numbers. Assume that u and v satisfy the separation condition

$$u<1<v. \tag{9}$$

Then the inequality

$$1 < \frac{\lambda}{\lambda+\mu}u^r + \frac{\mu}{\lambda+\mu}v^s \tag{10}$$

holds true if either

$$1 < u^{\alpha} v^{\beta}, \qquad s > 0 \qquad \text{and} \qquad r\lambda \leqslant s\mu\alpha/\beta \tag{11}$$

or if

$$u^{\alpha} v^{\beta} < 1, \qquad s < 0 \qquad \text{and} \qquad r\lambda \leqslant s\mu\alpha/\beta, \tag{12}$$

where $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$. If u and v satisfy the separation condition (9) together with

$$1 < \alpha\frac{1}{u} + \beta\frac{1}{v}, \tag{13}$$

then the inequality (2.6) is also valid if

$$r \leqslant s \leqslant -1 \qquad \text{and} \qquad \mu\alpha \leqslant \lambda\beta. \tag{14}$$

Proof. Validity of inequality (10), when conditions (11) are satisfied, are established in [12, Theorem 3.1]. For the proof of (12) we let

$$d = \frac{\lambda}{\lambda+\mu}u^r + \frac{\mu}{\lambda+\mu}v^s.$$

Application of the inequality of weighted arithmetic and geometric means gives

$$d^{\lambda+\mu} \geqslant u^{r\lambda} v^{s\mu}. \tag{15}$$

It follows from (12) that $v < u^{-\alpha/\beta}$. Taking into account that $s < 0$ we obtain $v^{s\mu} > u^{-s\mu\alpha/\beta}$. This and (15) give

$$d^{\lambda+\mu} > u^{r\lambda - s\mu\alpha/\beta} \geqslant 1,$$

where the last inequality is a consequence of $0 < u < 1$ and (12). Conditions (14) of validity of the inequality (10) are established in [12]. The proof is complete. □

We will call inequalities of the form (10) the Huygens-Wilker type inequalities.

3. Huygens-Wilker type inequalities for the *gtf* and *ghf*

This section is devoted to the study of the Huygens-Wilker type inequalities which involve either the *gtf* or the *ghf*. Our first result reads as follows.

THEOREM 2. *Let (u,v) be an ordered pair of functions defined as follows*

$$(u,v) = \left(\frac{f(x)}{x}, \frac{f^{-1}(x)}{x}\right), \tag{16}$$

where $x \in I$ and

$$f(x) = \sin_p x \qquad \text{or} \qquad f(x) = \tanh_p x. \tag{17}$$

Further, let $\lambda,\mu > 0$. *Then the inequality*

$$1 < \frac{\lambda}{\lambda+\mu}u^r + \frac{\mu}{\lambda+\mu}v^s \tag{18}$$

is satisfied if

$$s > 0 \qquad \textit{and} \qquad r\lambda \leqslant s\mu. \tag{19}$$

Proof. We shall prove first the assertion when $f(x) = \sin_p x$. Let

$$u = \frac{\sin_p x}{x} \qquad \text{and} \qquad v = \frac{\sin_p^{-1} x}{x}. \tag{20}$$

It follows from Lemma 3.32 in [8] and from (5) that

$$\frac{\sin_p x}{x} < 1 < \frac{\sin_p^{-1} x}{x}.$$

Thus with u and v as defined in (20) we see that condition (9) is satisfied. Moreover, inequality (11) is also satisfied with $\alpha = \beta = 1/2$ because

$$1 < \frac{\sin_p x}{x}\frac{\sin_p^{-1} x}{x}$$

(see [8, Lemma 3.32]). To complete the proof when $f(x) = \sin_p x$ we utilize Proposition 1. A proof of (18), when $f(x) = \tanh_p x$, goes along the lines introduced above. Let

$$u = \frac{\tanh_p x}{x} \qquad \text{and} \qquad v = \frac{\tanh_p^{-1} x}{x}. \tag{21}$$

Making use of [8, Lemma 3.32] and formula (8) we obtain

$$\frac{\tanh_p x}{x} < 1 < \frac{\tanh_p^{-1} x}{x}.$$

This shows that in the case under discussion condition (9) is satisfied. To complete the proof we appeal again to [8, Lemma 3.32] to claim that the first inequality in (11) is satisfied with $\alpha = \beta = 1/2$. Application of Proposition 1 yields the assertion. □

We shall establish now the following.

THEOREM 3. *Let* (u,v) *be an ordered pair of functions defined as follows*

$$(u,v) = \left(\frac{f^{-1}(x)}{x}, \frac{f(x)}{x}\right), \tag{22}$$

where $x \in I$ *and*

$$f(x) = \tan_p x \qquad \textit{or} \qquad f(x) = \sinh_p x. \tag{23}$$

Further, let $\lambda, \mu > 0$. *Then the inequality*

$$1 < \frac{\lambda}{\lambda+\mu} u^r + \frac{\mu}{\lambda+\mu} v^s \tag{24}$$

is valid provided

$$s < 0 \qquad \textit{and} \qquad r\lambda \leqslant s\mu. \tag{25}$$

Proof. For the later use let as record a couple of two-sided inequalities

$$\frac{\tan_p^{-1} x}{x} < 1 < \frac{\tan_p x}{x} \tag{26}$$

and

$$\frac{\sinh_p^{-1} x}{x} < 1 < \frac{\sinh_p x}{x}. \tag{27}$$

The first inequalities in (26) and (27) follow from (6) and (7), respectively. The second ones are obtained in [8, Lemma 3.32]. Also, it has been shown in [8, Lemma 3.32] that

$$\frac{\tan_p^{-1} x}{x} \frac{\tan_p x}{x} < 1$$

and

$$\frac{\sinh_p^{-1} x}{x} \frac{\sinh_p x}{x} < 1.$$

Thus in both cases the first inequality in (12) is satisfied with $\alpha = \beta = 1/2$ for the pairs of functions

$$(u, v) = \left(\frac{\tan_p^{-1} x}{x}, \frac{\tan_p x}{x} \right)$$

and

$$(u, v) = \left(\frac{\sinh_p^{-1} x}{x}, \frac{\sinh_p x}{x} \right).$$

Using Proposition 1 again we obtain the desired result. □

We close this section with the following.

THEOREM 4. *Let* $p \geqslant 2$ *and let* $x \in (0, a_p)$. *Also, let* λ, μ *and* s *be positive numbers. If the real number* r *satisfies the inequality*

$$r\lambda \leqslant s\mu, \tag{28}$$

then the inequality (10) is satisfied if either

$$(u, v) = \left(\frac{x}{\sinh_p x}, \frac{x}{\tanh_p x} \right) \tag{29}$$

or if

$$(u, v) = \left(\frac{x}{\sinh_p x}, \cosh_p x \right) \tag{30}$$

or if

$$(u,v) = \left(\frac{x}{\sinh_p x}, \frac{x}{\sin_p x}\right) \tag{31}$$

or if

$$(u,v) = \left(\frac{\sin_p x}{x}, \cosh_p x\right) \tag{32}$$

or if

$$(u,v) = \left(\frac{\sin_p x}{x}, \frac{x}{\tanh_p x}\right) \tag{33}$$

or if

$$(u,v) = \left(\frac{\tanh_p x}{x}, \cosh_p x\right) \tag{34}$$

Proof. Let u and v be defined in (29). It follows from [8, Lemma 3.32, parts (3) and (4)] that

$$u < 1 < v.$$

Moreover, the third and fifth members of [8, (3.11)] give the inequality

$$1 < uv.$$

Thus (9) and the first part of (11) are satisfied with $\alpha = \beta = 1/2$. Application of Proposition 1 shows that the inequality (10) is valid provided condition (28) is satisfied. The assertion when u and v are the same as in (30) can be established in a similar way. In this case one has $u < 1 < v$, where the second inequality follows from the definition of $\cosh_p$. The second and fifth members of [8, (3.11)] yield the inequality $1 < uv$. The asserted result is obtained with the aid of Proposition 1. The remaining cases when u and v are defined in (31)–(34) can be established in a similar manner. Recall that in all cases in question inequality (26) in [8] yields $1 < uv$. The proof is complete. □

4. Huygens' and Wilker's inequalities of the first and the second kind

The goal of this section is to obtain some additional results which pertain to the Huygens and Wilker inequalities for two families of functions discussed in this paper. The first three results deal with inequalities for the *gtf*.

THEOREM 5. *Let $p \geqslant 2$ and assume that $0 < x < a_p$. Then*

$$1 < \frac{1}{3}\left(2\frac{\sin_p x}{x} + \frac{\tan_p x}{x}\right) \tag{35}$$

and

$$1 < \frac{1}{2}\left[\left(\frac{\sin_p x}{x}\right)^2 + \frac{\tan_p x}{x}\right]. \tag{36}$$

Moreover, inequality (35) implies inequality (36).

Proof. Let

$$u = \frac{\sin_p x}{x} \qquad \text{and} \qquad v = \frac{\tan_p x}{x}$$

$(p > 1, 0 < x < a_p)$. It follows from Lemma 3.32 in [8] that u and v satisfy the separation condition (9), i.e., $u < 1 < v$. We shall show now that u and v satisfy the first inequality in (11) with $\alpha = p/(1+p)$ and $\beta = 1/(p+1)$. To this aim we utilize the inequality

$$(\cos_p x)^{\frac{1}{1+p}} < \frac{\sin_p x}{x}$$

(see [8, (3.7)]) which can be written as follows

$$1 < \Big(\frac{\sin_p x}{x}\Big)^{\frac{p}{1+p}} \Big(\frac{\tan_p x}{x}\Big)^{\frac{1}{1+p}}$$

or

$$1 < u^{\alpha} v^{\beta},$$

where α and β are defined above. Thus the last condition in (11) takes the form

$$r\lambda \leqslant s\mu p.$$

Inequality (35) now follows using Proposition 1 with $r = s = 1$, $\lambda = 2$ and $\mu = 1$. Similarly, inequality (36) is obtained with the aid of Proposition 1 with $r = 2, s = 1$ and $\lambda = \mu = 1$. For the proof of the last assertion we write (35) in the form

$$\frac{\sin_p x}{x} > \frac{3\cos_p x}{1 + 2\cos_p x}.$$

Then

$$\begin{aligned}\Big(\frac{\sin_p x}{x}\Big)^2 + \frac{\tan_p x}{x} &> \frac{9\cos_p^2 x}{(1+2\cos_p x)^2} + \frac{3}{1+2\cos_p x} \\ &= \frac{9\cos_p^2 x + 6\cos_p x + 3}{(1+2\cos_p x)^2} = 2 + \Big(\frac{1-\cos_p x}{1+2\cos_p x}\Big)^2 > 2.\end{aligned}$$

The assertion now follows. □

The second Huygens and the second Wilker inequalities are obtained in the following.

THEOREM 6. *Let $1 < p \leqslant 2$. If $x \in (0, a_p)$, then*

$$1 < \frac{1}{3}\Big(2\frac{x}{\sin_p x} + \frac{x}{\tan_p x}\Big) \tag{37}$$

and

$$1 < \frac{1}{2}\Big[\Big(\frac{x}{\sin_p x}\Big)^2 + \frac{x}{\tan_p x}\Big]. \tag{38}$$

Moreover, inequality (37) implies inequality (38).

Proof. The following inequality

$$\frac{x}{\sin_p x} > \frac{3}{2+\cos_p x} \tag{39}$$

was obtained in [8] (see Theorem 3.22). Multiplying both sides of (39) by $(2+\cos_p x)/3$ we obtain inequality (37). To prove inequality (38) and also in order to establish the last statement we utilize (39) again to obtain

$$\begin{aligned}\left(\frac{x}{\sin_p x}\right)^2 + \frac{x}{\tan_p x} &= \left(\frac{x}{\sin_p x}\right)^2 + \frac{x}{\sin_p x}\cos_p x \\ &> \frac{9}{(2+\cos_p x)^2} + \frac{3\cos_p x}{2+\cos_p x} = \frac{3\cos_p^2 x + 6\cos_p x + 9}{(2+\cos_p x)^2} \\ &= 2 + \left(\frac{\cos_p x - 1}{2+\cos_p x}\right)^2 > 2.\end{aligned}$$

The proof is complete. □

An inequality which connects the right sides of (37) and (38) is established in the following.

THEOREM 7. *Let* $1 < p \leqslant 2$ *and assume that* $0 < x < a_p$. *Then*

$$\frac{1}{2}\left[\left(\frac{x}{\sin_p x}\right)^2 + \frac{x}{\tan_p x}\right] > \frac{1}{3}\left(2\frac{x}{\sin_p x} + \frac{x}{\tan_p x}\right). \tag{40}$$

Proof. The following inequality

$$\cos_p x > 3\frac{\sin_p x}{x} - 2 \tag{41}$$

is a consequence of (39) and plays a crucial role in this proof. We have

$$\begin{aligned}&\frac{1}{2}\left[\left(\frac{x}{\sin_p x}\right)^2 + \frac{x}{\tan_p x}\right] - \frac{1}{3}\left(2\frac{x}{\sin_p x} + \frac{x}{\tan_p x}\right) \\ &= \frac{1}{6}\left(\frac{x}{\sin_p x}\right)^2\left(3 + \frac{\sin_p x}{x}\cos_p x - 4\frac{\sin_p x}{x}\right) \\ &> \frac{1}{6}\left(\frac{x}{\sin_p x}\right)^2\left[3 + \frac{\sin_p x}{x}\left(3\frac{\sin_p x}{x} - 2\right) - 4\frac{\sin_p x}{x}\right] \\ &= \frac{1}{6}\left(\frac{x}{\sin_p x}\right)^2\left[3\left(\frac{\sin_p x}{x}\right)^2 - 6\frac{\sin_p x}{x} + 3\right] \\ &= \frac{1}{2}\left(\frac{x}{\sin_p x}\right)^2\left(\frac{\sin_p x}{x} - 1\right)^2 > 0.\end{aligned}$$

The proof is complete. □

We shall demonstrate now that the results similar to these obtained in Theorems 5 - 7 also hold true for the *ghf*. The first Huygens inequality and the first Wilker inequality for the class of functions under discussion are obtained in the following.

THEOREM 8. *Let* $1 < p \leqslant 2$ *and let* $x > 0$. *Then*

$$1 < \frac{1}{3}\left(2\frac{\sinh_p x}{x} + \frac{\tanh_p x}{x}\right) \tag{42}$$

and

$$1 < \frac{1}{2}\left[\left(\frac{\sinh_p x}{x}\right)^2 + \frac{\tanh_p x}{x}\right]. \tag{43}$$

Moreover, inequality (42) implies inequality (43).

Proof. Let

$$u = \frac{\tanh_p x}{x} \qquad \text{and} \qquad v = \frac{\sinh_p x}{x}$$

$(x > 0)$. It follows from Lemma 3.32 in [8] that u and v satisfy the separation condition (9), i.e., $u < 1 < v$. We shall show now that u and v satisfy the first inequality in (11) with $\alpha = 1/(1+p)$ and $\beta = p/(p+1)$. To this aim we utilize the inequality

$$(\cosh_p x)^{\frac{1}{1+p}} < \frac{\sinh_p x}{x}$$

(see [8, (3.9)]) which can be written as follows

$$1 < \left(\frac{\tanh_p x}{x}\right)^{\frac{1}{1+p}} \left(\frac{\sinh_p x}{x}\right)^{\frac{p}{1+p}}$$

or

$$1 < u^{\alpha} v^{\beta},$$

where α and β are defined above. Thus the last condition in (11) takes the form

$$r\lambda p \leqslant s\mu.$$

Inequality (42) now follows using Proposition 2.1 with $r = s = \lambda = 1$, $\mu = 2$ and $p \leqslant 2$. Similarly, inequality (43) is obtained if $s = 2$, $r = \lambda = \mu = 1$ and $p \leqslant 2$.

The last assertion can be established in the same way as it was done in the proof of the corresponding statement in Theorem 5. We omit further details. □

The counterpart of Theorem 6 for the *ghf* reads as follows.

THEOREM 9. *Let* $p \geqslant 2$ *and assume that* $x > 0$. *Then*

$$1 < \frac{1}{3}\left(2\frac{x}{\sinh_p x} + \frac{x}{\tanh_p x}\right) \tag{44}$$

and

$$1 < \frac{1}{2}\left[\left(\frac{x}{\sinh_p x}\right)^2 + \frac{x}{\tanh_p x}\right]. \tag{45}$$

Moreover, inequality (44) implies inequality (45).

Proof. For the proof of (44) we utilize the following result [8, Theorem 3.24]

$$\frac{x}{\sinh_p x} > \frac{3}{2+\cosh_p x} \tag{46}$$

$(p \geqslant 2, x > 0)$ from which (44) follows. Inequality (45) and the last assertion can be proven using the same ideas as those employed in the proof of the corresponding part of Theorem 6. We omit further details. □

We close this section with the following.

THEOREM 10. *Let* $p \geqslant 2$ *and let* $x > 0$. *Then*

$$\frac{1}{2}\left[\left(\frac{x}{\sinh_p x}\right)^2 + \frac{x}{\tanh_p x}\right] > \frac{1}{3}\left(2\frac{x}{\sinh_p x} + \frac{x}{\tanh_p x}\right). \tag{47}$$

Proof. In order to obtain the desired result one can follow the lines of the proof of (40) utilizing the inequality

$$\cosh_p x > 3\frac{\sinh_p x}{x} - 2$$

which follows from (46). We leave it to the reader to complete the proof of (47). □

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