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LYAPUNOV–TYPE INEQUALITIES FOR HIGHER–ORDER DIFFERENTIAL EQUATIONS WITH ONE–DIMENSIONAL p–LAPLACIAN

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Abstract. In this paper, we establish Lyapunov-type inequalities for a single higher-order differential equation, a cycled system and a coupled system with one-dimensional p-Laplacian. Our result generalize some given results.

1. Introduction

The Lyapunov inequality for linear ordinary differential equation

$$u''(x)+r(x)u(x)=0, \quad x\in(a,b), \qquad u(a)=0=u(b),$$

where $r\in C([a,b],[0,\infty))$, gives a necessary condition for the existence of a positive solution as follows:

$$\frac{4}{b-a}\leqslant\int_a^b r(x)dx.$$

Lyapunov [1] initiated to estimate the above inequality. Since then, there have been several results to generalize the above linear ordinary differential equation from different viewpoints. To some latest results, the reader is referred to [7, 8, 9, 10, 11, 12, 13, 14, 15] and the references quoted therein. For example, Watanabe [16] obtained Lyapunov type inequality for the existence of the solution of the equation including p-Laplacian (2.1) under clamped boundary condition, and the usage of the best constant of L_p Sobolev inequality clarifies the process for obtaining such inequality.

In 2003, Yang [2] generalized above result to certain higher-order differential equations as follows. Consider the differential equation

$$u^{(2n)}(x)+r(x)u(x)=0, \qquad u^{(i)}(a)=0=u^{(i)}(b),$$

for $i=0,1,\ldots,n-1$, where $u(x)\neq 0$, $x\in(a,b)$, $r(x)\in C([a,b])$. Then

$$\int_a^b (x-a)^{2n-1}(b-x)^{2n-1}|r(x)|dx\geqslant(2n-1)[(n-1)!]^2(b-a)^{2n-1},$$

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especially,

$$\int_a^b |r(x)|dx > \frac{4^{2n-1}(2n-1)[(n-1)!]^2}{(b-a)^{2n-1}}.$$

In 2004, Pinasco [3] extended linear ordinary differential equations to the following one dimensional p-Laplacian problem:

$$-\varphi_p(u'(x))' = r(x)\varphi_p(u(x)), \quad x \in (a,b), \qquad u(a) = 0 = u(b), \tag{1.1}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $r \in C([a,b],[0,\infty))$. He obtained Lyapunov inequality as follows:

$$\frac{2^p}{(b-a)^{p/q}} \leqslant \int_a^b r(x)dx.$$

In 2010, Sim and Lee [4] obtained if u is a positive solution for (1.1), then

$$\frac{(b-a)^{p-1}}{2^{p-2}} \leqslant \int_a^b (t-a)^{p-1}(b-t)^{p-1}r(x)dx,$$

where $p = 2$, the above inequality was obtained by Hartman [5].

Motivated by above papers, the purpose of this paper is to get Lyapunov inequalities for single higher-order differential equations as well as systems with one-dimensional p-Laplacian. The idea of the proof of our results comes from that of Sim and Lee [4]. Our result generalize some given results.

2. Single equation

Consider higher-order differential equations with one-dimensional p-Laplacian

$$(-1)^n\varphi_p(u^{(n)}(x))^{(n)} = r(x)\varphi_p(u(x)), \quad x \in (a,b), \qquad u^{(i)}(a) = 0 = u^{(i)}(b), \tag{2.1}$$

for $i = 0,1,\dots,n-1$, where $n \in \mathbb{N}$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $r \in C([a,b],[0,\infty))$.

THEOREM 2.1. *If u is a positive solution for (2.1), then one have*

$$\frac{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}{2^{pn-2}(p-1)^{p-1}} \leqslant \int_a^b r(x)\left((x-a)(b-x)\right)^{pn-1}dx, \tag{2.2}$$

especially,

$$\frac{2^{pn}(pn-1)^{p-1}[(n-1)!]^p}{(p-1)^{p-1}(b-a)^{pn-1}} \leqslant \int_a^b r(x)dx. \tag{2.3}$$

Proof. From $u(a) = u'(a) = \cdots = u^{(n-1)}(a)$, we get

$$|u(x)| = \left|\frac{1}{(n-1)!}\int_a^x (x-s)^{n-1}u^{(n)}(s)ds\right| \leqslant \frac{1}{(n-1)!}\int_a^x (x-s)^{n-1}|u^{(n)}(s)|ds. \tag{2.4}$$

By using Höder inequality on the integral of the right-hand side of (2.4) with indices $p/(p-1)$ and p, we have

$$|u(x)| \leqslant \frac{1}{(n-1)!}\left(\int_a^x (x-s)^{p(n-1)/(p-1)}ds\right)^{(p-1)/p}\left(\int_a^x |u^{(n)}(s)|^p ds\right)^{1/p}$$
$$= \frac{1}{(n-1)!}\left(\frac{p-1}{pn-1}(x-a)^{(pn-1)/(p-1)}\right)^{(p-1)/p}\left(\int_a^x |u^{(n)}(s)|^p ds\right)^{1/p}.$$

For $a \leqslant x \leqslant (a+b)/2$, noting $x-a \leqslant (2/(b-a))(x-a)(b-x)$, we have

$$|u(x)| \leqslant \frac{1}{(n-1)!}\left(\frac{p-1}{pn-1}\left(\frac{2}{b-a}(x-a)(b-x)\right)^{(pn-1)/(p-1)}\right)^{(p-1)/p}$$
$$\times\left(\int_a^{(a+b)/2} |u^{(n)}(s)|^p ds\right)^{1/p}.$$

Thus, we have

$$|u(x)|^p \leqslant \frac{1}{[(n-1)!]^p}\left(\frac{p-1}{pn-1}\left(\frac{2}{b-a}(x-a)(b-x)\right)^{(pn-1)/(p-1)}\right)^{p-1}$$
$$\times\left(\int_a^{(a+b)/2} |u^{(n)}(s)|^p ds\right)$$
$$= \frac{2^{pn-1}(p-1)^{p-1}}{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}((x-a)(b-x))^{pn-1}$$
$$\times\left(\int_a^{(a+b)/2} |u^{(n)}(s)|^p ds\right). \tag{2.5}$$

Similarly, by using Höder inequality, we get

$$|u(x)| = \left|\frac{1}{(n-1)!}\int_x^b (s-x)^{n-1}u^{(n)}(s)ds\right| \leqslant \frac{1}{(n-1)!}\int_x^b (s-x)^{n-1}|u^{(n)}(s)|ds$$
$$\leqslant \frac{1}{(n-1)!}\left(\frac{p-1}{pn-1}(b-x)^{(pn-1)/(p-1)}\right)^{(p-1)/p}\left(\int_x^b |u^{(n)}(s)|^p ds\right)^{1/p}.$$

For $(a+b)/2 \leqslant x \leqslant b$, noting $b-x \leqslant (2/(b-a))(x-a)(b-x)$, we have

$$|u(x)|^p \leqslant \frac{2^{pn-1}(p-1)^{p-1}}{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}((x-a)(b-x))^{pn-1}$$
$$\times\left(\int_{(a+b)/2}^b |u^{(n)}(s)|^p ds\right). \tag{2.6}$$

Adding (2.5) and (2.6), we have

$$2|u(x)|^p \leqslant \frac{2^{pn-1}(p-1)^{p-1}}{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}((x-a)(b-x))^{pn-1}$$
$$\times\left(\int_a^b |u^{(n)}(s)|^p ds\right). \tag{2.7}$$

Multiplying both sides of (2.7) by $r(t)$ and rewriting, we get

$$\frac{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}{2^{pn-2}(p-1)^{p-1}}r(x)|u(x)|^p \leqslant r(x)\left((x-a)(b-x)\right)^{pn-1} \times \left(\int_a^b |u^{(n)}(s)|^p ds\right). \tag{2.8}$$

Since u is a solution for (2.1), we have

$$\int_a^b |u^{(n)}(x)|^p dx = \int_a^b r(x)|u(x)|^p dx. \tag{2.9}$$

Integrating (2.8) on $[a,b]$ and using (2.9), we have

$$\begin{aligned} &\frac{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}{2^{pn-2}(p-1)^{p-1}} \int_a^b |u^{(n)}(x)|^p dx \\ = &\frac{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}{2^{pn-2}(p-1)^{p-1}} \int_a^b r(x)|u(x)|^p dx \\ \leqslant &\int_a^b r(x)\left((x-a)(b-x)\right)^{pn-1}\left(\int_a^b |u^{(n)}(s)|^p ds\right) dx. \end{aligned}$$

Therefore, we get

$$\frac{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}{2^{pn-2}(p-1)^{p-1}} \leqslant \int_a^b r(x)\left((x-a)(b-x)\right)^{pn-1} dx.$$

For $a \leqslant x \leqslant b$, since $(x-a)(b-x) \leqslant (b-a)^2/4$, we have

$$\frac{2^{pn}(pn-1)^{p-1}[(n-1)!]^p}{(p-1)^{p-1}(b-a)^{pn-1}} \leqslant \int_a^b r(x)dx.$$

This completes the proof. □

REMARK 2.2. When $n=1$, the above result coincides with Theorem 2.1 in Sim and Lee [4]. When $n=1$ and $p=2$, the above result coincides with Hartman's estimate [5].

COROLLARY 2.3. *If u is a positive solution for (2.1) with $p=2$, then one have*

$$\frac{(2n-1)[(n-1)!]^2(b-a)^{2n-1}}{2^{2(n-1)}} \leqslant \int_a^b r(x)\left((x-a)(b-x)\right)^{2n-1} dx,$$

especially,

$$\frac{2^{2n}(2n-1)[(n-1)!]^2}{(b-a)^{2n-1}} \leqslant \int_a^b r(x)dx.$$

3. Cycled system

Consider a cycled system

$$\begin{aligned}(-1)^n\varphi_p(u_1^{(n)}(x))^{(n)} &= r_1(x)\varphi_p(u_2(x)), \quad x\in(a,b),\\ (-1)^n\varphi_p(u_2^{(n)}(x))^{(n)} &= r_2(x)\varphi_p(u_3(x)), \quad x\in(a,b),\\ &\cdots\\ (-1)^n\varphi_p(u_{m-1}^{(n)}(x))^{(n)} &= r_{m-1}(x)\varphi_p(u_m(x)), \quad x\in(a,b),\\ (-1)^n\varphi_p(u_m^{(n)}(x))^{(n)} &= r_m(x)\varphi_p(u_1(x)), \quad x\in(a,b),\\ u_j^{(i)}(a) &= 0 = u_j^{(i)}(b),\end{aligned} \tag{3.1}$$

for $i=0,1,\dots,n-1$ and $j=1,2,\dots,m$, where $m,n\in\mathbb{N}$.

THEOREM 3.1. *If $(u_1,u_2,\dots,u_m)$ is a positive solution for (3.1), then one have*

$$\left(\frac{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}{2^{pn-2}(p-1)^{p-1}}\right)^m \leqslant \prod_{j=1}^m\left(\int_a^b r_j(x)\left((x-a)(b-x)\right)^{pn-1}dx\right), \tag{3.2}$$

especially,

$$\left(\frac{2^{pn}(pn-1)^{p-1}[(n-1)!]^p}{(p-1)^{p-1}(b-a)^{pn-1}}\right)^m \leqslant \prod_{j=1}^m\left(\int_a^b r_j(x)dx\right). \tag{3.3}$$

Proof. We only show the case $m=2$. For the general case, we can prove it by repeating this procedure. As in (3.1), for $j=1,2$, we obtain from (2.7)

$$|u_j(x)|^p \leqslant \frac{2^{pn-2}(p-1)^{p-1}\left((x-a)(b-x)\right)^{pn-1}}{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}\left(\int_a^b |u_j^{(n)}(s)|^p ds\right).$$

Thus, for $j=1,2$, we get

$$|u_j(x)| \leqslant \left(\frac{2^{pn-2}(p-1)^{p-1}\left((x-a)(b-x)\right)^{pn-1}}{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}\right)^{1/p}\left(\int_a^b |u_j^{(n)}(s)|^p ds\right)^{1/p} \tag{3.4}$$

and

$$\begin{aligned}|u_j(x)|^{p-1} \leqslant &\left(\frac{2^{pn-2}(p-1)^{p-1}\left((x-a)(b-x)\right)^{pn-1}}{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}\right)^{(p-1)/p}\\ &\times\left(\int_a^b |u_j^{(n)}(s)|^p ds\right)^{(p-1)/p}.\end{aligned} \tag{3.5}$$

Multiplying the first equation of (3.1) by u_1 and integrating on $[a,b]$, we have by (3.4) and (3.5) that

$$\begin{aligned}\int_a^b |u_1^{(n)}(x)|^p dx &\leqslant \int_a^b r_1(x)|u_2(x)|^{p-1}|u_1(x)|dx\\ &\leqslant \left(\int_a^b \frac{2^{pn-2}(p-1)^{p-1}}{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}} r_1(x)\left((x-a)(b-x)\right)^{pn-1}dx\right)\\ &\quad\times\left(\int_a^b |u_2^{(n)}(x)|^p dx\right)^{(p-1)/p}\left(\int_a^b |u_1^{(n)}(s)|^p ds\right)^{1/p}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\left(\int_a^b |u_1^{(n)}(s)|^p ds\right)^{(p-1)/p} &\leqslant \frac{2^{pn-2}(p-1)^{p-1}}{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}\\ &\quad\times\left(\int_a^b r_1(x)\left((x-a)(b-x)\right)^{pn-1}dx\right)\\ &\quad\times\left(\int_a^b |u_2^{(n)}(x)|^p dx\right)^{(p-1)/p}.\end{aligned}$$

Similarly, for the second equation in (3.1), we have

$$\begin{aligned}\left(\int_a^b |u_2^{(n)}(s)|^p ds\right)^{(p-1)/p} &\leqslant \frac{2^{pn-2}(p-1)^{p-1}}{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}\\ &\quad\times\left(\int_a^b r_2(x)\left((x-a)(b-x)\right)^{pn-1}dx\right)\\ &\quad\times\left(\int_a^b |u_1^{(n)}(x)|^p dx\right)^{(p-1)/p}.\end{aligned}$$

Therefore, we obtain

$$\left(\frac{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}{2^{pn-2}(p-1)^{p-1}}\right)^2 \leqslant \prod_{j=1}^{2}\left(\int_a^b r_j(x)\left((x-a)(b-x)\right)^{pn-1}dx\right),$$

For $a\leqslant x\leqslant b$, since $(x-a)(b-x)\leqslant(b-a)^2/4$, we have from the above three inequalities

$$\left(\frac{2^{pn}(pn-1)^{p-1}[(n-1)!]^p}{(p-1)^{p-1}(b-a)^{pn-1}}\right)^2 \leqslant \prod_{j=1}^{2}\left(\int_a^b r_j(x)dx\right),$$

This completes the proof. □

REMARK 3.2. When $n=1$, the above result coincides with Theorem 3.1 in Sim and Lee [4].

COROLLARY 3.3. *If* $(u_1,u_2,\dots,u_m)$ *is a positive solution for (3.1) with* $p=2$, *then one have*

$$\left(\frac{(2n-1)[(n-1)!]^2(b-a)^{2n-1}}{2^{2(n-1)}}\right)^m \leqslant \prod_{j=1}^m \left(\int_a^b r_j(x)\,((x-a)(b-x))^{2n-1}\,dx\right),$$

especially,

$$\left(\frac{2^{2n}(2n-1)[(n-1)!]^2}{(b-a)^{2n-1}}\right)^m \leqslant \prod_{j=1}^m \left(\int_a^b r_j(x)dx\right).$$

COROLLARY 3.4. *If* $(u_1,u_2,\dots,u_m)$ *is a positive solution for (3.1) with* $r_1=r_2=\dots=r_m=r$, *then one have*

$$\frac{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}{2^{pn-2}(p-1)^{p-1}} \leqslant \int_a^b r(x)\,((x-a)(b-x))^{pn-1}\,dx.$$

especially,

$$\frac{2^{pn}(pn-1)^{p-1}[(n-1)!]^p}{(p-1)^{p-1}(b-a)^{pn-1}} \leqslant \int_a^b r(x)dx.$$

4. Strongly coupled system

Consider a strongly coupled system

$$\begin{aligned}
(-1)^n\varphi_p(u_1^{(n)}(x))^{(n)} &= r_1(x)\,(\varphi_p(u_1(x))+\varphi_p(u_2(x))+\varphi_p(u_m(x))), \quad x\in(a,b),\\
(-1)^n\varphi_p(u_2^{(n)}(x))^{(n)} &= r_2(x)\,(\varphi_p(u_1(x))+\varphi_p(u_2(x))+\varphi_p(u_m(x))), \quad x\in(a,b),\\
&\cdots \\
(-1)^n\varphi_p(u_m^{(n)}(x))^{(n)} &= r_m(x)\,(\varphi_p(u_1(x))+\varphi_p(u_2(x))+\varphi_p(u_m(x))), \quad x\in(a,b),\\
u_j^{(i)}(a) &= 0 = u_j^{(i)}(b),
\end{aligned} \tag{4.1}$$

for $i=0,1,\dots,n-1$ and $j=1,2,\dots,m$, where $m,n\in\mathbb{N}$.

THEOREM 4.1. *If* $(u_1,u_2,\dots,u_m)$ *is a positive solution for (4.1), then one have*

$$\frac{1}{m}\frac{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}{2^{pn-2}(p-1)^{p-1}} \leqslant \sum_{j=1}^m \left(\int_a^b r_j(x)\,((x-a)(b-x))^{pn-1}\,dx\right), \tag{4.2}$$

especially,

$$\frac{1}{m}\frac{2^{pn}(pn-1)^{p-1}[(n-1)!]^p}{(p-1)^{p-1}(b-a)^{pn-1}} \leqslant \sum_{j=1}^m \left(\int_a^b r_j(x)dx\right). \tag{4.3}$$

Proof. As in the proof of Theorem 3.1, we only show the case $m=2$. Multiplying u_1 to the first equation in (4.1) and integrating on $[a,b]$ and using (2.7), (3.4) and (3.5), we obtain from

$$\begin{aligned}\int_a^b |u_1^{(n)}(x)|^p dx &\leqslant \int_a^b r_1(x)|u_1(x)|^p dx + \int_a^b r_1(x)|u_2(x)|^{p-1}|u_1(x)|dx \\ &\leqslant \Omega\left(\int_a^b r_1(x)\left((x-a)(b-x)\right)^{pn-1}dx\right)\left(\int_a^b |u_1^{(n)}(x)|^p dx\right) \\ &\quad +\Omega\left(\int_a^b r_1(x)\left((x-a)(b-x)\right)^{pn-1}dx\right) \\ &\quad \times\left(\int_a^b |u_2^{(n)}(x)|^p dx\right)^{(p-1)/p}\left(\int_a^b |u_1^{(n)}(s)|^p ds\right)^{1/p}, \qquad (4.4)\end{aligned}$$

where

$$\Omega = \frac{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}{2^{pn-2}(p-1)^{p-1}}.$$

Similarly, from the second equation of (4.1), we have

$$\begin{aligned}\int_a^b |u_2^{(n)}(x)|^p dx &\leqslant \int_a^b r_2(x)|u_2(x)|^p dx + \int_a^b r_2(x)|u_1(x)|^{p-1}|u_2(x)|dx \\ &\leqslant \Omega\left(\int_a^b r_2(x)\left((x-a)(b-x)\right)^{pn-1}dx\right)\left(\int_a^b |u_2^{(n)}(x)|^p dx\right) \\ &\quad +\Omega\left(\int_a^b r_2(x)\left((x-a)(b-x)\right)^{pn-1}dx\right) \\ &\quad \times\left(\int_a^b |u_1^{(n)}(x)|^p dx\right)^{(p-1)/p}\left(\int_a^b |u_2^{(n)}(s)|^p ds\right)^{1/p}. \qquad (4.5)\end{aligned}$$

Let us set

$$\mathscr{X} = \int_a^b |u_1^{(n)}(x)|^p dx, \quad \mathscr{Y} = \int_a^b |u_2^{(n)}(x)|^p dx$$

$$\mathscr{C}_1 = \Omega\int_a^b r_1(x)\left((x-a)(b-x)\right)^{pn-1}dx, \quad \mathscr{C}_2 = \Omega\int_a^b r_2(x)\left((x-a)(b-x)\right)^{pn-1}dx.$$

Then from (4.4) and (4.5), we have

$$\begin{aligned}&\mathscr{X} \leqslant \mathscr{C}_1\mathscr{X} + \mathscr{C}_1\mathscr{X}^{1/p}\mathscr{Y}^{(p-1)/p}, \\ &\mathscr{Y} \leqslant \mathscr{C}_2\mathscr{Y} + \mathscr{C}_2\mathscr{Y}^{1/p}\mathscr{X}^{(p-1)/p}, \qquad (4.6)\end{aligned}$$

respectively. Equation (4.6) implies

$$\begin{aligned}&\mathscr{X} \leqslant \mathscr{C}_1(\mathscr{X}+\mathscr{Y}) + \mathscr{C}_1\left(\mathscr{X}^{1/p}\mathscr{Y}^{(p-1)/p} + \mathscr{Y}^{1/p}\mathscr{X}^{(p-1)/p}\right), \\ &\mathscr{Y} \leqslant \mathscr{C}_2(\mathscr{X}+\mathscr{Y}) + \mathscr{C}_2\left(\mathscr{X}^{1/p}\mathscr{Y}^{(p-1)/p} + \mathscr{Y}^{1/p}\mathscr{X}^{(p-1)/p}\right),\end{aligned}$$

respectively. Therefore, we have

$$\mathscr{X}+\mathscr{Y}\leqslant(\mathscr{C}_1+\mathscr{C}_2)(\mathscr{X}+\mathscr{Y})+(\mathscr{C}_1+\mathscr{C}_2)\left(\mathscr{X}^{1/p}\mathscr{Y}^{(p-1)/p}+\mathscr{Y}^{1/p}\mathscr{X}^{(p-1)/p}\right).$$

Since $\mathscr{X}^{1/p}\mathscr{Y}^{(p-1)/p}+\mathscr{Y}^{1/p}\mathscr{X}^{(p-1)/p}\leqslant\mathscr{X}+\mathscr{Y}$ [6, page 38], we obtain

$$\mathscr{X}+\mathscr{Y}\leqslant 2(\mathscr{C}_1+\mathscr{C}_2)(\mathscr{X}+\mathscr{Y}).$$

Hence, we have

$$\frac{1}{2}\leqslant(\mathscr{C}_1+\mathscr{C}_2).$$

That is,

$$\frac{1}{2}\frac{1}{m}\frac{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}{2^{pn-2}(p-1)^{p-1}}\leqslant\sum_{j=1}^{2}\left(\int_a^b r_j(x)\left((x-a)(b-x)\right)^{pn-1}dx\right),$$

For $a\leqslant x\leqslant b$, since $(x-a)(b-x)\leqslant(b-a)^2/4$, we have

$$\frac{1}{2}\frac{2^{pn}(pn-1)^{p-1}[(n-1)!]^p}{(p-1)^{p-1}(b-a)^{pn-1}}\leqslant\sum_{j=1}^{2}\left(\int_a^b r_j(x)dx\right).$$

This completes the proof. □

REMARK 4.2. When $n=1$, the above result coincides with Theorem 4.1 in Sim and Lee [4].

COROLLARY 4.3. *If $(u_1,u_2,\dots,u_m)$ is a positive solution for (4.1) with $p=2$, then one have*

$$\frac{1}{m}\frac{(2n-1)[(n-1)!]^2(b-a)^{2n-1}}{2^{2(n-1)}}\leqslant\sum_{j=1}^{m}\left(\int_a^b r_j(x)\left((x-a)(b-x)\right)^{2n-1}dx\right),$$

especially,

$$\frac{1}{m}\frac{2^{2n}(2n-1)[(n-1)!]^2}{(b-a)^{2n-1}}\leqslant\sum_{j=1}^{m}\left(\int_a^b r_j(x)dx\right).$$

COROLLARY 4.4. *If $(u_1,u_2,\dots,u_m)$ is a positive solution for (4.1) with $r_1=r_2=\cdots=r_m=r$, then one have*

$$\frac{1}{m^2}\frac{(pn-1)^{p-1}[(n-1)!]^p(b-a)^{pn-1}}{2^{pn-2}(p-1)^{p-1}}\leqslant\int_a^b r(x)\left((x-a)(b-x)\right)^{pn-1}dx.$$

especially,

$$\frac{1}{m^2}\frac{2^{pn}(pn-1)^{p-1}[(n-1)!]^p}{(p-1)^{p-1}(b-a)^{pn-1}}\leqslant\int_a^b r(x)dx.$$

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