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OPERATOR INEQUALITIES INVOLVING THE ARITHMETIC, GEOMETRIC, HEINZ AND HERON MEANS

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Abstract. In this paper we present two improved arithmetic-geometric inequalities with Kantorovich constant (Lemma 3 and Lemma 5), based on which we provide some refinements in the operator case and then finally refer to the operator inequalities involving Heinz and Heron means.

1. Introduction

Throughout, let $B(H)$ be the set of all bounded linear operators on a complex Hilbert space H. For A, $B \in B(H)$, A^* denotes the conjugate operator of A; if A and B are self-adjoint operators, the order relation $A \geqslant B$ means, as usual, that $A-B$ is a positive operator.

Let a, b be two nonnegative real numbers, $v \in [0,1]$.

The v-weighted arithmetic, geometric mean of a and b, denoted by $A_v(a,b)$, $G_v(a,b)$, respectively, are defined as

$$A_v(a,b) = (1-v)a + vb, \quad G_v(a,b) = a^{1-v}b^v.$$

It is worth to mention that $A_v(a,b) \geqslant G_v(a,b)$ for all $v \in [0,1]$. This inequality is well known as Young's inequality.

In particular, if $v = \frac{1}{2}$, then $A_{\frac{1}{2}}(a,b) = \frac{a+b}{2}$, $G_{\frac{1}{2}}(a,b) = \sqrt{ab}$ are arithmetic and geometric mean respectively.

Parameterized family of means-Heinz means of a and b are defined as

$$H_v(a,b) = \frac{a^v b^{1-v} + a^{1-v} b^v}{2},$$

for $v \in [0,1]$. Concerns Heinz means: for $v = 0,\ 1$ this is equal to arithmetic mean and for $v = \frac{1}{2}$ to the geometric mean.

Heron means [1] are the family of means defined as a convex combination of the arithmetic and the geometric mean, that is

$$F_\alpha(a,b) = (1-\alpha)A_{\frac{1}{2}}(a,b) + \alpha G_{\frac{1}{2}}(a,b)$$

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for $0 \leqslant \alpha \leqslant 1$.

Since $H_v(a,b)$ is convex as a function of v and attains its minimum at $v = \frac{1}{2}$, it is obvious that

$$\sqrt{ab} \leqslant H_v(a,b) \leqslant \frac{a+b}{2}, \quad 0 \leqslant v \leqslant 1,$$

similarly,

$$\sqrt{ab} \leqslant F_\alpha(a,b) \leqslant \frac{a+b}{2}, \quad 0 \leqslant \alpha \leqslant 1. \tag{1}$$

Let A, $B \in B(H)$ be two positive operators, $v \in [0,1]$.

v-weighted arithmetic mean of A and B, denoted by $A\nabla_v B$, is defined as

$$A\nabla_v B = (1-v)A + vB.$$

If A is invertible, v-geometric mean of A and B, denoted by $A\sharp_v B$, is defined as

$$A\sharp_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}}.$$

In addition, if both A and B are invertible, v-harmonic mean of A and B, denoted by $A!_v B$, is defined as

$$A!_v B = ((1-v)A^{-1} + vB^{-1})^{-1}.$$

For more details, see F. Kubo and T. Ando [11].

When $v = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$, $A!B$ for brevity, respectively.

The operator version of the Heinz means, denoted by $H_v(A,B)$, is defined as

$$H_v(A,B) = \frac{A\sharp_v B + A\sharp_{1-v} B}{2},$$

for $0 \leqslant v \leqslant 1$.

The operator version of the Heron means, denoted by $F_\alpha(A,B)$, is defined as

$$F_\alpha(A,B) = (1-\alpha)A\nabla B + \alpha A\sharp B,$$

for $0 \leqslant \alpha \leqslant 1$.

It is well known that if A and B are positive invertible operators, then

$$A\nabla_v B \geqslant A\sharp_v B \geqslant A!_v B, \tag{2}$$

for $0 < v < 1$. A simple and elegant proof of inequalities (2) was given in [2] and in [3] one can find a detailed approach.

S. Furuichi [6] gave a refinement of (2) as follows (Theorem 2),

$$\begin{aligned} A\nabla_v B &\geqslant S(h^r)A\sharp_v B \\ &\geqslant A\sharp_v B \\ &\geqslant S(h^r)A!_v B \\ &\geqslant A!_v B, \end{aligned} \tag{3}$$

where $r = min\{v, 1-v\}$ and $S(\cdot)$ is the Specht's ratio, denoted by:

$$S(t) = \frac{t^{\frac{1}{t-1}}}{e \ln t^{\frac{1}{t-1}}},$$

for $t > 0$, $t \neq 1$ and $S(1) = \lim_{t \to 1} S(t) = 1$.

Zuo [14] et al. gave a refinement of the first inequality (3) as follows (Theorem 7),

$$K(h,2)^r A \sharp_v B \leqslant A \nabla_v B, \tag{4}$$

where $K(\cdot,2)$ is Kantorovich constant, defined as

$$K(t,2) = \frac{(t+1)^2}{4t},$$

for $t > 0$.

In [5] (Theorem 2.1), S. Furuichi gave another refinement of (2):

$$\begin{aligned} A\nabla_v B &\geqslant A\sharp_v B + 2r(A\nabla B - A\sharp B) \\ &\geqslant A\sharp_v B \\ &\geqslant \{A^{-1}\sharp B^{-1} + 2r(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\}^{-1} \\ &\geqslant A!_v B. \end{aligned}$$

Recently, F. Kittaneh et al. [10], M. Krnić et al. [12] and O. Hirzalleh et al [7]. also studied the similar topics.

In [13], we presented some inequalities related to $A\nabla_v B$, $A\sharp_v B$, $A!_v B$. In this paper we proceed with the results on this topic, obtaining the operator order relations among arithmetic, geometric, Heinz and Heron means. The paper is organized in the following way: in Section 2, two improvements of the Young inequality are given (Lemma 3 and Lemma 5). Based on these two lemmas, several refinements of the operator arithmetic-geometric inequality are obtained. In Section 3, related operator inequalities involving Heinz and Heron means are presented.

2. On the operator arithmetic-geometric mean inequality

In this section we give refinements of the operator arithmetic-geometric inequality on the related ones. The techniques are based on the monotonicity property of operator functions, described in the following lemma (for more details, the reader is referred to [4].)

LEMMA 1. *Let $X \in B(H)$ be self-adjoint and let f and g be continuous real functions such that $f(t) \geqslant g(t)$ for all $t \in Sp(X)$ (the spectrum of X). Then $f(X) \geqslant g(X)$.*

In the sequel, we make use of Lemma 2.1 from [13] which we cite here.

LEMMA 2. *If* $a,\ b>0$*, then for any* $v\in[0,1]-\frac{1}{2}$*, the inequality*

$$K(\sqrt{h},2)^{r'}a^{1-v}b^{v}+r(\sqrt{a}-\sqrt{b})^{2}\leqslant(1-v)a+vb \tag{5}$$

holds, where $h=\frac{b}{a}$*,* $r=\min\{v,1-v\}$ *and* $r'=\min\{2r,1-2r\}$*.*

Replacing a and b by a^2 and b^2 in (5), respectively, we have

$$K(h,2)^{r'}(a^{1-v}b^{v})^{2}+r(a-b)^{2}\leqslant(1-v)a^{2}+vb^{2}. \tag{6}$$

LEMMA 3. *If* $a,\ b>0$*, then for any* $v\in[0,1]-\frac{1}{2}$*, the inequality*

$$K(h,2)^{r'}(a^{1-v}b^{v})^{2}+r^{2}(a-b)^{2}\leqslant((1-v)a+vb)^{2} \tag{7}$$

holds, where $h=\frac{b}{a}$*,* $r=\min\{v,1-v\}$ *and* $r'=\min\{2r,1-2r\}$*.*

Proof. If $v<\frac{1}{2}$, by (6), we have

$$\begin{aligned}((1-v)a+vb)^{2}\ -\ r^{2}(a-b)^{2}&\\ &=((1-v)a+vb)^{2}-v^{2}(a-b)^{2}\\ &=(1-2v)a^{2}+2vab\\ &=(1-v)a^{2}+vb^{2}+2vab-va^{2}-vb^{2}\\ &=(1-v)a^{2}+vb^{2}-v(a-b)^{2}\\ &\geqslant K(h,2)^{r'}(a^{1-v}b^{v})^{2}.\end{aligned}$$

Similarly, if $v>\frac{1}{2}$, by (6), we also have

$$\begin{aligned}((1-v)a+vb)^{2}\ -\ r^{2}(a-b)^{2}&\\ &\geqslant K(h,2)^{r'}(a^{1-v}b^{v})^{2}.\end{aligned}$$

This completes the proof. □

REMARK 1. Since $K(t,2)=\frac{(t+1)^{2}}{4t}\geqslant 1$ for all $t>0$, the inequality (7) is a refinement of the following inequality

$$(a^{1-v}b^{v})^{2}+r^{2}(a-b)^{2}\leqslant((1-v)a+vb)^{2},$$

which is due to Hirzallah and Kittaneh [8].

In the sequel, we make use of Lemma 2.2 from [13] which we cite here.

LEMMA 4. *If* $a,\ b>0$*, then for any* $v\in[0,1]-\frac{1}{2}$*, the inequality*

$$K(\sqrt{h},2)^{-r'}a^{1-v}b^{v}+s(\sqrt{a}-\sqrt{b})^{2}\geqslant(1-v)a+vb \tag{8}$$

holds, where $h=\frac{b}{a}$*,* $r=\min\{v,1-v\}$*,* $r'=\min\{2r,1-2r\}$ *and* $s=\max\{v,1-v\}$*.*

Replacing a and b by a^2 and b^2 in (8), respectively, we have

$$K(h,2)^{-r'}(a^{1-v}b^v)^2 + s(a-b)^2 \geqslant (1-v)a^2 + vb^2. \tag{9}$$

LEMMA 5. *If $a,\ b > 0$, then for any $v \in [0,1] - \frac{1}{2}$, the inequality*

$$K(h,2)^{-r'}(a^{1-v}b^v)^2 + s^2(a-b)^2 \geqslant ((1-v)a+vb)^2 \tag{10}$$

holds, where $h = \frac{b}{a}$, $r = \min\{v, 1-v\}$, $r' = \min\{2r, 1-2r\}$ and $s = \max\{v, 1-v\}$.

Proof. If $v < \frac{1}{2}$, by (9), we have

$$\begin{aligned}((1-v)a+vb)^2 \; - \; s^2(a-b)^2 &\\ &= ((1-v)a+vb)^2 - (1-v)^2(a-b)^2\\ &= (2v-1)b^2 + 2(1-v)ab\\ &= (1-v)a^2 + vb^2 + 2(1-v)ab - (1-v)a^2 - (1-v)b^2\\ &= (1-v)a^2 + vb^2 - (1-v)(a-b)^2\\ &\leqslant K(h,2)^{-r'}(a^{1-v}b^v)^2.\end{aligned}$$

Similarly, if $v > \frac{1}{2}$, by (9), we also have

$$\begin{aligned}((1-v)a+vb)^2 \; - \; s^2(a-b)^2 &\\ &\leqslant K(h,2)^{-r'}(a^{1-v}b^v)^2.\end{aligned}$$

This completes the proof. $\square$

REMARK 2. Since $K(t,2) = \frac{(t+1)^2}{4t} \geqslant 1$ for all $t > 0$, the inequality (10) is a refinement of the following inequality

$$(a^{1-v}b^v)^2 + s^2(a-b)^2 \geqslant ((1-v)a+vb)^2$$

which is due to He and Zou [9].

Now, we present our first main result in this section based on Lemma 3. Concerning Theorem 1, it would be appropriate to mention here the results of Furuichi (with Specht's ratio) and of Zuo (with Kantorovich constant) as well as two recent papers mentioned in the beginning of the report (Kittaneh, Krnić, Lovričević, Pečarić, see [10], [12]), of which have the same assumptions and similar type of refinements.

THEOREM 1. *Let $A,\ B$ be two positive operators and positive real numbers m, m', M, M' satisfy either of the following conditions:*

(i) $0 < m'I \leqslant A \leqslant mI < MI \leqslant B \leqslant M'I$

(ii) $0 < m'I \leqslant B \leqslant mI < MI \leqslant A \leqslant M'I$.

Then

$$K(\sqrt{h},2)^{r'}A\sharp_v B + 2r^2(A\nabla B - A\sharp B) \leqslant (1-v)^2A+v^2B+2v(1-v)A\sharp B \tag{11}$$

holds for $v\in[0,1]-\frac{1}{2}$, *where* $r=\min\{v,1-v\}$, $r'=\min\{2r,1-2r\}$, $h=\frac{M}{m}$ *and* $h'=\frac{M'}{m'}$.

Proof. We shall use the same procedure as in the proof of Theorem 3.1 [13]. By Lemma 3, we have

$$K(\sqrt{b},2)^{r'}b^v+r^2(1-\sqrt{b})^2\leqslant((1-v)+v\sqrt{b})^2, \tag{12}$$

for any $b>0$, which for $X=A^{-\frac{1}{2}}BA^{\frac{1}{2}}$ and thus for $Sp(X)\subseteq(0,+\infty)$ then provides

$$K(\sqrt{t},2)^{r'}t^v+r^2(1-\sqrt{t})^2\leqslant((1-v)+v\sqrt{t})^2, \tag{13}$$

for any $t\in Sp(X)$.

In the first case, we have $I<hI\leqslant X\leqslant h'I$, thus $Sp(X)=[h,h']\subset(1,+\infty)$.

By (13), we obtain

$$(\min_{h\leqslant t\leqslant h'}K(\sqrt{t},2)^{r'})t^v+r^2(1-\sqrt{t})^2\leqslant((1-v)+v\sqrt{t})^2. \tag{14}$$

Since Kantorovich constant $K(t,2)=\frac{(t+1)^2}{4t}$ is an increasing function on $(1,\infty)$, it follows that

$$\min_{h\leqslant t\leqslant h'}K(\sqrt{t},2)^{r'}=K(\sqrt{h},2)^{r'}. \tag{15}$$

Combining (14) with (15), we obtain

$$K(\sqrt{h},2)^{r'}t^v+r^2(1-\sqrt{t})^2\leqslant((1-v)+v\sqrt{t})^2. \tag{16}$$

Similarly, in the second case, we have $0<\frac{1}{h'}I\leqslant X\leqslant\frac{1}{h}I$, thus $Sp(X)=[\frac{1}{h'},\frac{1}{h}]\subset(0,1)$.

By(13), we obtain

$$(\min_{\frac{1}{h'}\leqslant t\leqslant\frac{1}{h}}K(\sqrt{t},2)^{r'})t^v+r^2(1-\sqrt{t})^2\leqslant((1-v)+v\sqrt{t})^2, \tag{17}$$

Since Kantorovich constant $K(t,2)=\frac{(t+1)^2}{4t}$ is an decreasing function on $(0,1)$, it follows that

$$\min_{\frac{1}{h'}\leqslant t\leqslant\frac{1}{h}}K(\sqrt{t},2)^{r'}=K(\sqrt{\frac{1}{h}},2)^{r'}. \tag{18}$$

Since $K(t,2)=K(\frac{1}{t},2)$, (17) and (18) provides

$$K(\sqrt{h},2)^{r'}t^v+r^2(1-\sqrt{t})^2\leqslant((1-v)+v\sqrt{t})^2. \tag{19}$$

It is worth to mention that (19) is the same as (16).

According to Lemma 1, by (16), we get the order relation,

$$K(\sqrt{h},2)^{r'}X^{v}+r^{2}(I-X^{\frac{1}{2}})^{2}\leqslant((1-v)I+vX^{\frac{1}{2}})^{2}. \tag{20}$$

Multiplying both sides of (20) by $A^{\frac{1}{2}}$, we get (11).

This completes the proof. □

Since $A\nabla B-A\sharp B\geqslant 0$ and $K(\sqrt{h},2)^{r'}\geqslant 1$, relation (11) provides

$$\begin{aligned}
0 &< A\sharp_{v}B\\
&\leqslant K(\sqrt{h},2)^{r'}A\sharp_{v}B\\
&\leqslant K(\sqrt{h},2)^{r'}A\sharp_{v}B+2r^{2}(A\nabla B-A\sharp B)\\
&\leqslant (1-v)^{2}A+v^{2}B+2v(1-v)A\sharp B\\
&\leqslant (1-v)^{2}A+v^{2}B+2v(1-v)A\nabla B\\
&= A\nabla_{v}B.
\end{aligned} \tag{21}$$

Replacing A and B by A^{-1} and B^{-1} in (21), respectively, then we obtain

$$\begin{aligned}
0 &< A^{-1}\sharp_{v}B^{-1}\\
&\leqslant K(\sqrt{h},2)^{r'}A^{-1}\sharp_{v}B^{-1}\\
&\leqslant K(\sqrt{h},2)^{r'}A^{-1}\sharp_{v}B^{-1}+2r^{2}(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\\
&\leqslant (1-v)^{2}A^{-1}+v^{2}B^{-1}+2v(1-v)A^{-1}\sharp B^{-1}\\
&\leqslant (1-v)^{2}A^{-1}+v^{2}B^{-1}+2v(1-v)A^{-1}\nabla B^{-1}\\
&= A^{-1}\nabla_{v}B^{-1}.
\end{aligned} \tag{22}$$

Taking inverse in (22), we have

$$\begin{aligned}
A!_{v}B &\leqslant \{(1-v)^{2}A^{-1}+v^{2}B^{-1}+2v(1-v)A^{-1}\sharp B^{-1}\}^{-1}\\
&\leqslant \{K(\sqrt{h},2)^{r'}A^{-1}\sharp_{v}B^{-1}+2r^{2}(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\}^{-1}\\
&\leqslant K(\sqrt{h},2)^{-r'}A\sharp_{v}B\\
&\leqslant A\sharp_{v}B.
\end{aligned} \tag{23}$$

REMARK 3. Combining (21) with (23), we obtain a refinement of (2),

$$\begin{aligned}
A!_{v}B &\leqslant \{(1-v)^{2}A^{-1}+v^{2}B^{-1}+2v(1-v)A^{-1}\sharp B^{-1}\}^{-1}\\
&\leqslant \{K(\sqrt{h},2)^{r'}A^{-1}\sharp_{v}B^{-1}+2r^{2}(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\}^{-1}\\
&\leqslant K(\sqrt{h},2)^{-r'}A\sharp_{v}B\\
&\leqslant A\sharp_{v}B\\
&\leqslant K(\sqrt{h},2)^{r'}A\sharp_{v}B\\
&\leqslant K(\sqrt{h},2)^{r'}A\sharp_{v}B+2r^{2}(A\nabla B-A\sharp B)\\
&\leqslant (1-v)^{2}A+v^{2}B+2v(1-v)A\sharp B\\
&\leqslant A\nabla_{v}B.
\end{aligned}$$

REMARK 4. Replacing v by $1-v$ in (21), then combining (21) we have

$$\begin{aligned}0 &< H_v(A,B)\\ &\leqslant K(\sqrt{h},2)^{r'} H_v(A,B)\\ &\leqslant K(\sqrt{h},2)^{r'} H_v(A,B) + 2r^2(A\nabla B - A\sharp B)\\ &\leqslant \frac{(1-v)^2+v^2}{2}A + \frac{(1-v)^2+v^2}{2}B + 2v(1-v)A\sharp B\\ &\leqslant \frac{(1-v)^2+v^2}{2}A + \frac{(1-v)^2+v^2}{2}B + 2v(1-v)A\nabla B\\ &= A\nabla B,\end{aligned}$$

which is a refinement of $H_v(A,B) \leqslant A\nabla B$.

The following result is Based on Lemma 5.

THEOREM 2. *Let A, B be two positive operators and positive real numbers m, m', M, M' satisfy either of the following conditions:*

(i) $0 < m'I \leqslant A \leqslant mI < MI \leqslant B \leqslant M'I$

(ii) $0 < m'I \leqslant B \leqslant mI < MI \leqslant A \leqslant M'I$.

Then

$$K(\sqrt{h},2)^{-r'} A\sharp_v B \quad + 2s^2(A\nabla B - A\sharp B) \geqslant (1-v)^2 A + v^2 B + 2v(1-v)A\sharp B \tag{24}$$

holds for $v \in [0,1] - \frac{1}{2}$, *where* $r = \min\{v, 1-v\}$, $s = \max\{v, 1-v\}$, $r' = \min\{2r, 1-2r\}$, $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$.

Proof. By Lemma 5, we have

$$((1-v)+v\sqrt{b})^2 \leqslant K(\sqrt{b},2)^{-r'} b^v + s^2(1-\sqrt{b})^2 \tag{25}$$

for any $b > 0$.

With $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and thus $Sp(X) \subseteq (0,+\infty)$, relation (25) provides

$$((1-v)+v\sqrt{t})^2 \leqslant K(\sqrt{t},2)^{-r'} t^v + s^2(1-\sqrt{t})^2 \tag{26}$$

for any $t \in Sp(X)$.

By the same procedure as in Theorem 1, we can obtain

$$((1-v)+v\sqrt{t})^2 \leqslant K(\sqrt{h},2)^{-r'} t^v + s^2(1-\sqrt{t})^2 \tag{27}$$

By Lemma 1, we have

$$((1-v)I+vX^{\frac{1}{2}})^2 \leqslant K(\sqrt{h},2)^{-r'} X^v + s^2(I-X^{\frac{1}{2}})^2 \tag{28}$$

Multiplying both sides of (28) by $A^{\frac{1}{2}}$, the proof is completed. □

3. Operator inequalities involving the Heinz and the Heron means

In the last section, we present some inequalities related to the Heron means. It is easy to see that the following inequalities hold

$$A\sharp B \leqslant F_\alpha(A,B) \leqslant A\nabla B, \tag{29}$$

for $0 \leqslant \alpha \leqslant 1$, where A, B are positive and invertible operators. (29) is a refinement of the following inequality,

$$A\sharp B \leqslant A\nabla B,$$

which is due to T. Ando [11].

In [1], Bhatia gave the following inequality between the Heinz and Heron means,

$$H_v(a,b) \leqslant F_{\alpha(v)}(a,b), \tag{30}$$

for $v \in [0,1]$, where $\alpha(v) = 1-4(v-v^2)$.

Basing on (30), we have the following Theorem.

THEOREM 3. *Let A and B be two positive and invertible operators. Then*

$$H_v(A,B) \leqslant F_{\alpha(v)}(A,B), \tag{31}$$

for $v \in [0,1]$, where $\alpha(v) = 1-4(v-v^2)$.

Proof. Let $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Then $Sp(X) = (0,+\infty)$. Thus for $t \in Sp(X)$ we have, by (30):

$$\frac{t^v + t^{1-v}}{2} \leqslant (1-\alpha(v))t^{\frac{1}{2}} + \alpha(v)\frac{t+1}{2}. \tag{32}$$

By Lemma 1, we have

$$\frac{X^v + X^{1-v}}{2} \leqslant (1-\alpha(v))X^{\frac{1}{2}} + \alpha(v)\frac{X+I}{2}. \tag{33}$$

Multiplying both sides of inequality (33) by $A^{\frac{1}{2}}$, the proof is completed. □

REMARK 5. Because of $A\sharp B \leqslant A\nabla B$, (31) is a refinement of the inequality $H_v(A,B) \leqslant A\nabla B$.

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