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OPIAL–TYPE INEQUALITIES FOR TWO FUNCTIONS WITH GENERAL KERNELS AND APPLICATIONS

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Abstract. In this paper, we establish new Opial-type inequalities for general kernels. We prove the converse of our general result and related extreme case. As applications of our main results we extract the results of [12]. We provide the related applications for Widder's derivative and linear differential operator. At the end, we give the discrete analogue corresponding to our main results.

1. Introduction

Mathematical inequalities which involve derivatives and integrals of functions is of great interest. Opial's inequality [17] is of great importance in mathematics with respect to the applications in theory of differential equations and difference equations. Many mathematicians gave the improvements and generalizations in last few decades to add the considerable contribution in the literature and it has attracted a great deal of attention in the recent literature (see, for instance, [1], [2], [3], [6], [8], [13], [18]).

Let us recall that the original Opial's inequality [17] (see also [16, p. 114]) states the following:

THEOREM 1.1. *Let $a > 0$. If $f \in C^1[0,a]$ with $f(0) = f(a) = 0$ and $f(t) > 0$ on $(0,a)$, then*

$$\int_0^a |f(t)f'(t)|dt \leqslant \frac{a}{4}\int_0^a (f'(t))^2 dt.$$

The constant $\frac{a}{4}$ is the best possible.

Agarwal, Alzer and Pang [2, 3, 5] study the Opial-type inequalities involving ordinary derivatives and their applications in differential equations and difference equations. Here our main purpose is to give the Opial-type inequalities for general kernels. We also provide connection between our results in this paper with [12]. We provide the fractional versions of known Opial-type inequalities and they will include three main types of fractional derivatives: Riemann-Liouville, Caputo and Canavati type.

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By $C^n[a,b]$ we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order n, and $AC[a,b]$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^n[a,b]$ we denote the space of all functions $f \in C^{n-1}[a,b]$ with $f^{(n-1)} \in AC[a,b]$.

By $L_p[a,b]$, $1 \leqslant p < \infty$, we denote the space of all Lebesgue measurable functions f for which $|f^p|$ is Lebesgue integrable on $[a,b]$, and by $L_\infty[a,b]$ the set of all functions measurable and essentially bounded on $[a,b]$. Clearly, $L_\infty[a,b] \subset L_p[a,b]$ for all $p \geqslant 1$.

We say that a function $g:[a,b] \to \mathbb{R}$ belongs to the class $U(f,k)$ if it admits the representation

$$|g(t)| \leqslant \int_a^t k(t,\tau)\,|f(\tau)|\,d\tau,$$

where f is a continuous function and k is an arbitrary non-negative kernel such that $f(t)>0$ implies $g(\tau)>0$ for every $t \in [a,b]$. We also assume that all integrals under consideration exist and that they are finite.

The paper is organized in the following way: In Section 2, we prove the Opial-type inequalities involving two functions for general kernel with related extreme case. Also we prove the converse of our main result. In Section 3, we give an application of our main results and we will extract results of [12] from our main results with general kernels as applications of fractional derivative. In Section 4, we give results for Widder's derivatives. Section 5 is dedicated to results for linear differential operators. At the end, we conclude this paper by providing the discrete analogue of results given in Section 2.

2. Main results

The proofs of our results are similar to the proofs in [12] but for completeness of results and for the reader's convenience we will also give short version of proofs which resulted new inequalities for general kernels.

Our first main result is given in the following theorem.

THEOREM 2.1. *Let $g_1 \in U(f_1,k)$, $g_2 \in U(f_2,k)$. Let $\varphi > 0$, $w \geqslant 0$ be measurable functions on $[a,x]$, and k be a non-negative measurable kernel. Let $r>1$, $r>q>0$ and $p \geqslant 0$. Let $f_1, f_2 \in L_r[a,b]$. Then the following inequality holds:*

$$\int_a^x w(t)(|g_1(t)|^p|f_2(t)|^q + |g_2(t)|^p|f_1(t)|^q)dt \leqslant 2^{1-\frac{q}{r}}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}}$$
$$\times\left(\int_a^x [h(t)]^{\frac{r}{r-q}}dt\right)^{\frac{r-q}{r}}\left(\int_a^x \varphi(\tau)\left[|f_1(\tau)|^r + |f_2(\tau)|^r\right]d\tau\right)^{\frac{p+q}{r}}, \quad (2.1)$$

where

$$h(t) = w(t)\left[\int_a^t k(t,\tau)^{\frac{r}{r-1}}\varphi(\tau)^{\frac{1}{1-r}}\,d\tau\right]^{\frac{p(r-1)}{r}}[\varphi(t)]^{-\frac{q}{r}}, \quad (2.2)$$

and

$$d_{\frac{p}{q}} = \begin{cases} 2^{1-\frac{p}{q}}, \ 0 \leqslant p \leqslant q; \\ 1, \qquad p \geqslant q. \end{cases}$$

Proof. Since $g_1 \in U(f_1,k)$, $\varphi(\tau) > 0$, and using the Hölder inequality for $\{\frac{r}{r-1}, r\}$, we get that

$$\begin{aligned} |g_1(t)| &\leqslant \left(\int_a^t k(t,\tau)^{\frac{r}{r-1}} \varphi(\tau)^{\frac{1}{1-r}}\, d\tau\right)^{\frac{r-1}{r}} \left(\int_a^t \varphi(\tau)\,|f_1(\tau)|^r\, d\tau\right)^{\frac{1}{r}} \\ &\leqslant [P(t)]^{\frac{r-1}{r}} [G(t)]^{\frac{1}{r}}. \end{aligned} \tag{2.3}$$

Let

$$F(t) = \int_a^t \varphi(\tau)\,|f_2(\tau)|^r\, d\tau. \tag{2.4}$$

Then

$$|f_2(t)|^q = [\varphi(t)]^{-\frac{q}{r}} [F'(t)]^{\frac{q}{r}}. \tag{2.5}$$

Now (2.3) and (2.5) implies that for $w \geqslant 0$,

$$w(t)|g_1(t)|^p |f_2(t)|^q \leqslant h(t)[G(t)]^{\frac{p}{r}} [F'(t)]^{\frac{q}{r}},$$

where

$$h(t) = w(t)[P(t)]^{\frac{p(r-1)}{r}} [\varphi(t)]^{-\frac{q}{r}}. \tag{2.6}$$

Now integrating over $[a,x]$ and using Hölder's inequality for $\{\frac{r}{r-q}, \frac{r}{q}\}$, we obtain

$$\int_a^x w(t)|g_1(t)|^p |f_2(t)|^q dt \leqslant \left(\int_a^x [h(t)]^{\frac{r}{r-q}} dt\right)^{\frac{r-q}{r}} \left(\int_a^x [G(t)]^{\frac{p}{q}} F'(t) dt\right)^{\frac{q}{r}}. \tag{2.7}$$

Similarly we can write

$$\int_a^x w(t)|g_2(t)|^p |f_1(t)|^q dt \leqslant \left(\int_a^x [h(t)]^{\frac{r}{r-q}} dt\right)^{\frac{r-q}{r}} \left(\int_a^x [F(t)]^{\frac{p}{q}} G'(t) dt\right)^{\frac{q}{r}}. \tag{2.8}$$

Now we need the simple inequalities to complete our result:

$$c_\varepsilon (A+B)^\varepsilon \leqslant A^\varepsilon + B^\varepsilon \leqslant d_\varepsilon (A+B)^\varepsilon, \quad (A,B \geqslant 0), \tag{2.9}$$

where

$$c_\varepsilon = \begin{cases} 1, \qquad 0 \leqslant \varepsilon \leqslant 1; \\ 2^{1-\varepsilon}, \ \varepsilon \geqslant 1, \end{cases} \quad \textit{and} \quad d_\varepsilon = \begin{cases} 2^{1-\varepsilon}, \ 0 \leqslant \varepsilon \leqslant 1; \\ 1, \qquad \varepsilon \geqslant 1. \end{cases}$$

Therefore from (2.7), (2.8) and (2.9), with $r > q$, we conclude that

$$\int_a^x w(t)[|g_1(t)|^p|f_2(t)|^q + |g_2(t)|^p|f_1(t)|^q]dt$$

$$\leqslant 2^{1-\frac{q}{r}} \left(\int_a^x [h(t)]^{\frac{r}{r-q}} dt\right)^{\frac{r-q}{r}} \left(\int_a^x [G(t)]^{\frac{p}{q}} F'(t) + [F(t)]^{\frac{p}{q}} G'(t) dt\right)^{\frac{q}{r}}. \quad (2.10)$$

Since $G(a) = F(a) = 0$, then with (2.9) follows that

$$\int_a^x \left[[G(t)]^{\frac{p}{q}} F'(t) + [F(t)]^{\frac{p}{q}} G'(t)\right] dt \leqslant \frac{q}{p+q}(d_{\frac{p}{q}} - 2^{-\frac{p}{q}})[G(x) + F(x)]^{\frac{p}{q}+1}. \quad (2.11)$$

Using (2.11) in (2.10), we can obtain (2.1). □

The upcoming theorem is the extreme case of the Theorem 2.1.

THEOREM 2.2. *Let* $g_i \in U(f_1, k_i)$, $\widetilde{g}_i \in U(f_2, k_i)$, $(i = 1, 2)$. *Let* $w \geqslant 0$ *be measurable function on* $[a, x]$ *and* $p, q_1, q_2 \geqslant 0$ *and* $f_1, f_2 \in Ł_\infty[a, b]$. *Then the following inequality holds:*

$$\int_a^x w(t)\,[|g_1(t)|^{q_1}|\widetilde{g}_2(t)|^{q_2}|f_1(t)|^p + |g_2(t)|^{q_2}|\widetilde{g}_1(t)|^{q_1}|f_2(t)|^p]\,dt$$

$$\leqslant \|w\|_\infty \int_a^x \left(\int_a^t k_1(t,\tau)d\tau\right)^{q_1} \left(\int_a^t k_2(t,\tau)d\tau\right)^{q_2} dt$$

$$\times \frac{1}{2}\left[\|f_1\|_\infty^{2(q_1+p)} + \|f_1\|_\infty^{2q_2} + \|f_2\|_\infty^{2q_2} + \|f_2\|_\infty^{2(q_1+p)}\right]. \quad (2.12)$$

Proof. Since $g_i \in U(f_1, k_i)$, and $q_i \geqslant 0$ for $(i = 1, 2)$, we have

$$|g_i(t)|^{q_i} \leqslant \left(\int_a^t k_i(t,\tau)d\tau\right)^{q_i} \|f_1\|_\infty^{q_i}.$$

By analogy we get

$$|\widetilde{g}_i(t)|^{q_i} \leqslant \left(\int_a^t k_i(t,\tau)d\tau\right)^{q_i} \|f_2\|_\infty^{q_i}.$$

Hence

$$|g_1(t)|^{q_1}|\widetilde{g}_2(t)|^{q_2}|f_1(t)|^p \leqslant \left(\int_a^t k_1(t,\tau)d\tau\right)^{q_1} \left(\int_a^t k_2(t,\tau)d\tau\right)^{q_2} \|f_1\|_\infty^{q_1+p}\,\|f_2\|_\infty^{q_2}. \quad (2.13)$$

Likewise we can write

$$|g_2(t)|^{q_2}|\widetilde{g}_1(t)|^{q_1}|f_2(t)|^p \leqslant \left(\int_a^t k_2(t,\tau)d\tau\right)^{q_2}\left(\int_a^t k_1(t,\tau)d\tau\right)^{q_1}\|f_1\|_\infty^{q_2}\|f_2\|_\infty^{q_1+p}. \tag{2.14}$$

From (2.13) and (2.14), we have the inequality (2.12). □

Now we present a counterpart of the Theorem 2.1 for the case $r<0$. Conditions on r and q allow us to apply reverse Hölder's inequalities, first with parameters $\{\frac{r}{r-1}\in(0,1), r<0\}$, then with $\{\frac{r}{r-q}\in(0,1), \frac{r}{q}<0\}$.

THEOREM 2.3. *Let $g_i \in U(f_i,k)$, $(i=1,2)$. Let $\varphi>0$, $w\geqslant 0$ be measurable functions on $[a,x]$ and k be a non-negative measurable kernel. Let $r<0$, $q>0$ and $p\geqslant 0$. Let $f_1,f_2\in L_r[a,b]$, each of which is of fixed sign a.e. on $[a,b]$, with $\frac{1}{f_1},\frac{1}{f_2}\in L_r[a,b]$. Then*

$$\int_a^x w(t)(|g_1(t)|^p|f_2(t)|^q+|g_2(t)|^p|f_1(t)|^q)dt \geqslant 2^{1-\frac{q}{r}}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}$$
$$\times\left(\int_a^x [h(t)]^{\frac{r}{r-q}}dt\right)^{\frac{r-q}{r}}\left(\int_a^x \varphi(\tau)\left[|f_1(\tau)|^r+|f_2(\tau)|^r\right]d\tau\right)^{\frac{p+q}{r}}, \tag{2.15}$$

where h is defined by (2.2).

Proof. Since $g_1\in U(f_1,k)$ and $t\in[a,x]$ for fixed sign of f_1 on $[a,b]$, $\varphi(\tau)>0$ and using reverse Hölder's inequality for $\{\frac{r}{r-1},r\}$ we have

$$|g_1(t)| \geqslant \left(\int_a^t k(t,\tau)^{\frac{r}{r-1}}\varphi(\tau)^{\frac{1}{1-r}}d\tau\right)^{\frac{r-1}{r}}\left(\int_a^t \varphi(\tau)|f_1(\tau)|^r d\tau\right)^{\frac{1}{r}}$$
$$= [P(t)]^{\frac{r-1}{r}}[G(t)]^{\frac{1}{r}}.$$

Let F be defined by (2.4). Then

$$w(t)|g_1(t)|^p|f_2(t)|^q \geqslant h(t)[G(t)]^{\frac{p}{r}}[F'(t)]^{\frac{q}{r}}$$

where h is defined by (2.6). Now integrating over $[a,x]$ and again using reverse Hölder's inequality for $\{\frac{r}{r-q},\frac{r}{q}\}$, we obtain

$$\int_a^x w(t)|g_1(t)|^p|f_2(t)|^q dt \geqslant \left(\int_a^x [h(t)]^{\frac{r}{r-q}}dt\right)^{\frac{r-q}{r}}\left(\int_a^x [G(t)]^{\frac{p}{q}}F'(t)dt\right)^{\frac{q}{r}}, \tag{2.16}$$

and

$$\int_a^x w(t)|g_2(t)|^p|f_1(t)|^q dt \geqslant \left(\int_a^x [h(t)]^{\frac{r}{r-q}} dt\right)^{\frac{r-q}{r}} \left(\int_a^x [F(t)]^{\frac{p}{q}} G'(t) dt\right)^{\frac{q}{r}}. \tag{2.17}$$

For negative power we use the following inequality

$$A^\delta + B^\delta \geqslant 2^{1-\delta}(A+B)^\delta, \quad (\delta < 0;\ A, B \geqslant 0) \tag{2.18}$$

Therefore from (2.16), (2.17) and (2.18), with $\frac{q}{r} < 0$, we conclude that

$$\int_a^x w(t)[|g_1(t)|^p|f_2(t)|^q + |g_2(t)|^p|f_1(t)|^q] dt$$
$$\geqslant 2^{1-\frac{q}{r}} \left(\int_a^x [h(t)]^{\frac{r}{r-q}} dt\right)^{\frac{r-q}{r}} \left(\int_a^x [G(t)]^{\frac{p}{q}} F'(t) + [F(t)]^{\frac{p}{q}} G'(t) dt\right)^{\frac{q}{r}}. \tag{2.19}$$

For $\frac{p}{q} > 0$, we use (2.9) with $G(a) = F(a) = 0$, we have

$$\int_a^x \left[[G(t)]^{\frac{p}{q}} F'(t) + [F(t)]^{\frac{p}{q}} G'(t)\right] dt \geqslant \frac{q}{p+q}\left(c_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)[G(x)+F(x)]^{\frac{p}{q}+1}. \tag{2.20}$$

Using (2.20) in (2.19), we get (2.15). □

3. Applications for fractional derivatives

First we survey some facts about fractional derivatives needed in this paper. For more details see the monographs [14, Chapter 2] and [19, Chapter 1]. We start with the application of our main results for Riemann-Liouville fractional derivative, Caputo fractional derivative and Canavati fractional derivative. Also we will show that the results in this paper are the generalization of the results given in [12].

Let $x \in [a,b]$, $\alpha > 0$, $n = [\alpha] + 1$ ($[\cdot]$ is the integral part) and Γ is the gamma function $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. For $f \in L_1[a,b]$ the *Riemann-Liouville fractional integral* $J^\alpha f$ of order α is defined by

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt.$$

For $f : [a,b] \to \mathbb{R}$ the *Riemann-Liouville fractional derivative* $D^\alpha f$ of order α is defined by

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt = \frac{d^n}{dx^n} J^{n-\alpha} f(x).$$

In addition, we stipulate $J^0 f := f =: D^0 f$ and $J^{-\alpha} f := D^\alpha f$ if $\alpha > 0$.

Next, define n as

$$n = [\alpha]+1\,, \text{ for } \alpha \notin \mathbb{N}_0; \quad n = \alpha\,, \text{ for } \alpha \in \mathbb{N}_0\,. \tag{3.1}$$

For n given by (3.1) and $f \in AC^n[a,b]$ the *Caputo fractional derivative* ${}^C D^\alpha f$ of order α is defined by

$${}^C D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt = J^{n-\alpha} f^{(n)}(x).$$

A third fractional derivative, the *Canavati fractional derivative* ${}^{\bar{C}} D^\alpha f$ of order α, is defined for $f \in C^\alpha[a,b] = \left\{ f \in C^{n-1}[a,b] : J^{n-\alpha} f^{(n-1)} \in C^1[a,b] \right\}$ by

$${}^{\bar{C}} D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{n-\alpha-1} f^{(n-1)}(t)\,dt = \frac{d}{dx} J^{n-\alpha} f^{(n-1)}(x)\,.$$

If $\alpha \in \mathbb{N}$ then $D^\alpha f = {}^C D^\alpha f = {}^{\bar{C}} D^\alpha f = f^{(\alpha)}$, the ordinary α-order derivatives.

The next theorem is composition identity for the Riemann-Liouville fractional derivatives. For details see [11].

THEOREM 3.1. *Let* $\alpha > \beta \geqslant 0$, $n = [\alpha]+1$, $m = [\beta]+1$ *and let* $f \in AC^n[a,b]$ *be such that* $D^\alpha f, D^\beta f \in L_1[a,b]$.

(*i*) *If* $\alpha - \beta \notin \mathbb{N}$ *and* f *is such that* $D^{\alpha-k} f(a) = 0$ *for* $k = 1,\ldots,n$ *and* $D^{\beta-k} f(a) = 0$ *for* $k = 1,\ldots,m$, *then*

$$D^\beta f(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-t)^{\alpha-\beta-1} D^\alpha f(t)\,dt\,, \quad x \in [a,b]\,. \tag{3.2}$$

(*ii*) *If* $\alpha - \beta = l \in \mathbb{N}$ *and* f *is such that* $D^{\alpha-k} f(a) = 0$ *for* $k = 1,\ldots,l$, *then* (3.2) *holds.*

COROLLARY 3.2. [11, Corollary 1] *Let* $\alpha > \beta \geqslant 0$, $n = [\alpha]+1$, $m = [\beta]+1$. *Composition identity* (3.2) *is valid if one of the following conditions holds:*

(*i*) $f \in J^\alpha(L_1[a,b]) = \{f : f = J^\alpha \varphi, \varphi \in L_1[a,b]\}$.

(*ii*) $J^{n-\alpha} f \in AC^n[a,b]$ *and* $D^{\alpha-k} f(a) = 0$ *for* $k = 1,\ldots n$.

(*iii*) $D^{\alpha-1} f \in AC[a,b]$, $D^{\alpha-k} f \in C[a,b]$ *and* $D^{\alpha-k} f(a) = 0$ *for* $k = 1,\ldots n$.

(*iv*) $f \in AC^n[a,b]$, $D^\alpha f, D^\beta f \in L_1[a,b]$, $\alpha - \beta \notin \mathbb{N}$, $D^{\alpha-k} f(a) = 0$ *for* $k = 1,\ldots,n$ *and* $D^{\beta-k} f(a) = 0$ *for* $k = 1,\ldots,m$.

(*v*) $f \in AC^n[a,b]$, $D^\alpha f, D^\beta f \in L_1[a,b]$, $\alpha - \beta = l \in \mathbb{N}$, $D^{\alpha-k} f(a) = 0$ *for* $k = 1,\ldots,l$.

(vi) $f \in AC^n[a,b]$, $D^\alpha f, D^\beta f \in L_1[a,b]$ *and* $f^{(k)}(a) = 0$ *for* $k = 0, \dots, n-2$.

(vii) $f \in AC^n[a,b]$, $D^\alpha f, D^\beta f \in L_1[a,b]$, $\alpha \notin \mathbb{N}$ *and* $D^{\alpha-1} f$ *is bounded in a neighborhood of* $t = a$.

In upcoming remarks we extract the results of [12] from our general results.

REMARK 3.3. Let $\alpha > \beta \geqslant 0$. Suppose that one of the conditions (i)-(vii) of the Corollary 3.2 holds for $\{\alpha, \beta, f\}$ and $\{\alpha, \beta, g\}$. Let φ and $w \geqslant 0$ be measurable function $[a,x]$. Let $r > 1$, $r > q > 0$ and $p \geqslant 0$. Let $D^\alpha f, D^\alpha g \in L_r[a,b]$. Then by replacing g_1 by $D^\beta g$, f_1 by $D^\alpha g$, g_2 by $D^\beta f$, f_2 by $D^\alpha f$, with the kernel

$$k(t,\tau) = \begin{cases} \frac{(t-\tau)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}, & a \leqslant \tau \leqslant t; \\ 0, & t < \tau \leqslant b, \end{cases} \tag{3.3}$$

in Theorem 2.1, we obtain the Theorem 2.1 of [12].

REMARK 3.4. Let $\alpha > \beta_1$, $\beta_2 \geqslant 0$. Suppose that one of the conditions (i) to (iiv) of the Corollary 3.2 holds for $\{\alpha, \beta_i, f\}$ and $\{\alpha, \beta_i, g\}$, $(i = 1,2)$. Let φ and $w \geqslant 0$ be measurable function $[a,x]$. Let p, q_1 $q_2 \geqslant 0$. Let $D^\alpha f, D^\alpha g \in L_r[a,b]$. Then by replacing g_1 by $D^{\beta_1} f$, $\widetilde{g}_2$ by $D^{\beta_2} g$, f_1 by $D^\alpha f$, g_2 by $D^{\beta_2} f$, $\widetilde{g}_1$ by $D^{\beta_1} g$, f_2 by $D^\alpha g$ with the kernel

$$k_i(t,\tau) = \begin{cases} \frac{(t-\tau)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)}, & a \leqslant \tau \leqslant t; \\ 0, & t < \tau \leqslant b, \end{cases} \tag{3.4}$$

for $(i = 1,2)$ in Theorem 2.2, we obtain the [12, Theorem 2.2].

REMARK 3.5. Let $\alpha > \beta \geqslant 0$. Suppose that one of the conditions (i) to (iiv) of the Corollary 3.2 holds for $\{\alpha, \beta, f\}$ and $\{\alpha, \beta, g\}$. Let φ and $w \geqslant 0$ be measurable function $[a,x]$. Let $r < 0$, $q > 0$ and $p \geqslant 0$. Let $D^\alpha f, D^\alpha g \in L_r[a,b]$. Then by replacing g_1 by $D^\beta g$, f_1 by $D^\alpha g$, g_2 by $D^\beta f$, f_2 by $D^\alpha f$, and taking the kernel $k(t,\tau)$ defined by (3.3) in Theorem 2.3, we obtain the Theorem 2.3 of [12].

The upcoming theorem is composition identity for the Caputo fractional derivatives. For details see [10, Theorem 2.1].

THEOREM 3.6. *Let* $\alpha > \beta \geqslant 0$ *with* n *and* m *are defined by* (3.1). *Let* $f \in AC^n[a,b]$ *be such that* $f^{(i)}(a) = 0$ *for* $i = m, m+1, \dots, n-1$. *Let* ${}^C D^\alpha f, {}^C D^\beta f \in L_1[a,b]$. *Then*

$${}^C D^\beta f(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-t)^{\alpha-\beta-1} \, {}^C D^\alpha f(t)\, dt, \quad x \in [a,b].$$

REMARK 3.7. Let $\alpha > \beta \geqslant 0$ with n and m are defined by (3.1). Let $f, g \in AC^n[a,b]$ be such that $f^{(i)}(a) = g^{(i)}(a) = 0$ for $i = m, \dots, n-1$. Let $\varphi > 0$ and $\omega \geqslant 0$ be measurable functions on $[a,x]$. Let $r > 1$, $r > q > 0$ and $p \geqslant 0$. Let ${}^C D^\alpha f, {}^C D^\alpha g \in L_r[a,b]$. Then by replacing g_1 by ${}^C D^\beta g$, f_1 by ${}^C D^\alpha g$, g_2 by ${}^C D^\beta f$, f_2 by ${}^C D^\alpha f$ with the kernel defined by (3.3) in Theorem 2.1, we obtain the Theorem 2.4 of [12].

REMARK 3.8. Let $\alpha > \beta_1, \beta_2 \geqslant 0$ with n, m_1 and m_2 given by (3.1). Let $m = \min\{m_1, m_2\}$ and $f, g \in AC^n[a,b]$ be such that $f^{(i)}(a) = g^{(i)}(a) = 0$ for $i = m, \dots, n-1$. Let $w \geqslant 0$ be measurable function on $[a,x]$. Let $p, q_1, q_2 \geqslant 0$ and let ${}^C D^\alpha f, {}^C D^\alpha g \in L_\infty[a,b]$. Then by replacing g_1 by ${}^C D^{\beta_1} f$, $\widetilde{g}_2$ by ${}^C D^{\beta_2} g$, f_1 by ${}^C D^\alpha f$, g_2 by ${}^C D^{\beta_2} f$, $\widetilde{g}_1$ by ${}^C D^{\beta_1} g$, f_2 by ${}^C D^\alpha g$ with the kernel defined by (3.4) in Theorem 2.2, we obtain the Theorem 2.5 of [12].

REMARK 3.9. Let $\alpha > \beta \geqslant 0$ with n and m given by (3.1). Let $f, g \in AC^n[a,b]$ be such that $f^{(i)}(a) = g^{(i)}(a) = 0$ for $i = m, \dots, n-1$. Let $\varphi > 0$ and $\omega \geqslant 0$ be measurable functions on $[a,x]$. Let $r < 0$, $q > 0$ and $p \geqslant 0$. Let ${}^C D^\alpha f, {}^C D^\alpha g \in L_r[a,b]$, each of which is of fixed sign a.e. on $[a,b]$ with $1/{}^C D^\alpha f, 1/{}^C D^\alpha g \in L_r[a,b]$. Then by replacing g_1 by ${}^C D^\beta g$, f_1 by ${}^C D^\alpha g$, g_2 by ${}^C D^\beta f$, f_2 by ${}^C D^\alpha f$, and the kernel $k(t,\tau)$ defined by (3.3) in Theorem 2.3, we obtain Theorem 2.6 of [12].

The following theorem gives the conditions in the composition rule for Canavati fractional derivatives. For details see [9, Theorem 2.1].

THEOREM 3.10. *Let $\alpha > \beta > 0$, $n = [\alpha] + 1$, $m = [\beta] + 1$. Let $f \in C^\alpha[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1, m, \dots, n-2$. Then $f \in C^\beta[a,b]$ and*

$$ {}^{\bar{C}} D^\beta f(x) = \frac{1}{\Gamma(\alpha - \beta)} \int_a^x (x-t)^{\alpha - \beta - 1} \, {}^{\bar{C}} D^\alpha f(t) \, dt, \quad x \in [a,b]. $$

REMARK 3.11. Let $\alpha > \beta \geqslant 0$, $n = [\alpha] + 1$ and $m = [\beta] + 1$. Let $f, g \in C^\alpha[a,b]$ be such that $f^{(i)}(a) = g^{(i)}(a) = 0$ for $i = m-1, \dots, n-2$. Let $\varphi > 0$ and $\omega \geqslant 0$ be measurable functions on $[a,x]$. Let $r > 1$, $r > q > 0$ and $p \geqslant 0$. Let ${}^{\bar{C}} D^\alpha f, {}^{\bar{C}} D^\alpha g \in L_r[a,b]$. Then by replacing g_1 by ${}^{\bar{C}} D^\beta g$, f_1 by ${}^{\bar{C}} D^\alpha g$, g_2 by ${}^{\bar{C}} D^\beta f$, f_2 by ${}^{\bar{C}} D^\alpha f$, and the kernel $k(t,\tau)$ defined by (3.3), we obtain Theorem 2.7 of [12].

REMARK 3.12. Let $\alpha > \beta_1, \beta_2 \geqslant 0$, $n = [\alpha] + 1$ and $m = \min\{[\beta_1] + 1, [\beta_2] + 1\}$. Let $f, g \in C^\alpha[a,b]$ be such that $f^{(i)}(a) = g^{(i)}(a) = 0$ for $i = m-1, \dots, n-2$. Let $w \geqslant 0$ be measurable function on $[a,x]$. Let $p, q_1, q_2 \geqslant 0$ and let ${}^{\bar{C}} D^\alpha f, {}^{\bar{C}} D^\alpha g \in L_\infty[a,b]$. Then by replacing g_1 by ${}^{\bar{C}} D^{\beta_1} f$, $\widetilde{g}_2$ by ${}^{\bar{C}} D^{\beta_2} g$, f_1 by ${}^{\bar{C}} D^\alpha f$, g_2 by ${}^{\bar{C}} D^{\beta_2} f$, $\widetilde{g}_1$ by ${}^{\bar{C}} D^{\beta_1} g$, f_2 by ${}^{\bar{C}} D^\alpha g$ and the kernel $k(t,\tau)$ defined by (3.4), we obtain Theorem 2.8 of [12].

REMARK 3.13. Let $\alpha > \beta \geqslant 0$, $n = [\alpha] + 1$ and $m = [\beta] + 1$. Let $f, g \in C^\alpha[a,b]$ be such that $f^{(i)}(a) = g^{(i)}(a) = 0$ for $i = m-1, \dots, n-2$. Let $\varphi > 0$ and $\omega \geqslant 0$ be measurable functions on $[a,x]$. Let $r < 0$, $q > 0$ and $p \geqslant 0$. Let ${}^{\bar{C}} D^\alpha f, {}^{\bar{C}} D^\alpha g \in L_r[a,b]$, each of which is of fixed sign a.e. on $[a,b]$ with $1/{}^{\bar{C}} D^\alpha f, 1/{}^{\bar{C}} D^\alpha g \in L_r[a,b]$. Then by replacing g_1 by ${}^{\bar{C}} D^\beta g$, f_1 by ${}^{\bar{C}} D^\alpha g$, g_2 by ${}^{\bar{C}} D^\beta f$, f_2 by ${}^{\bar{C}} D^\alpha f$, and the kernel $k(t,\tau)$ defined by (3.3) in Theorem 2.3, we obtain Theorem 2.9 of [12].

4. Applications for Widder's derivatives

Here we give the application of our main results for Widder's derivative. First it is necessary to give some important details about Widder's derivatives (see [20]). Let $f, u_0, u_1, \ldots, u_n \in C^{n+1}([a,b])$, $n \geqslant 0$, and the Wronskians

$$W_i(x) := W[u_0(x), u_1(x), \ldots, u_i(x)] = \begin{vmatrix} u_0(x) & \cdots & u_i(x) \\ u_0'(x) & \cdots & u_i'(x) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ u_0^{(i)}(x) & \cdots & u_i^{(i)}(x) \end{vmatrix},$$

$i = 0, 1, \ldots, n$. Here $W_0(x) = u_0(x)$. Assume $W_i(x) > 0$ over $[a,b]$, $i = 0, 1, \ldots, n$. For $i \geqslant 0$, the differential operator of order i (Widder's derivative):

$$L_i f(x) := \frac{W[u_0(x), u_1(x), \ldots, u_{i-1}(x), f(x)]}{W_{i-1}(x)},$$

$i = 1, \ldots, n+1; L_0 f(x) := f(x)$ for all $x \in [a,b]$. Consider also

$$g_i(x,t) := \frac{1}{W_i(t)} \begin{vmatrix} u_0(t) & \cdots & u_i(t) \\ u_0'(t) & \cdots & u_i'(t) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ u_0^{(i)}(x) & \cdots & u_i^{(i)}(x) \end{vmatrix},$$

$i = 1, 2, \ldots, n$; $g_0(x,t) := \frac{u_0(x)}{u_0(t)}$ for all $x, t \in [a,b]$.

EXAMPLE 4.1. [20] Sets of the form $\{u_0, u_1, u_2, \ldots, u_n\}$ are $\{1, x, x^2, \ldots, x^n\}$, $\{1, \sin x, \cos x, -\sin 2x, \cos 2x, \ldots, (-1)^{n-1} \sin nx, (-1)^{n-1} \cos nx\}$, etc.

We also mention the generalized Widder-Talylor's formula, see [20] (see also [7]).

THEOREM 4.2. *Let the functions $f, u_0, u_1, \ldots, u_n \in C^{n+1}([a,b])$, and the Wronskians $W_0(x), W_1(x), \ldots, W_n(x) > 0$ on $[a,b], x \in [a,b]$. Then for $t \in [a,b]$ we have*

$$f(x) = f(t)\frac{u_0(x)}{u_0(t)} + L_1 f(t) g_1(x,t) + \ldots + L_n f(t) g_n(x,t) + R_n(x)$$

where

$$R_n(x) := \int_t^x g_n(x,t) L_{n+1} f(t)\, dt.$$

For example (see [20]) one could take $u_0(x) = c > 0$. If $u_i(x) = x^i$, $i = 0,1,...,n$, defined on [a,b], then

$$L_i f(t) = f^{(i)}(t) \ \ and \ \ g_i(x,t) = \frac{(x-t)^i}{i!}, \quad t \in [a,b].$$

We need

COROLLARY 4.3. *By additionally assuming for fixed $x_0 \in [a,b]$ that $L_i f(x_0) = 0$, $i = 0,1,...,n$, we get that*

$$f(x) := \int_{x_0}^{x} g_n(x,t) L_{n+1} f(t) dt, \quad for\, all\, x \in [a,b]. \tag{4.1}$$

Now we give the application for the Widder's derivative in upcoming theorem.

THEOREM 4.4. *Let the functions $f, g, u_0, u_1, ..., u_n \in C^{n+1}([a,b])$, and the Wronkians $W_0(x), W_1(x), ..., W_n(x) > 0$ on $[a,b]$, $x \in [a,b]$. Suppose that that the (4.1) holds for the functions f and g. Let $r > 1$, $r > q > 0$ and $p \geqslant 0$. Let $L_{n+1} f, L_{n+1} g \in L_r[a,b]$, $L_i f(a) = 0$, and $L_i g(a) = 0$ $(i = 0,1,...,n)$. Then*

$$\int_a^x w(t)(|f(t)|^p |L_{n+1} g(t)|^q + |g(t)|^p |L_{n+1} f(t)|^q) dt \leqslant 2^{1-\frac{q}{r}} \left(\frac{q}{p+q}\right)^{\frac{q}{r}} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}}$$
$$\times \left(\int_a^x [h(t)]^{\frac{r}{r-q}} dt\right)^{\frac{r-q}{r}} \left(\int_a^x \varphi(\tau) \left[|L_{n+1} f(\tau)|^r + |L_{n+1} g(\tau)|^r\right] d\tau\right)^{\frac{p+q}{r}}, \tag{4.2}$$

where h is defined by (2.2).

Proof. Applying Theorem 2.1 with $g_1 = f$, $f_2 = L_{n+1} g$, $g_2 = g$, $f_1 = L_{n+1} f$ and $k(t,\tau) = g_n(t,\tau)$, we get the inequality (4.2). □

EXAMPLE 4.5. If we take $u_0(x) = c > 0$ and $u_n(x) = x^n$, $n = 0,1,2,...,n$ defined on $[a,b]$, then $L_n f(t) = f^{(n)}(t)$ and $g_n(t,\tau) = \frac{(t-\tau)^n}{n!}$, $\tau \in [a,b]$, and the inequality (4.2) becomes

$$\int_a^x w(t)(|f(t)|^p |g^{(n+1)}(t)|^q + |g(t)|^p |f^{(n+1)}(t)|^q) dt \leqslant 2^{1-\frac{q}{r}} \left(\frac{q}{p+q}\right)^{\frac{q}{r}} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}}$$
$$\times \left(\int_a^x [h(t)]^{\frac{r}{r-q}} dt\right)^{\frac{r-q}{r}} \left(\int_a^x \varphi(\tau) \left[\left|f^{(n+1)}(\tau)\right|^r + \left|g^{(n+1)}(\tau)\right|^r\right] d\tau\right)^{\frac{p+q}{r}},$$

where h is defined by (2.2).

The upcoming theorem is the converse of the Theorem 4.4.

THEOREM 4.6. *Suppose that that the* (4.1) *holds for the functions* f *and* g. *Let* $r<0,\ q>0$ *and* $p\geqslant 0$. *Let* $f_1,f_2\in L_r[a,b]$. *Then*

$$\int_a^x w(t)(|f(t)|^p|L_{n+1}g(t)|^q+|g(t)|^p|L_{n+1}f(t)|^q)dt \geqslant 2^{1-\frac{q}{r}}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}$$
$$\times\left(\int_a^x [h(t)]^{\frac{r}{r-q}}dt\right)^{\frac{r-q}{r}}\left(\int_a^x \varphi(\tau)\left[|L_{n+1}f(\tau)|^r+|L_{n+1}g(\tau)|^r\right]d\tau\right)^{\frac{p+q}{r}},$$

where h *is defined by* (2.2).

Proof. The proof is similar to the proof of Theorem 4.4. □

5. Applications for linear differential operator

The following hypotheses are assumed throughout this section: Let I be a closed interval in $\mathbb{R}$, a a fixed point in I, let Φ be a continuous function nonnegative on $I\times I$, and let $y,\bar{h}\in C(I)$. We assume that the following condition involving Φ, h and y is satisfied (for details see [15]):

$$|y(x)|\leqslant\left|\int_a^x \Phi(x,t)|\bar{h}(t)|dt\right|,\qquad x\in I. \tag{5.1}$$

Some typical example of (5.1) is given below.

EXAMPLE 5.1. Let K be a continuous function on $I\times I$ and let y be defined by

$$y(x)=\int_a^s K(s,t)\bar{h}(t)dt,\qquad s\in I.$$

Then (5.1) holds with $\Phi(s,t)=|K(s,t)|$. A useful modification of this example – easier to attain in practice – is obtained when a function $z\in C(I)$ defined by

$$z(s)=\int_a^s K(s,t)\bar{h}(t)dt$$

satisfies a inequality $|z(t)|\geqslant|y(t)|$. Again (5.1) holds with $\Phi(s,t)=|K(s,t)|$.

In [6], the results yields the Opial-type inequalities for linear differential operator (see [1], [2], [8]).

EXAMPLE 5.2. Let

$$L=\sum_{j=0}^{n-1} a_j(t)D^j+D^n, \quad t\in I,$$

be the linear differential operator with $a_j \in C(I)$, let $\bar{h}\in C(I)$. Let $y_1(x),\dots y_n(x)$ be the set of lineary independent solution of $L_y=0$ and here is the associated Green's function for L is

$$G(x,t):=\frac{\begin{vmatrix} y_1(t) & \cdots & y_n(t) \\ y_1'(t) & \cdots & y_n'(t) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ y_1^{(n-2)}(t) & \cdot & y_n^{(n-2)}(t) \\ y_1(x) & \cdots & y_n(x) \end{vmatrix}}{\begin{vmatrix} y_1(t) & \cdots & y_n(t) \\ y_1'(t) & \cdots & y_n'(t) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ y_1^{(n-2)}(t) & \cdot & y_n^{(n-2)}(t) \\ y_1(t) & \cdots & y_n(t) \end{vmatrix}},$$

which is continuous function on I^2. It is known that

$$y(x)=\int_a^x G(x,t)\bar{h}(t)dt$$

is the unique solution to the initial value problem

$$L_y=\bar{h}, \quad y^{(j)}(a)=0, \quad j=0,1,\dots,n-1.$$

Then (5.1) is satisfied for y and $\bar{h}$ with with $\Phi(x,t)=|G(x,t)|$.

THEOREM 5.3. *Let $y_i \in U(\bar{h}_i,G)$ $(i=1,2)$. Let $\varphi>0$, $w\geqslant 0$ be measurable functions on $[a,x]$. Let $r>1$, $r>q>0$ and $p\geqslant 0$. Let $h_1,h_2\in L_r[a,b]$. Then*

$$\int_a^x w(t)(|y_1(t)|^p|\bar{h}_2(t)|^q+|y_2(t)|^p|\bar{h}_1(t)|^q)dt \leqslant 2^{1-\frac{q}{r}}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(d_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}$$
$$\times\left(\int_a^x [h(t)]^{\frac{r}{r-q}}dt\right)^{\frac{r-q}{r}}\left(\int_a^x \varphi(\tau)\left[\left|\bar{h}_1(\tau)\right|^r+\left|\bar{h}_2(\tau)\right|^r\right]d\tau\right)^{\frac{p+q}{r}}, \quad (5.2)$$

where h is defined by (2.2).

Proof. Applying Theorem 2.1 with $g_1 = y_1, f_2 = \bar{h}_2, g_2 = y_2, f_1 = \bar{h}_1$ and $k(t,\tau) = G(t,\tau)$, we get the inequality (5.2). □

THEOREM 5.4. *Let $y_i \in U(\bar{h}_i, G)$ $(i = 1,2)$. Let $\varphi > 0$, $w \geqslant 0$ be measurable functions on $[a,x]$. Let $r < 0$, $q > 0$ and $p \geqslant 0$. Let $h_1, h_2 \in L_r[a,b]$. Then*

$$\int_a^x w(t)(|y_1(t)|^p |\bar{h}_2(t)|^q + |y_2(t)|^p |\bar{h}_1(t)|^q) dt \geqslant 2^{1-\frac{q}{r}} \left(\frac{q}{p+q}\right)^{\frac{q}{r}} \left(c_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}}$$
$$\times \left(\int_a^x [h(t)]^{\frac{r}{r-q}} dt\right)^{\frac{r-q}{r}} \left(\int_a^x \varphi(\tau) \left[\left|\bar{h}_1(\tau)\right|^r + \left|\bar{h}_2(\tau)\right|^r\right] d\tau\right)^{\frac{p+q}{r}}.$$

where h is defined by (2.2).

Proof. The proof is similar to the proof of Theorem 5.3. □

6. Discrete analogues to main results

This section deals with discrete analogues of main results in Section 2.

THEOREM 6.1. *For any $i = 1,2$ and $\alpha, \beta = 0, 1, \cdots, m-1$, let $k_{\alpha\beta} > 0$ and a^i_α, b^i_β be real numbers such that*

$$|a^i_\alpha| \leqslant \sum_{\beta=0}^{\alpha-1} k_{\alpha\beta} |b^i_\beta|.$$

Then for any constants $p \geqslant 0, q > 0, r > \max\{1,q\}$, $\phi_\alpha, \phi_\beta > 0$ and for any $\omega_\alpha \geqslant 0$, we have

$$\sum_{\alpha=0}^{m-1} \omega_\alpha \left(|a^1_\alpha|^p |b^2_\alpha|^q + |a^2_\alpha|^p |b^1_\alpha|^q\right) \leqslant 2^{1-\frac{q}{r}} (d_{\frac{p}{q}} + 1)^{\frac{q}{r}} C_m^{\frac{q}{r}} X_m^{\frac{p+q}{r}} \left(\sum_{\alpha=0}^{m-1} h_\alpha^{\frac{r}{r-q}}\right)^{\frac{r-q}{r}}, \tag{6.1}$$

where

$$C_1 = 0,$$
$$C_\alpha = \frac{q}{p+q} \left(\frac{1}{p+q}\right)^{\frac{p}{q}} (1 - C_{\alpha-1})^{-\frac{p}{q}}, \alpha = 2,3,4 \cdots, m,$$

$$X_m = \sum_{\beta=0}^{m-1} \phi_\beta |b^1_\beta|^r + \sum_{\beta=0}^{m-1} \phi_\beta |b^2_\beta|^r,$$
$$h_\alpha = \omega_\alpha (P_\alpha)^{\frac{p(r-1)}{r}} (\phi_\alpha)^{-\frac{q}{r}},$$
$$P_\alpha = \sum_{\beta=0}^{\alpha-1} k_{\alpha\beta}^{\frac{r}{r-1}} \phi_\beta^{-\frac{1}{r-1}}. \tag{6.2}$$

Proof. For $i = 1,2$, $\alpha = 0,1,\cdots,m-1$, $\phi_\beta > 0$, and using Hölder's inequality for $\{\frac{r}{r-1}, r\}$ and $i = 1$, we obtain

$$|a_\alpha^1| \leqslant \left(\sum_{\beta=0}^{\alpha-1} k_{\alpha\beta}^{\frac{r}{r-1}} \phi_\beta^{-\frac{1}{r-1}}\right)^{\frac{r-1}{r}} \left(\sum_{\beta=0}^{\alpha-1} \phi_\beta |b_\beta^1|^r\right)^{\frac{1}{r}} \leqslant [P_\alpha]^{\frac{r-1}{r}} [G_\alpha]^{\frac{1}{r}}. \tag{6.3}$$

Let

$$F_\alpha = \sum_{\beta=0}^{\alpha-1} \phi_\beta |b_\beta^2|^r, \tag{6.4}$$

then

$$\triangle F_\alpha = F_{\alpha+1} - F_\alpha = \phi_\alpha |b_\alpha^2|^r.$$

This implies

$$|b_\alpha^2|^q = \phi_\alpha^{-\frac{q}{r}} (\triangle F_\alpha)^{\frac{q}{r}}. \tag{6.5}$$

Now from inequality (6.3) for $p \geqslant 0$, and applying Hölder's inequality for $\{\frac{r}{r-q}, \frac{r}{q}\}$, we get

$$\sum_{\alpha=0}^{m-1} \omega_\alpha |a_\alpha^1|^p |b_\alpha^2|^q \leqslant \left(\sum_{\alpha=0}^{m-1} h_\alpha^{\frac{r}{r-q}}\right)^{\frac{r-q}{r}} \left(\sum_{\alpha=0}^{m-1} G_\alpha^{\frac{p}{q}} \triangle F_\alpha\right)^{\frac{q}{r}}. \tag{6.6}$$

Similarly, we can write

$$\sum_{\alpha=0}^{m-1} \omega_\alpha |a_\alpha^2|^p |b_\alpha^1|^q \leqslant \left(\sum_{\alpha=0}^{m-1} h_\alpha^{\frac{r}{r-q}}\right)^{\frac{r-q}{r}} \left(\sum_{\alpha=0}^{m-1} F_\alpha^{\frac{p}{q}} \triangle G_\alpha\right)^{\frac{q}{r}}. \tag{6.7}$$

Adding inequalities (6.6) and (6.7), and using inequalities given in (2.9) we get

$$\sum_{\alpha=0}^{m-1} \omega_\alpha \left[|a_\alpha^1|^p |b_\alpha^2|^q + |a_\alpha^2|^p |b_\alpha^1|^q\right] \leqslant 2^{1-\frac{q}{r}} \left(\sum_{\alpha=0}^{m-1} h_\alpha^{\frac{r}{r-q}}\right)^{\frac{r-q}{r}} \left[\sum_{\alpha=0}^{m-1} G_\alpha^{\frac{p}{q}} \triangle F_\alpha + \sum_{\alpha=0}^{m-1} F_\alpha^{\frac{p}{q}} \triangle G_\alpha\right]^{\frac{q}{r}}. \tag{6.8}$$

Now we take

$$\begin{aligned}
&\sum_{\alpha=0}^{m-1} G_\alpha^{\frac{p}{q}} \triangle F_\alpha + \sum_{\alpha=0}^{m-1} F_\alpha^{\frac{p}{q}} \triangle G_\alpha \\
&\leqslant d_{\frac{p}{q}} \sum_{\alpha=0}^{m-1} (G_\alpha + F_\alpha)^{\frac{p}{q}} (\triangle G_\alpha + \triangle F_\alpha) - \sum_{\alpha=0}^{m-1} (G_\alpha^{\frac{p}{q}} \triangle G_\alpha + F_\alpha^{\frac{p}{q}} \triangle F_\alpha) \\
&\leqslant C_m \left(d_{\frac{p}{q}} (G_m + F_m)^{\frac{p}{q}+1} + \left(G_m^{\frac{p}{q}+1} + F_m^{\frac{p}{q}+1}\right)\right) \text{ (By Lemma 2 in [4])} \\
&\leqslant C_m X_m^{\frac{p}{q}+1} \left(d_{\frac{p}{q}} + 1\right).
\end{aligned} \tag{6.9}$$

Now using inequality (6.9) in (6.8), we obtain (6.1). □

The upcoming theorem is the extreme case of Theorem 6.1.

THEOREM 6.2. *For any* $i = 1,2$ *and* $\alpha, \beta = 0,1,\cdots,m-1$, *let* $k_{\alpha\beta} > 0$ *and* a^i_α, b^i_β *and* $\tilde{a}^i_\alpha$ *be real numbers such that*

$$|a^i_\alpha| \leqslant \sum_{\beta=0}^{\alpha-1} k^i_{\alpha\beta} |b^1_\beta|, \text{ and } |\tilde{a}^i_\alpha| \leqslant \sum_{\beta=0}^{\alpha-1} k^i_{\alpha\beta} |b^2_\beta|.$$

Then for any constants $p, q_i \geqslant 0$, $\omega_\alpha \geqslant 0$, *we have the following inequality:*

$$\sum_{\alpha=0}^{m-1} \omega_\alpha \left[|a^1_\alpha|^{q_1} |\tilde{a^2_\alpha}|^{q_2} |b^1_\alpha|^p + |a^2_\alpha|^{q_2} |\tilde{a^1_\alpha}|^{q_1} |b^2_\alpha|^p \right] \leqslant \frac{1}{2} \|\omega\|_\infty \sum_{\alpha=0}^{m-1} \left(\sum_{\beta=0}^{\alpha-1} k^1_{\alpha\beta} \right)^{q_1} \times \left(\sum_{\beta=0}^{\alpha-1} k^2_{\alpha\beta} \right)^{q_2} \left[\|b^1\|_\infty^{2(q_1+p)} + \|b^2\|_\infty^{2q_2} + \|b^1\|_\infty^{2q_2} + \|b^2\|_\infty^{2(p+q_1)} \right]. \quad (6.10)$$

Proof. For any $i = 1,2$ and $\alpha, \beta = 0,1,\cdots,m-1$, we have

$$|a^i_\alpha|^{q_i} \leqslant \left(\sum_{\beta=0}^{\alpha-1} k^i_{\alpha\beta} \right)^{q_i} \|b^1\|_\infty^{q_i}.$$

By analogy we have

$$|\tilde{a}^i_\alpha|^{q_i} \leqslant \left(\sum_{\beta=0}^{\alpha-1} k^i_{\alpha\beta} \right)^{q_i} \|b^2\|_\infty^{q_i}.$$

Hence

$$|a^1_\alpha|^{q_1} \, |\tilde{a}^2_\alpha|^{q_2} \, |b^1_\alpha|^p \leqslant \left(\sum_{\beta=0}^{\alpha-1} k^1_{\alpha\beta} \right)^{q_1} \left(\sum_{\beta=0}^{\alpha-1} k^2_{\alpha\beta} \right)^{q_2} \|b^1\|_\infty^{q_1} \, \|b^2\|_\infty^{q_2} \, \|b^1\|_\infty^{p}. \quad (6.11)$$

Likewise

$$|a^2_\alpha|^{q_2} \, |\tilde{a}^1_\alpha|^{q_1} \, |b^2_\alpha|^p \leqslant \left(\sum_{\beta=0}^{\alpha-1} k^1_{\alpha\beta} \right)^{q_1} \left(\sum_{\beta=0}^{\alpha-1} k^2_{\alpha\beta} \right)^{q_2} \|b^1\|_\infty^{q_2} \, \|b^2\|_\infty^{q_1} \, \|b^2\|_\infty^{p}. \quad (6.12)$$

Adding inequalities given in (6.11) and (6.12), and after short calculations we can obtain (6.10). □

The upcoming theorem is the converse of the Theorem 6.1.

THEOREM 6.3. *For any* $i=1,2$ *and* $\alpha,\beta=0,1,\cdots,m-1$, *let* $k_{\alpha\beta}>0$ *and* a^i_α, b^i_β *be real numbers such that*

$$a^i_\alpha \leqslant \sum_{\beta=0}^{\alpha-1} k_{\alpha\beta}|b_\beta|.$$

Then for any constants $p,q>0, r<0$, *and for any* $\omega_\alpha \geqslant 0$ *we have*

$$\sum_{\alpha=0}^{m-1} \omega_\alpha \left(|a^1_\alpha|^p|b^2_\alpha|^q+|a^2_\alpha|^p|b^1_\alpha|^q\right) \geqslant 2^{1-\frac{q}{r}}(d_{\frac{p}{q}}+1)^{\frac{q}{r}} C_m^{\frac{q}{r}} X_m^{\frac{p+q}{q}} \left(\sum_{\alpha=0}^{m-1} h_\alpha^{\frac{r}{r-q}}\right)^{\frac{r-q}{r}}, \quad (6.13)$$

where, C_α, X_m, h_α *are given in* (6.2).

Proof. Since $\phi_\beta>0$, for $i=1,2$, $\alpha=0,1,\cdots,m-1$ and for fixed sign of b^1_β, $\phi(\beta)>0$ and using reverse Hölder's inequality for $\{\frac{r}{r-1},r\}$ and for $i=1$, we obtain

$$\begin{aligned} |a^{(1)}_\alpha| &\geqslant \left(\sum_{\beta=0}^{\alpha-1} k_{\alpha\beta}^{\frac{r}{r-1}} \phi_\beta^{-\frac{1}{r-1}}\right)^{\frac{r-1}{r}} \left(\sum_{\beta=0}^{\alpha-1} \phi_\beta |b^1_\beta|^r\right)^{\frac{1}{r}} \\ &\geqslant (P_\alpha)^{\frac{r-1}{r}}(G_\alpha)^{\frac{1}{r}}. \end{aligned} \quad (6.14)$$

Then we have

$$\sum_{\alpha=0}^{m-1} \omega_\alpha |a^1_\alpha|^p|b^2_\alpha|^q \geqslant \sum_{\alpha=0}^{m-1} h_\alpha G_\alpha^{\frac{p}{r}} \triangle F_\alpha^{\frac{q}{r}}. \quad (6.15)$$

Applying reverse Hölder's inequality for $\{\frac{r}{r-q},\frac{r}{q}\}$, we get

$$\sum_{\alpha=0}^{m-1} \omega_\alpha |a^1_\alpha|^p|b^2_\alpha|^q \geqslant \left(\sum_{\alpha=0}^{m-1} h_\alpha^{\frac{r}{r-q}}\right)^{\frac{r-q}{r}} \left(\sum_{\alpha=0}^{m-1} G_\alpha^{\frac{p}{q}} \triangle F_\alpha\right)^{\frac{q}{r}}. \quad (6.16)$$

Similarly, we can write

$$\sum_{\alpha=0}^{m-1} \omega_\alpha |a^2_\alpha|^p|b^1_\alpha|^q \geqslant \left(\sum_{\alpha=0}^{m-1} h_\alpha^{\frac{r}{r-q}}\right)^{\frac{r-q}{r}} \left(\sum_{\alpha=0}^{m-1} F_\alpha^{\frac{p}{q}} \triangle G_\alpha\right)^{\frac{q}{r}}. \quad (6.17)$$

Adding inequalities given in (6.16) and (6.17), and using the inequalities given in (2.9), we have

$$\begin{aligned} &\sum_{\alpha=0}^{m-1} \omega_\alpha \left[|a^1_\alpha|^p|b^2_\alpha|^q+|a^2_\alpha|^p|b^1_\alpha|^q\right] \\ &\qquad \geqslant 2^{1-\frac{q}{r}} \left(\sum_{\alpha=0}^{m-1} h_\alpha^{\frac{r}{r-q}}\right)^{\frac{r-q}{r}} \left[\sum_{\alpha=0}^{m-1} G_\alpha^{\frac{p}{q}} \triangle F_\alpha + \sum_{\alpha=0}^{m-1} F_\alpha^{\frac{p}{q}} \triangle G_\alpha\right]^{\frac{q}{r}}. \end{aligned} \quad (6.18)$$

From the inequality (6.9) and for $\frac{q}{r} < 0$, we have

$$\left(\sum_{\alpha=0}^{m-1} G_{\alpha}^{\frac{p}{q}} \triangle F_{\alpha} + \sum_{\alpha=0}^{m-1} F_{\alpha}^{\frac{p}{q}} \triangle G_{\alpha}\right)^{\frac{q}{r}} \geqslant C_m^{\frac{q}{r}} X_m^{\frac{p+q}{r}} \left(d_{\frac{p}{q}} + 1\right)^{\frac{q}{r}}. \tag{6.19}$$

Using inequality (6.19) in (6.18), we obtain (6.13). □

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