

SOME INEQUALITIES OF OPERATOR MONOTONE FUNCTIONS

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Abstract. Let $f(t)$ be any non-constant operator monotone function on $[0, \infty)$ and also let A and B be strictly positive operators:

(i) If $A > B$, then

$$f(A^\alpha) - f(B^\alpha) \geq f(\|A^\alpha\|) - f\left(\|A^\alpha\| - \frac{1}{\|(A^\alpha - B^\alpha)^{-1}\|}\right) > 0$$

for all $\alpha \in (0, 1]$.

(ii) If $\log A > \log B$, then there exists $\beta \in (0, 1]$ such that

$$f(A^\alpha) - f(B^\alpha) \geq f(\|A^\alpha\|) - f\left(\|A^\alpha\| - \frac{1}{\|(A^\alpha - B^\alpha)^{-1}\|}\right) > 0$$

for all $\alpha \in (0, \beta]$.

1. Introduction

A capital letter means a bounded linear operator on a complex Hilbert space H , and $B(H)$ denotes the algebra of all bounded linear operators on H equipped with operator norm $\|\cdot\|$.

An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. Chaotic order is defined by $\log A \geq \log B$ for strictly positive operators A and B .

A continuous real-valued function f defined on an interval J is called operator monotone if $A \geq B$ implies that $f(A) \geq f(B)$ for all self-adjoint operators A, B with spectra in J . The well known celebrated Löwner-Heinz inequality asserts that if $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$, which means that $t \mapsto t^\alpha$ is operator monotone. Another well known example of operator monotone function is $t \mapsto \log t$ on $(0, \infty)$.

Recently, Furuta [5] showed that if $A > B$, then $f(A^\alpha) > f(B^\alpha)$ for all $\alpha \in (0, 1]$ and if $\log A > \log B$, then there exists $\beta \in (0, 1]$ such that $f(A^\alpha) > f(B^\alpha)$ for all $\alpha \in (0, \beta]$, where $f(t)$ is a non-constant operator monotone function on $[0, \infty)$. Comprehensive survey on related order preserving operator inequalities were given in [2, 4]. In this paper, we will obtain an improvement which leads to the result of Moslehian and Najafi [6].

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2. Results

THEOREM A. [5, Theorem 2.1] *Let A and B be strictly positive operators on a Hilbert space H . If $A > B$, then the following inequality holds:*

$$f(A) > f(B) \quad (2.1)$$

for any non-constant operator monotone function f on $[0, \infty)$.

LEMMA B. [6, Lemma 2.1] *Let $A, B \in B(H)$ be invertible positive operators such that $A - B \geq m > 0$. Then*

$$B^{-1} - A^{-1} \geq \frac{m}{(\|A\| - m)\|A\|}. \quad (2.2)$$

THEOREM 1. *Let A and B be strictly positive operators on a Hilbert space H . If $A > B$, then the following inequality holds:*

$$f(A) - f(B) \geq f(\|A\|) - f\left(\|A\| - \frac{1}{\|(A-B)^{-1}\|}\right) > 0 \quad (2.3)$$

for any non-constant operator monotone function f on $[0, \infty)$.

Proof. At first we state the following (2.4) and (2.5) which act very important roles in the proof:

$$A - B \geq \frac{1}{\|(A-B)^{-1}\|} > 0 \quad (2.4)$$

$$(B+s)^{-1} - (A+s)^{-1} \geq \frac{\frac{1}{\|(A-B)^{-1}\|}}{\left(\|A+s\| - \frac{1}{\|(A-B)^{-1}\|}\right)\|A+s\|} \text{ for any } s \geq 0 \quad (2.5)$$

because (2.4) is obvious and (2.5) follows by (2.2) of Lemma B and (2.4).

It is well known (for examples, [1], [7]) that an operator monotone function f on $[0, \infty)$ has the representation

$$\begin{aligned} f(t) &= a + bt + \int_0^\infty \frac{ts}{t+s} dm(s) \\ &= a + bt + \int_0^\infty \left(s - \frac{s^2}{t+s}\right) dm(s) \end{aligned} \quad (2.6)$$

with $a \in \mathbb{R}$, $b \geq 0$ and a positive measure m on $[0, \infty)$.

Then we have

$$f(A) - f(B) = b(A - B) + \int_0^\infty \{(B+s)^{-1} - (A+s)^{-1}\} s^2 dm(s). \quad (2.7)$$

By replacing A by $\|A\|$ and B by $\|A\| - \frac{1}{\|(A-B)^{-1}\|}$ in (2.7), we have

$$\begin{aligned} & f(\|A\|) - f\left(\|A\| - \frac{1}{\|(A-B)^{-1}\|}\right) \\ &= b\left(\|A\| - \|A\| + \frac{1}{\|(A-B)^{-1}\|}\right) \\ & \quad + \int_0^\infty \left\{ \left(\|A\| - \frac{1}{\|(A-B)^{-1}\|} + s\right)^{-1} - (\|A\| + s)^{-1} \right\} s^2 dm(s). \end{aligned} \tag{2.8}$$

The equality (2.7) minus the equality (2.8), we have

$$\begin{aligned} & (f(A) - f(B)) - \left(f(\|A\|) - f\left(\|A\| - \frac{1}{\|(A-B)^{-1}\|}\right) \right) \\ &= b\left(A - B - \frac{1}{\|(A-B)^{-1}\|}\right) + \int_0^\infty \left\{ ((B+s)^{-1} - (A+s)^{-1}) \right. \\ & \quad \left. - \left(\left(\|A\| - \frac{1}{\|(A-B)^{-1}\|} + s\right)^{-1} - (\|A\| + s)^{-1} \right) \right\} s^2 dm(s) \\ &\geq b\left(A - B - \frac{1}{\|(A-B)^{-1}\|}\right) + \int_0^\infty \left\{ \left(\frac{\frac{1}{\|(A-B)^{-1}\|}}{\left(\|A\| + s\| - \frac{1}{\|(A-B)^{-1}\|}\right)\|A\| + s\|} \right) \right. \\ & \quad \left. - \left(\frac{1}{\|A\| + s - \frac{1}{\|(A-B)^{-1}\|}} - \frac{1}{\|A\| + s} \right) \right\} s^2 dm(s) \quad \text{by (2.5)} \\ &= b\left(A - B - \frac{1}{\|(A-B)^{-1}\|}\right) + \int_0^\infty \left\{ \left(\frac{\frac{1}{\|(A-B)^{-1}\|}}{\left(\|A\| + s\| - \frac{1}{\|(A-B)^{-1}\|}\right)\|A\| + s\|} \right) \right. \\ & \quad \left. - \left(\frac{\frac{1}{\|(A-B)^{-1}\|}}{\left(\|A\| + s - \frac{1}{\|(A-B)^{-1}\|}\right)(\|A\| + s)} \right) \right\} s^2 dm(s) \\ &= b\left[A - B - \frac{1}{\|(A-B)^{-1}\|}\right] \geq 0. \end{aligned} \tag{2.9}$$

since $\|A + s\| = \|A\| + s$ for any $s \geq 0$ and (2.4).

Therefore

$$f(A) - f(B) \geq f(\|A\|) - f\left(\|A\| - \frac{1}{\|(A-B)^{-1}\|}\right) > 0 \tag{2.3}$$

since the first inequality of (2.3) follows by (2.9) and the last one of (2.3) follows by (2.1) of Theorem A since $\|A\| > \|A\| - \frac{1}{\|(A-B)^{-1}\|} > 0$ holds. \square

COROLLARY 2. [6, Corollary 2.5] *Let $A, B \in B(H)$ such that $A > B > 0$, we have*

$$(i) \quad A^r - B^r \geq \|A\|^r - \left(\|A\| - \frac{1}{\|(A-B)^{-1}\|}\right)^r > 0 \quad \text{for all } 0 < r \leq 1.$$

$$(ii) \quad \log A - \log B \geq \log \|A\| - \log\left(\|A\| - \frac{1}{\|(A-B)^{-1}\|}\right) > 0.$$

Proof. It follows from Theorem 1 since t^r ($0 < r \leq 1$) and $\log t$ are both operator monotone on $(0, \infty)$. \square

COROLLARY 3. *Let A and B be positive operators on a Hilbert space H such that $A - B \geq m > 0$, then*

$$f(A) - f(B) \geq f(\|A\|) - f(\|A\| - m) > 0 \quad (2.10)$$

for any non-constant operator monotone function f on $[0, \infty)$.

Proof. Since $\frac{1}{\|(A-B)^{-1}\|} \geq m > 0$ holds, the monotonicity of $f(t)$ yields

$$f(\|A\|) - f\left(\|A\| - \frac{1}{\|(A-B)^{-1}\|}\right) \geq f(\|A\|) - f(\|A\| - m) \quad (2.11)$$

and by virtue of (2.3) of Theorem 1 and (2.11), we have the first inequality of (2.10) and the last one of (2.10) follows by (2.1) of Theorem A since $\|A\| > \|A\| - m$ holds. \square

COROLLARY 4. [6] *Let A and B be strictly positive operators on a Hilbert space H such that $A - B \geq m > 0$, then*

$$(i) \quad A^r - B^r \geq \|A\|^r - (\|A\| - m)^r > 0 \text{ for all } 0 < r \leq 1.$$

$$(ii) \quad \log A - \log B \geq \log \|A\| - \log(\|A\| - m) > 0.$$

Proof. It follows from Corollary 3, t^r ($0 < r \leq 1$) and $\log t$ are both operator monotone on $(0, \infty)$.

We remark that (i) of Corollary 4 is [6, Theorem 2.3] and (ii) of Corollary 4 is [6, Corollary 2.4]. \square

THEOREM 5. *Let $f(t)$ be any non-constant operator monotone function on $[0, \infty)$ and A and B be strictly positive operators. Then*

(i) *If $A > B$, then*

$$f(A^\alpha) - f(B^\alpha) \geq f(\|A^\alpha\|) - f\left(\|A^\alpha\| - \frac{1}{\|(A^\alpha - B^\alpha)^{-1}\|}\right) > 0 \quad (2.12)$$

for all $\alpha \in (0, 1]$.

(ii) *If $\log A > \log B$, then there exists $\beta \in (0, 1]$ such that*

$$f(A^\alpha) - f(B^\alpha) \geq f(\|A^\alpha\|) - f\left(\|A^\alpha\| - \frac{1}{\|(A^\alpha - B^\alpha)^{-1}\|}\right) > 0 \quad (2.13)$$

for all $\alpha \in (0, \beta]$.

Proof. We have only to trace the proof of (i) of Remark 3.1 in [5] for the proof of (i), also we have only to trace the proof of Theorem 2.2 in [5] for the proof of (ii) and we shall state both proofs in detail as follows.

Recall that the following obvious relation

$$X > Y > 0 \implies X^r > Y^r \quad \text{for any } r \in (0, 1]. \quad (2.14)$$

In fact $X > Y > 0$ ensures $X \geq Y + \xi I > Y > 0$ for some $\xi > 0$, then $X^r \geq (Y + \xi I)^r > Y^r$ for any $r \in (0, 1]$ by Löwner-Heinz inequality and we have (2.14).

For the proof of (i), $A > B$ implies $A^\alpha > B^\alpha$ for all $\alpha \in (0, 1]$ by (2.14) and we have (2.12) by (2.3) in Theorem 1.

Next we show the proof of (ii). By [3, Corollary 4] we have

$$\log A > \log B \implies \text{there exists } \beta \in (0, 1] \text{ such that } A^\beta > B^\beta. \quad (2.15)$$

Applying (2.14) for $r = \frac{\alpha}{\beta} \in (0, 1]$ to (2.15), then $A^\alpha > B^\alpha$ for any $\alpha \in (0, \beta]$. So we have (2.13) by (2.3) Theorem 1. \square

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