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SOME INEQUALITIES OF OPERATOR MONOTONE FUNCTIONS

HONGLIANG ZUO AND GUANGCAI DUAN

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Abstract. Let $f(t)$ be any non-constant operator monotone function on $[0,\infty)$ and also let A and B be strictly positive operators:
(i) If $A > B$, then

$$f(A^\alpha) - f(B^\alpha) \geqslant f(\| A^\alpha \|) - f\left(\| A^\alpha \| - \frac{1}{\| (A^\alpha - B^\alpha)^{-1} \|}\right) > 0$$

for all $\alpha \in (0,1]$.
(ii) If $\log A > \log B$, then there exists $\beta \in (0,1]$ such that

$$f(A^\alpha) - f(B^\alpha) \geqslant f(\| A^\alpha \|) - f\left(\| A^\alpha \| - \frac{1}{\| (A^\alpha - B^\alpha)^{-1} \|}\right) > 0$$

for all $\alpha \in (0,\beta]$.

1. Introduction

A capital letter means a bounded liner operator on a complex Hilbert space H, and $B(H)$ denotes the algebra of all bounded linear operators on H equipped with operator norm $\| \cdot \|$.

An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in H$ and an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. Chaotic order is defined by $\log A \geqslant \log B$ for strictly positive operators A and B.

A continuous real-valued function f defined on an interval J is called operator monotone if $A \geqslant B$ implies that $f(A) \geqslant f(B)$ for all self-adjoint operators A, B with spectra in J. The well known celebrated Löwer-Heinz inequality asserts that if $A \geqslant B \geqslant 0$, then $A^\alpha \geqslant B^\alpha$ for any $\alpha \in [0,1]$, which means that $t \mapsto t^\alpha$ is operator monotone. Another well known example of operator monotone function is $t \mapsto \log t$ on $(0,\infty)$.

Recently, Furuta [5] showed that if $A > B$, then $f(A^\alpha) > f(B^\alpha)$ for all $\alpha \in (0,1]$ and if $\log A > \log B$, then there exists $\beta \in (0,1]$ such that $f(A^\alpha) > f(B^\alpha)$ for all $\alpha \in (0,\beta]$, where $f(t)$ is a non-constant operator monotone function on $[0,\infty)$. Comprehensive survey on related order preserving operator inequalities were given in [2,4]. In this paper, we will obtain an improvement which leads to the result of Moslehian and Najafi [6].

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2. Results

THEOREM A. [5, Theorem 2.1] *Let A and B be strictly positive operators on a Hilbert space H. If $A > B$, then the following inequality holds:*

$$f(A) > f(B) \tag{2.1}$$

for any non-constant operator monotone function f on $[0,\infty)$.

LEMMA B. [6, Lemma 2.1] *Let $A,B \in B(H)$ be invertible positive operators such that $A - B \geqslant m > 0$. Then*

$$B^{-1} - A^{-1} \geqslant \frac{m}{(\|A\| - m)\|A\|}. \tag{2.2}$$

THEOREM 1. *Let A and B be strictly positive operators on a Hilbert space H. If $A > B$, then the following inequality holds:*

$$f(A) - f(B) \geqslant f(\|A\|) - f\left(\|A\| - \frac{1}{\|(A-B)^{-1}\|}\right) > 0 \tag{2.3}$$

for any non-constant operator monotone function f on $[0,\infty)$.

Proof. At first we state the following (2.4) and (2.5) which act very important roles in the proof:

$$A - B \geqslant \frac{1}{\|(A-B)^{-1}\|} > 0 \tag{2.4}$$

$$(B+s)^{-1} - (A+s)^{-1} \geqslant \frac{\frac{1}{\|(A-B)^{-1}\|}}{\left(\|A+s\| - \frac{1}{\|(A-B)^{-1}\|}\right)\|A+s\|} \quad \text{for any } s \geqslant 0 \tag{2.5}$$

because (2.4) is obvious and (2.5) follows by (2.2) of Lemma B and (2.4).

It is well known (for examples, [1], [7]) that an operator monotone function f on $[0,\infty)$ has the representation

$$\begin{aligned} f(t) &= a + bt + \int_0^\infty \frac{ts}{t+s} dm(s) \\ &= a + bt + \int_0^\infty \left(s - \frac{s^2}{t+s}\right) dm(s) \end{aligned} \tag{2.6}$$

with $a \in \mathbb{R}$, $b \geqslant 0$ and a positive measure m on $[0,\infty)$.

Then we have

$$f(A) - f(B) = b(A-B) + \int_0^\infty \{(B+s)^{-1} - (A+s)^{-1}\} s^2 dm(s). \tag{2.7}$$

By replacing A by $\|A\|$ and B by $\|A\|-\frac{1}{\|(A-B)^{-1}\|}$ in (2.7), we have

$$\begin{aligned}
&f(\|A\|)-f\left(\|A\|-\frac{1}{\|(A-B)^{-1}\|}\right)\\
&=b\left(\|A\|-\|A\|+\frac{1}{\|(A-B)^{-1}\|}\right)\\
&\quad+\int_0^\infty\left\{\left(\|A\|-\frac{1}{\|(A-B)^{-1}\|}+s\right)^{-1}-(\|A\|+s)^{-1}\right\}s^2dm(s). \qquad (2.8)
\end{aligned}$$

The equality (2.7) minus the equality (2.8), we have

$$\begin{aligned}
&(f(A)-f(B))-\Big(f(\|A\|)-f\Big(\|A\|-\frac{1}{\|(A-B)^{-1}\|}\Big)\Big)\\
&=b\Big(A-B-\frac{1}{\|(A-B)^{-1}\|}\Big)+\int_0^\infty\Big\{((B+s)^{-1}-(A+s)^{-1})\\
&\quad-\Big(\Big(\|A\|-\frac{1}{\|(A-B)^{-1}\|}+s\Big)^{-1}-(\|A\|+s)^{-1}\Big)\Big\}s^2dm(s)\\
&\geqslant b\Big(A-B-\frac{1}{\|(A-B)^{-1}\|}\Big)+\int_0^\infty\Bigg\{\Bigg(\frac{\frac{1}{\|(A-B)^{-1}\|}}{\Big(\|A+s\|-\frac{1}{\|(A-B)^{-1}\|}\Big)\|A+s\|}\Bigg)\\
&\quad-\Bigg(\frac{1}{\|A\|+s-\frac{1}{\|(A-B)^{-1}\|}}-\frac{1}{\|A\|+s}\Bigg)\Bigg\}s^2dm(s) \quad \text{by (2.5)}\\
&=b\Big(A-B-\frac{1}{\|(A-B)^{-1}\|}\Big)+\int_0^\infty\Bigg\{\Bigg(\frac{\frac{1}{\|(A-B)^{-1}\|}}{(\|A+s\|-\frac{1}{\|(A-B)^{-1}\|})\|A+s\|}\Bigg)\\
&\quad-\Bigg(\frac{\frac{1}{\|(A-B)^{-1}\|}}{\Big(\|A\|+s-\frac{1}{\|(A-B)^{-1}\|}\Big)(\|A\|+s)}\Bigg)\Bigg\}s^2dm(s)\\
&=b\Big[(A-B)-\frac{1}{\|(A-B)^{-1}\|}\Big]\geqslant 0. \qquad (2.9)
\end{aligned}$$

since $\|A+s\|=\|A\|+s$ for any $s\geqslant 0$ and (2.4).

Therefore

$$f(A)-f(B)\geqslant f(\|A\|)-f\Big(\|A\|-\frac{1}{\|(A-B)^{-1}\|}\Big)>0 \qquad (2.3)$$

since the first inequality of (2.3) follows by (2.9) and the last one of (2.3) follows by (2.1) of Theorem A since $\|A\|>\|A\|-\frac{1}{\|(A-B)^{-1}\|}>0$ holds. □

COROLLARY 2. [6, Corollary 2.5] *Let $A,B\in B(H)$ such that $A>B>0$, we have*

(i) $A^r-B^r\geqslant\|A\|^r-\Big(\|A\|-\frac{1}{\|(A-B)^{-1}\|}\Big)^r>0$ *for all* $0<r\leqslant 1$.

(ii) $\log A - \log B \geqslant \log \|A\| - \log(\|A\| - \frac{1}{\|(A-B)^{-1}\|}) > 0$.

Proof. It follows from Theorem 1 since t^r $(0 < r \leqslant 1)$ and $\log t$ are both operator monotone on $(0, \infty)$. □

COROLLARY 3. *Let A and B be positive operators on a Hilbert space H such that $A - B \geqslant m > 0$, then*

$$f(A) - f(B) \geqslant f(\|A\|) - f(\|A\| - m) > 0 \tag{2.10}$$

for any non-constant operator monotone function f on $[0, \infty)$.

Proof. Since $\frac{1}{\|(A-B)^{-1}\|} \geqslant m > 0$ holds, the monotonicity of $f(t)$ yields

$$f(\|A\|) - f\Big(\|A\| - \frac{1}{\|(A-B)^{-1}\|}\Big) \geqslant f(\|A\|) - f(\|A\| - m) \tag{2.11}$$

and by virtue of (2.3) of Theorem 1 and (2.11), we have the frist inequality of (2.10) and the last one of (2.10) follows by (2.1) of Theorem A since $\|A\| > \|A\| - m$ holds. □

COROLLARY 4. [6] *Let A and B be strictly positive operators on a Hilbert space H such that $A - B \geqslant m > 0$, then*

(i) $A^r - B^r \geqslant \|A\|^r - (\|A\| - m)^r > 0$ *for all* $0 < r \leqslant 1$.

(ii) $\log A - \log B \geqslant \log \|A\| - \log(\|A\| - m) > 0$.

Proof. It follows from Corollary 3, t^r $(0 < r \leqslant 1)$ and $\log t$ are both operator monotone on $(0, \infty)$.

We remark that (i) of Corollary 4 is [6, Theorem 2.3] and (ii) of Corollary 4 is [6, Corollary 2.4]. □

THEOREM 5. *Let $f(t)$ be any non-constant operator monotone function on $[0, \infty)$ and A and B be strictly positive operators. Then*

(i) If $A > B$, then

$$f(A^\alpha) - f(B^\alpha) \geqslant f(\|A^\alpha\|) - f\Big(\|A^\alpha\| - \frac{1}{\|(A^\alpha - B^\alpha)^{-1}\|}\Big) > 0 \tag{2.12}$$

for all $\alpha \in (0, 1]$.

(ii) If $\log A > \log B$, then there exists $\beta \in (0, 1]$ such that

$$f(A^\alpha) - f(B^\alpha) \geqslant f(\|A^\alpha\|) - f\Big(\|A^\alpha\| - \frac{1}{\|(A^\alpha - B^\alpha)^{-1}\|}\Big) > 0 \tag{2.13}$$

for all $\alpha \in (0, \beta]$.

Proof. We have only to trace the proof of (i) of Remark 3.1 in [5] for the proof of (i), also we have only to trace the proof of Theorem 2.2 in [5] for the proof of (ii) and we shall state both proofs in detail as follows.

Recall that the following obvious relation

$$X > Y > 0 \Longrightarrow X^r > Y^r \quad \text{for any} \quad r \in (0,1]. \tag{2.14}$$

In fact $X > Y > 0$ ensures $X \geqslant Y + \xi I > Y > 0$ for some $\xi > 0$, then $X^r \geqslant (Y+\xi I)^r > Y^r$ for any $r \in (0,1]$ by Löwer-Heinz inequality and we have (2.14).

For the proof of (i), $A > B$ implies $A^\alpha > B^\alpha$ for all $\alpha \in (0,1]$ by (2.14) and we have (2.12) by (2.3) in Theorem 1.

Next we show the proof of (ii). By [3, Corollary 4] we have

$$\log A > \log B \Rightarrow \text{there exists} \quad \beta \in (0,1] \quad \text{such that} \quad A^\beta > B^\beta. \tag{2.15}$$

Applying (2.14) for $r = \frac{\alpha}{\beta} \in (0,1]$ to (2.15), then $A^\alpha > B^\alpha$ for any $\alpha \in (0,\beta]$. So we have (2.13) by (2.3) Theorem 1. □

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REFERENCES

[1] R. BHATIA, *Matrix Analysis*, New York: Springer Verlag, 1997.

[2] M. FUJII, *Furuta inequality and its related topics*, Ann. Funct. Anal. **1**, 1 (2010), 28–45.

[3] M. FUJII, J. F. JIANG AND E. KAMEI, *Characterization of chaotic order and its application to Furuta inequality*, Proc. Amer. Math. Soc. **125** (1997), 3655–3658.

[4] T. FURUTA, *Comprehensive survey on an order preserving operator inequality*, Banach J. Math. Anal. **7**, 1 (2013), 14–40.

[5] T. FURUTA, *Operator monotone functions, $A > B > 0$ and $\log A > \log B$*, J. Math. Inequal. **7**, 1 (2013), 93–96.

[6] M. MOSLEHIAN AND H. NAJAFI, *An extension of the Löwner-Heinz inequality*, Linear Algebra Appl. **437** (2012), 2359–2365.

[7] X. ZHAN, *Matrix Inequalities*, Berlin: Springer Verlag, 2002.

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Hongliang Zuo
College of Mathematics and Information Science
Henan Normal University
Xinxiang, Henan, 453007
China
e-mail: zuodke@yahoo.com

Guangcai Duan
College of Mathematics and Information Science
Henan Normal University
Xinxiang, Henan, 453007
China
e-mail: duanguangcai159@sina.com