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DIFFERENTIAL HARNACK INEQUALITY FOR NONLINEAR HEAT EQUATIONS

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Abstract. In this paper, we establish some differential Harnack inequalities for positive solutions to nonlinear heat equations coupled with the Bernhard List's flow. We also give some applications of these inequalities.

1. Introduction

The study of differential Harnack estimates for parabolic equations originated in Li and Yau's work [6], in which they proved a Harnack inequality for positive solutions to the heat equation. Later, Yau generalized this result to the Harnack inequalities for some nonlinear heat-type equation [7]. Since then, the Harnack estimates for positive solutions to the heat equation coupled with various geometric flows have been widely studied. Recently, Cao and Zhang [2] proved an interesting differential Harnack inequality for a positive solution to the forward nonlinear heat equation

$$\frac{\partial f}{\partial t} = \triangle f - f\ln f + Rf \tag{1.1}$$

coupled with the Ricci flow. They got the following result.

THEOREM 1.1. ([2]) *Let $(M, g(t))$, $t \in [0,T)$, be a solution to the Ricci flow on a closed manifold, and suppose that $g(0)$ has weakly positive curvature operator. Let f be a positive solution to the heat equation (1.1), $u = -\ln f$ and*

$$H = 2\triangle u - |\nabla u|^2 - 3R - \frac{2n}{t}.$$

Then in some time interval $(0,T)$,

$$H \leqslant \frac{n}{4}.$$

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If the equation (1.1) is changed into

$$\frac{\partial f}{\partial t} = \triangle f - \frac{f\ln f}{1+\frac{t}{2}} + Rf, \tag{1.2}$$

under the same assumption as Theorem 1.1, they got

$$H \leqslant 0.$$

Very recently, Fang [3] studied the linear heat equations with potentials on closed Riemannian manifolds evolving by the Bernhard List's flow

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2R_{ij} + 2\alpha_n d\psi \otimes d\psi, \\ \dfrac{\partial \psi}{\partial t} = \triangle_{g(t)}\psi, \end{cases}$$

where $\psi : M \to R$ is a smooth function and $\alpha_n = \frac{n-1}{n-2}$. Let f be a positive solution of the time-dependent nonlinear heat equation with potential, i.e.,

$$\frac{\partial f}{\partial t} = \triangle f + c(R - \alpha_n|\nabla\psi|^2)f \tag{1.3}$$

where c is any constant. Under the Bernhard List's flow, we have $\frac{d}{dt}\int_M f dv = 0$. The flow was first introduced by Bernhard List [1] and Fang [3] proved some differential Harnack estimate for positive solutions to the linear heat equations with potentials under this flow.

THEOREM 1.1. (Fang [3]) *Let $(g(t),\psi(t))$, $t \in [0,T)$, be a solution to the Bernhard List's flow on a closed manifold M, and suppose that $\mathscr{H}(\mathscr{S}, X)$ is nonnegative for $\forall X \in \Gamma(TM)$ and all times $t \in [0,T)$. Let f be a positive solution to the heat equation (1.3), $u = -\ln f$, and*

$$P = 2\triangle u - |\nabla u|^2 - 3S - \frac{2n}{t}.$$

Then for $\forall t \in (0,T)$

$$P \leqslant 0.$$

REMARK 1.1. In this paper, we will use the same symbol as in [3]. $\mathscr{S}$ represents a two-tensor on $(M,g(t))$ with components $S_{ij} = R_{ij} - \alpha_n\nabla_i\psi\nabla_j\psi$ and its trace $S = g^{ij}S_{ij} = R - \alpha_n|\nabla\psi|^2$. Moreover,

$$\mathscr{H}(\mathscr{S},X) = \frac{\partial S}{\partial t} + 2\langle\nabla S, X\rangle + 2\mathscr{S}(X,X) + \frac{S}{t}.$$

In this paper, we will extend this result to the nonlinear heat equation

$$\frac{\partial f}{\partial t} = \triangle f - f\ln f + c(R - \alpha_n|\nabla\psi|^2)f, \tag{1.4}$$

coupled with Bernhard List's flow. Now, we give our first main result.

THEOREM 1.2. *Let $(g(t),\psi(t))$, $t\in[0,T)$, be a solution to the Bernhard List's flow on a closed manifold M. Suppose $R\geqslant\alpha_n|\nabla\psi|^2$ and $\mathscr{H}(\mathscr{S},X)$ is nonnegative for $\forall X\in\Gamma(TM)$ and all times $t\in[0,T)$. Let f be a positive solution to the heat equation (1.4), $u=-\ln f$, and*

$$P=2\triangle u-|\nabla u|^2-3S-\frac{2n}{t}.$$

Then for $\forall t\in(0,T)$

$$P\leqslant 0.$$

REMARK 1.2. We can see that the condition on the scalar curvature in Theorem 1.2 means that $S\geqslant 0$. From the evolution equation of S in Lemma 2.1 under Bernhard List's flow, we know the condition is preserved under this flow, and in fact we only need the condition is valid at the initial time $t=0$.

REMARK 1.3. The author [9] got $P\leqslant\frac{n}{4}$ under the condition $\mathscr{H}(\mathscr{S},X)$ is nonnegative. If we strengthen the assumption, we get the prior bound $P\leqslant 0$ in the Theorem 1.2 using a different computation method with [9].

Secondly, we will consider the positive solution $f(x,t)<1$ to the nonlinear heat equation without any potential, i.e.,

$$\frac{\partial f}{\partial t}=\triangle f-f\ln f \qquad (1.5).$$

This equation has ever been considered by Hsu [4] and Ma [5]. Note that $0<f<1$ is preserved under equation (1.5) as time t evolves (see Section 4 in Wu [8]). Let us consider $u=-\ln f$ in equation (1.5), then

$$\frac{\partial u}{\partial t}=\triangle u-|\nabla u|^2-u. \qquad (1.6).$$

We get our second main result.

THEOREM 1.3. *Let $(g(t),\psi(t))$, $t\in[0,T)$, be a solution to the Bernhard List's flow on a closed manifold M, and suppose that $0<f<1$ be a positive solution to the nonlinear equation (1.5). Let $u=-\ln f$ and $P'=|\nabla u|^2-\frac{u}{t}$. Then for $\forall t\in(0,T)$,*

$$P'\leqslant 0.$$

Hence for $\forall t\in(0,T)$,

$$|\nabla u|^2\leqslant\frac{-f^2\ln f}{t}.$$

The rest of the paper is organized as follows. In Section 2, we will first calculate a general evolution equation for the function H which is associated to the Harnack quantity P, then we will prove Theorem 1.2. As an application of Theorem 1.1, we will get an integral Harnack inequality. In Section 3, we will define one functional which is associated with the Harnack quantity, and show it is monotone under Bernhard List's flow. We will consider a special nonlinear equation and prove Theorem 1.3 in the last section.

2. Harnack estimate

In this section, we will derive a general evolution formula for the Harnack quantity H and then prove Theorem 1.2. First we recall some evolution equations of geometric quantities under Bernhard List's flow which can be found in [3].

LEMMA 2.1. *Let $(g(t),\psi(t))$, $t\in[0,T)$, be a solution to the Bernhard List's flow, then the following evolution equation hold*

(1) $$\frac{\partial}{\partial t}\Gamma_{ij}^{k}=g^{kl}(-\nabla_i R_{jl}-\nabla_j R_{il}+\nabla_l R_{ij}+2\alpha_n\nabla_i\nabla_j\psi\nabla_l\psi),$$

(2) $$\frac{\partial}{\partial t}S_{ij}=\triangle S_{ij}-R_{ip}S_{pj}-R_{jp}S_{pi}-2R_{pijq}S_{pq}+2\alpha_n\nabla_i\nabla_j\psi\triangle\psi,$$

(3) $$\frac{\partial}{\partial t}S=\triangle S+2|S_{ij}|^2+2\alpha_n(\triangle\psi)^2,$$

(4) $$\frac{\partial}{\partial t}dv=-Sdv.$$

Let us consider positive solutions to equation (1.3), assume $f=e^{-u}$, then we have

$$\frac{\partial u}{\partial t}=\triangle u-|\nabla u|^2-u-cS. \qquad (2.1).$$

LEMMA 2.2. *Let $(g(t),\psi(t))$, $t\in[0,T)$, be a solution to the Bernhard List's flow, and let u satisfy (2.1). Set*

$$H=\alpha\triangle u-\beta|\nabla u|^2+aS-b\frac{u}{t}-d\frac{n}{t},$$

where α,β,a,b,d are constants. Then H satisfies the following evolution equation

$$\begin{aligned}\frac{\partial}{\partial t}H =& \triangle H-2\nabla H\cdot\nabla u-2(\beta-\alpha)\Big|u_{ij}+\frac{\alpha}{2(\beta-\alpha)}S_{ij}-\frac{\lambda}{2t}g_{ij}\Big|^2+AH\\ &+\Big[A\beta-2\beta-\frac{b}{t}\Big]|\nabla u|^2+\Big(\frac{\lambda\alpha}{t}+\frac{bc}{t}-Aa\Big)S+(Ab+b)\frac{u}{t}+Ad\frac{n}{t}\\ &+\frac{bn+dn}{t^2}-c\alpha\triangle S-2\alpha R_{ij}u_iu_j+2\beta\alpha_n(\nabla\psi\nabla u)^2+(2a-2\beta)\nabla u\cdot\nabla S\\ &+\Big(2\beta-\frac{b}{t}\Big)|\nabla u|^2+2a\alpha_n(\triangle\psi)^2-2\alpha\alpha_n\triangle\psi\nabla\psi\cdot\nabla u,\end{aligned}$$

where $A=\frac{2\lambda(\beta-\alpha)}{\alpha t}-1$.

Proof. We first calculate the first two terms in H,

$$\frac{\partial}{\partial t}(\triangle u)=\triangle(\triangle u)-\triangle(|\nabla u|^2)-\triangle S+2S_{ij}u_{ij}-2\alpha_n\triangle\psi\nabla\psi\cdot\nabla u-\triangle u,$$

and

$$\begin{aligned}\frac{\partial}{\partial t}(|\nabla u|^2) &= 2\nabla(\triangle u)\cdot\nabla u - 2\nabla(|\nabla u|^2)\cdot\nabla u - 2|\nabla u|^2 - 2\nabla u\cdot\nabla S + 2S_{ij}u_iu_j \\ &= \triangle(|\nabla u|^2) - 2|\nabla\nabla u|^2 - 2\nabla(|\nabla u|^2)\cdot\nabla u - 2|\nabla u|^2 - 2\nabla u\cdot\nabla S \\ &\quad -2\alpha_n(\nabla\psi\cdot\nabla u)^2,\end{aligned}$$

here we used

$$\triangle(|\nabla u|^2) = 2\nabla u\cdot\triangle\nabla u + 2|\nabla\nabla u|^2,$$

and

$$\triangle\nabla u = \nabla u\cdot\triangle\nabla u + Ric(\nabla u,\cdot).$$

We still need another equality

$$\begin{aligned}-2\nabla H\cdot\nabla u &= -\alpha\triangle(|\nabla u|^2) + 2\alpha R_{ij}u_iu_j + 2\alpha|\nabla\nabla u|^2 \\ &\quad +2\beta\nabla(|\nabla u|^2)\cdot\nabla u - 2a\nabla u\cdot\nabla S + \frac{2b}{t}|\nabla u|^2.\end{aligned}$$

Using the evolution of S in Lemma 2.1 and (2.1), we have

$$\begin{aligned}\frac{\partial H}{\partial t} &= \triangle H - 2\nabla H\cdot\nabla u + 2(\beta-\alpha)|\nabla\nabla u|^2 - 2\alpha R_{ij}u_iu_j + 2(a+c\beta)\nabla u\cdot\nabla S \\ &\quad +\left(2\beta-\frac{b}{t}\right)|\nabla u|^2 + 2\alpha S_{ij}u_{ij} - c\alpha S - \alpha\triangle u + 2a|S_{ij}|^2 + 2a\alpha_n(\triangle\psi)^2 \\ &\quad +\frac{bu}{t} + \frac{bcS}{t} + \frac{bu}{t^2} + \frac{dn}{t^2} + 2\beta\alpha_n(\nabla\psi\cdot\nabla u)^2 - 2\alpha\alpha_n\triangle\psi\nabla\psi\cdot\nabla u - \triangle u \\ &= \triangle H - 2\nabla H\cdot\nabla u + 2(\beta-\alpha)|u_{ij} + \frac{\alpha}{2(\beta-\alpha)}S_{ij} - \frac{\lambda}{2t}g_{ij}|^2 \\ &\quad +\left[\frac{2(\beta-\alpha)\lambda}{t} - \alpha\right]\triangle u - c\alpha\triangle S - 2\alpha R_{ij}u_iu_j + 2(a+c\beta)\nabla u\cdot\nabla S \\ &\quad +\left(2\beta-\frac{b}{t}\right)|\nabla u|^2 + 2a\alpha_n(\triangle\psi)^2 + \left(\frac{bc}{t} + \frac{\alpha\lambda}{t}\right)S + \frac{bcS}{t} + \frac{bu}{t^2} + \frac{dn}{t^2} \\ &\quad -2\alpha_n\triangle\psi\nabla\psi\cdot\nabla u + 2\beta\alpha_n(\nabla\psi\cdot\nabla u)^2 + \left[2a - \frac{\alpha^2}{2(\beta-\alpha)}\right]|S_{ij}|^2 \\ &= \triangle H - 2\nabla H\cdot\nabla u - 2(\beta-\alpha)|u_{ij} + \frac{\alpha}{2(\beta-\alpha)}S_{ij} - \frac{\lambda}{2t}g_{ij}|^2 + AH \\ &\quad +\left[A\beta - 2\beta - \frac{b}{t}\right]|\nabla u|^2 + \left(\frac{\lambda\alpha}{t} + \frac{bc}{t} - Aa\right)S + (Ab+b)\frac{u}{t} + Ad\frac{n}{t} \\ &\quad +\frac{bn+dn}{t^2} - c\alpha\triangle S - 2\alpha R_{ij}u_iu_j + 2\beta\alpha_n(\nabla\psi\nabla u)^2 + (2a-2\beta)\nabla u\cdot\nabla S \\ &\quad +\left(2\beta-\frac{b}{t}\right)|\nabla u|^2 + 2a\alpha_n(\triangle\psi)^2 - 2\alpha\alpha_n\triangle\psi\nabla\psi\cdot\nabla u,\end{aligned}$$

We complete the proof. □

In the above Lemma, we choose $\alpha = 2$, $\beta = 1$, $a = -3$, $b = 0$, $c = 1$, $d = \lambda = 2$, then we get the following result

COROLLARY 2.3. *Let $(g(t),\psi(t))$, $t\in[0,T)$, be a solution to the Bernhard List's flow on a closed manifold M and f be positive solutions to nonlinear equation (1.4). Assume $u=-\ln f$ and*

$$P=2\triangle u-|\nabla u|^2-3S-\frac{2n}{t}.$$

Then

$$\begin{aligned}\frac{\partial P}{\partial t}&=\triangle P-2\nabla P\cdot\nabla u-2|u_{ij}-S_{ij}-\frac{1}{t}g_{ij}|^2-\Big(\frac{2}{t}+1\Big)P-2\alpha_n(\triangle\psi+\nabla\psi\cdot\nabla u)^2\\&\quad-\Big(\frac{2}{t}+1\Big)|\nabla u|^2-\frac{n}{t}-3S-2[\triangle S+2|S_{ij}|^2+2\alpha_n(\triangle\psi)^2+\frac{S}{t}+2\mathscr{S}(\nabla u\nabla u)]\\&=\triangle P-2\nabla P\cdot\nabla u-2|u_{ij}-S_{ij}-\frac{1}{t}g_{ij}|^2-\Big(\frac{2}{t}+1\Big)P-2\alpha_n(\triangle\psi+\nabla\psi\cdot\nabla u)^2\\&\quad-\Big(\frac{2}{t}+1\Big)|\nabla u|^2-\frac{n}{t}-3S-2\mathscr{H}(\mathscr{S},\nabla u).\end{aligned}$$

Now we are at the position to prove Theorem 1.2.

Proof. For t is small enough we can easily see that $P<0$. Since $S\geqslant 0$ and $\mathscr{H}(\mathscr{S}$, X) is nonnegative for $\forall X\in\Gamma(TM)$ and all times $t\in[0,T)$, By Corollary 2.3 and the maximum principle we have

$$P\leqslant 0, \tag{2.2}$$

for all time $t\in(0,T)$. □

Integrating (2.2) along a space-time path, and we have the following results.

THEOREM 2.4. *Let $(g(t),\psi(t))$, $t\in[0,T)$, be a solution to the Bernhard List's flow on a closed manifold M and f be positive solutions to nonlinear equation (1.4). Suppose that $R\geqslant\alpha_n|\nabla\psi|^2$ and $\mathscr{H}$ $(\mathscr{S},X)$ is nonnegative for $\forall X\in\Gamma(TM)$ and all times $t\in[0,T)$. Let (x_1,t_2) and (x_2,t_2), $0<t_1<t_2$, be two points in $M\times(0,T)$, and $\Gamma=inf_\gamma\int_{t_1}^{t_2}e^t(|\dot\gamma|^2+S+\frac{2n}{t})dt$, where γ is any space time path joining (x_1,t_2) and (x_2,t_2). Then we have*

$$e^{t_2}\ln f(x_2,t_2)-e^{t_1}\ln f(x_1,t_1)\leqslant\frac{1}{2}\Gamma.$$

Proof. Let us consider the solutions to

$$\frac{\partial u}{\partial t}=\triangle u-|\nabla u|^2-u-S,$$

combing with

$$P=2\triangle u-|\nabla u|^2-3S-\frac{2n}{t}.$$

We have

$$2\frac{\partial u}{\partial t}+|\nabla u|^2+2u-S-\frac{2n}{t}\leqslant 0.$$

Pick a space-time curve connecting (x_1,t_2) and (x_2,t_2) with $0<t_1<t_2$. Along the space-time path γ, we have

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial t}+\nabla u\cdot\dot\gamma\\ &\leqslant -\frac{1}{2}|\nabla u|^2-u+\frac{S}{2}+\frac{n}{t}+\nabla u\cdot\dot\gamma\\ &\leqslant \frac{1}{2}\Big(|\dot\gamma|^2+S+\frac{2n}{t}\Big)-u.\end{aligned}$$

For any space-time curve, we arrives at

$$\frac{d(e^t\cdot u)}{dt}\leqslant\frac{e^t}{2}\Big(|\dot\gamma|^2+S+\frac{2n}{t}\Big).$$

Therefore, the desired Harnack inequality is proved by the integrating from t_1 and t_2. □

3. Entropy formula and monotonicity

In this section, we will define one functional which is associated with the Harnack quantity and show that it is monotone under Bernhard List's flow.

THEOREM 3.1. *Let $(g(t),\psi(t))$, $t\in[0,T)$, be a solution to the Bernhard List's flow on a closed manifold M and f be positive solutions to nonlinear equation (1.4). Suppose that $R\geqslant\alpha_n|\nabla\psi|^2$ and $\mathscr{H}(\mathscr{S},X)$ is nonnegative for $\forall X\in\Gamma(TM)$ and all times $t\in[0,T)$. Moreover, assume that $u=-\ln f$ is a positive solution to (2.1). Set*

$$F:=\int_M t^2Pe^{-u}dv,$$

then for $\forall t\in(0,T)$, we have $F\leqslant 0$ and $\frac{dF}{dt}\leqslant 0$.

Proof.

$$\begin{aligned}\frac{dF}{dt} &= \int_M\Big(2tPe^{-u}+t^2e^{-u}\frac{\partial P}{\partial t}-t^2Pe^{-u}\frac{\partial u}{\partial t}-St^2Pe^{-u}\Big)dv\\ &= \int_M\Big[2tPe^{-u}+t^2e^{-u}\Big(\triangle P-2\nabla P\cdot\nabla u-2\Big|u_{ij}-S_{ij}-\frac{1}{t}g_{ij}\Big|^2-\Big(\frac{2}{t}+1\Big)P\\ &\quad -2\alpha_n(\triangle\psi+\nabla\psi\cdot\nabla u)^2-\Big(\frac{2}{t}+1\Big)|\nabla u|^2-\frac{n}{t}-3S-2\mathscr{H}(\mathscr{S},\nabla u)\Big)\Big]dv\\ &\quad -\int_M t^2Pe^{-u}(\triangle u-|\nabla u|^2-u-S)dv-\int_M St^2Pe^{-u}dv\\ &= \int_M t^2e^{-u}Pudv+\int_M\Big[\triangle(t^2Pe^{-u})-2t^2e^{-u}\Big|u_{ij}-S_{ij}-\frac{1}{t}g_{ij}\Big|^2\\ &\quad -(2t+t^2)e^{-u}|\nabla u|^2-t^2e^{-u}P-nte^{-u}-3t^2e^{-u}S-2t^2e^{-u}\mathscr{H}(\mathscr{S},\nabla u))\\ &\quad -2t^2e^{-u}\alpha_n(\triangle\psi+\nabla\psi\cdot\nabla u)^2\Big]dv\\ &\leqslant 0. \quad \square\end{aligned}$$

4. Gradient estimate for the nonlinear heat equations

In this section, we will consider the nonlinear heat equation (1.5) and applying the maximum principle to prove Theorem 1.3. Our method is similar to J.-Y. Wu's [8].

Proof of Theorem 1.3. Set $H = |\nabla u|^2 - \frac{u}{t}$. In Lemma 2.2, we choose $\alpha = 0$, $\beta = -1$, $a = 0$, $b = 1$, $d = 0$, $\lambda = 0$, $c = 0$. We have

$$\begin{aligned}\frac{\partial H}{\partial t} &= \triangle H - 2\nabla H \cdot \nabla u - 2|\nabla\nabla u|^2 - \frac{1}{t}H - 2|\nabla u|^2 + \frac{u}{t}\\ &= \triangle H - 2\nabla H \cdot \nabla u - 2|\nabla\nabla u|^2 - \left(\frac{1}{t}+1\right)H - |\nabla u|^2.\end{aligned}$$

Notice that if t is small enough, then $H < 0$.

Now we can get the result from the maximum principle. □

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