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MARCINKIEWICZ INTEGRALS ASSOCIATED WITH SCHRÖDINGER OPERATOR ON GENERALIZED MORREY SPACES

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Abstract. Let $L=-\Delta+V$ be a Schrödinger operator, where Δ is the Laplacian on $\mathbb{R}^n$, while nonnegative potential V belongs to the reverse Hölder class. In this paper, we study the boundedness of the Marcinkiewicz operator associated with Schrödinger operator μ_j^L on generalized Morrey spaces $M_{p,\varphi}$. We find the sufficient conditions on the pair (φ_1,φ_2) which ensures the boundedness of the operators μ_j^L from one generalized Morrey space M_{p,φ_1} to another M_{p,φ_2}, $1<p<\infty$ and from the space M_{1,φ_1} to the weak space WM_{1,φ_2}.

1. Introduction

The classical Morrey spaces were originally introduced by Morrey in [20] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [5, 6, 12, 13, 14, 20].

Suppose that $S^{n-1}=\{x\in\mathbb{R}^n : |x|=1\}$ is the unit sphere of $\mathbb{R}^n$ $(n\geqslant 2)$ equipped with the normalized Lebesgue measure $d\sigma=d\sigma(x')$.

In [23], Stein defined the Marcinkiewicz integral for higher dimensions. Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx)=\Omega(x) \tag{1.1}$$

for all $t>0$ and $x\in\mathbb{R}^n$.

(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}}\Omega(x')d\sigma(x')=0, \tag{1.2}$$

where $x'=x/|x|$ for any $x\neq 0$.

(iii) $\Omega\in L_1(S^{n-1})$.

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The Marcinkiewicz integral operator of higher dimension μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

REMARK 1.1. We easily see that the Marcinkiewicz integral operator of higher dimension μ_Ω can be regarded as a generalized version of the classical Marcinkiewicz integral in the one dimension case. Also, it is easy to see that μ_Ω is a special case of the Littlewood-Paley g-function if we take

$$g(x) = \Omega(x')|x|^{-n+1}\chi_{|x|\leqslant 1}(|x|).$$

We say that $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leqslant 1$ if there exists a constant $C > 0$ such that $|\Omega(x') - \Omega(y')| \leqslant C|x'-y'|^\alpha$ for all $x', y' \in S^{n-1}$.

In [23], Stein proved the following results.

THEOREM 1.1. (E. M. Stein) *Suppose that* Ω *satisfies* (1.1).

(a) *If* $\Omega \in L_1(S^{n-1})$ *and* Ω *is odd, then* μ_Ω *is bounded on* $L_p(\mathbb{R}^n)$ *for* $1 < p < \infty$.

(b) *If* Ω *satisfies* (1.2) *and* $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leqslant 1$, *then* μ_Ω *is of weak type* $(1,1)$. *That is, there exists a constant* C *such that for any* $t > 0$ *and* $f \in L_1(\mathbb{R}^n)$,

$$|\{x \in \mathbb{R}^n : \ \mu_\Omega(f)(x) > t\}| \leqslant \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|dx.$$

(c) *If* Ω *satisfies* (1.2) *and* $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leqslant 1$, *then* μ_Ω *is of type* (p,p) *for* $1 < p \leqslant 2$. *That is, there exists a constant* A_p *such that for any* $f \in L_p(\mathbb{R}^n)$,

$$\|\mu_\Omega(f)\|_{L_p} \leqslant A_p \|f\|_{L_p}.$$

The L_p boundedness of μ_Ω has been studied extensively. See [3, 17, 23, 24], among others. A survey of past studies can be found in [9]. Recently the following result was obtained in [2].

THEOREM 1.2. *Suppose that* Ω *satisfies* (1.1) *and* (1.2). *If*

$$\Omega \in L(\log^+ L)^{1/2}(S^{n-1}), \tag{1.3}$$

then μ_Ω *is bounded on* $L_p(\mathbb{R}^n)$ *for* $1 < p < \infty$. *The exponent* $1/2$ *is the best possible.*

On the other hand, the study of Schrödinger operator $L = -\Delta + V$ recently attracted much attention. In particular, Shen [7] considered Lp estimates for Schrödinger operators L with certain potentials which include Schrödinger Riesz transforms $R_j^L =$

$\frac{\partial}{\partial x_j}L^{-\frac{1}{2}}$, $j=1,\ldots,n$. Then, Dziubanński and Zienkiewicz [8] introduced the Hardy type space $H_L^1(\mathbb{R}^n)$ associated with the Schrödinger operator L, which is larger than the classical Hardy space $H^1(\mathbb{R}^n)$.

Similar to the classical Marcinkiewicz function, we define the Marcinkiewicz functions μ_j associated with the Schrödinger operator L by

$$\mu_j^L f(x) = \left(\int_0^\infty \left|\int_{|x-y|\leqslant t} K_j^L(x,y)f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2},$$

where $K_j^L(x,y) = \widetilde{K_j^L}(x,y)|x-y|$ and $\widetilde{K_j^L}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j}L^{-\frac{1}{2}}$, $j = 1,\ldots,n$. In particular, when $V=0$, $K_j^\Delta(x,y) = \widetilde{K_j^\Delta}(x,y)|x-y| = \frac{(x-y)_j/|x-y|}{|x-y|^{n-1}}$ and $\widetilde{K_j^\Delta}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j}\Delta^{-\frac{1}{2}}$, $j=1,\ldots,n$. In this paper, we write $K_j(x,y) = K_j^\Delta(x,y)$ and

$$\mu_j f(x) = \left(\int_0^\infty \left|\int_{|x-y|\leqslant t} K_j(x,y)f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Obviously, μ_j are classical Marcinkiewicz functions. Therefore, it will be an interesting thing to study the property of μ_j^L. The main purpose of this paper is to show that Marcinkiewicz integrals associated with Schrödinger operators are bounded from one generalized Morrey space M_{p,φ_1} to another M_{p,φ_2}, $1<p<\infty$, and from the space M_{1,φ_1} to the weak space WM_{1,φ_2}.

Note that a nonnegative locally L_q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to B_q $(1<q<\infty)$ if there exists $C>0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)dy\right)^{1/q} \leqslant C\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)dy\right) \tag{1.4}$$

holds for every ball $x\in\mathbb{R}^n$ and $r>0$, where $B(x,r)$ denotes the open ball centered at x with radius r; see [21]. It is worth pointing out that the B_q class is that, if $V\in B_q$ for some $q>1$, then there exists $\varepsilon>0$, which depends only n and the constant C in (1.4), such that $V\in B_{q+\varepsilon}$. Throughout this paper, we always assume that $0\neq V\in B_n$.

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \quad 0<t<\infty,$$

where w is a weight.

The following theorem in the case $w=1$ was proved in [4].

THEOREM 1.3. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t)H_w^* g(t) \leqslant C \operatorname*{ess\,sup}_{t>0} v_1(t)g(t) \tag{1.5}$$

holds for some $C > 0$ *for all non-negative and non-decreasing* g *on* $(0,\infty)$ *if and only if*

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{1.6}$$

Moreover, the value $C = B$ *is the best constant for* (1.5).

REMARK 1.2. In (1.5) and (1.6) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

By $A \lesssim B$ we mean that $A \leqslant CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Generalized Morrey spaces

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey in [20] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [11, 18].

We denote by $M_{p,\lambda} \equiv M_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $1 \leqslant p < \infty$ and $0 \leqslant \lambda \leqslant n$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

DEFINITION 2.1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \leqslant p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space consisting of all measurable functions f for which

$$\|f\|_{WL_p(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} < \infty.$$

Also the spaces $L_p^{\rm loc}(\mathbb{R}^n)$ and $WL_p^{\rm loc}(\mathbb{R}^n)$ endowed with the natural topology are defined as the set of all functions f such that $f\chi_B \in L_p(\mathbb{R}^n)$ and $f\chi_B \in WL_p(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$, respectively.

According to this definition, we recover the space $M_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}},$$

$$WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|dy.$$

Suppose that T represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|Tf(x)| \leqslant c_0 \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \tag{2.1}$$

where c_0 is independent of f and x.

We point out that the condition (2.1) was first introduced by Soria and Weiss in [22]. The condition (2.1) are satisfied by many interesting operators in harmonic analysis, such as the Calderón–Zygmund operators, Carleson's maximal operator, Hardy–Littlewood maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci–Stein's oscillatory singular integrals, the Bochner–Riesz means and so on (see [19], [22] for details).

The following statement, was proved in [15].

THEOREM 2.4. *Let $1 \leqslant p < \infty$ and (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \leqslant C\varphi_2(x,r), \tag{2.2}$$

where C does not depend on x and r. Let T be a sublinear operator satisfies the condition (2.1) bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Then the operator T is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}. Moreover, for $p > 1$

$$\|Tf\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}},$$

and for $p = 1$

$$\|Tf\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

COROLLARY 2.1. [16] *Let* $1 \leqslant p < \infty$, Ω *satisfies the conditions* (1.1), (1.2) *and* (1.3). *Let also* (φ_1, φ_2) *satisfies the condition* (2.2). *Then the operator* μ_Ω *and the operator* μ_j *are bounded from* M_{p,φ_1} *to* M_{p,φ_2} *for* $p > 1$ *and from* M_{1,φ_1} *to* WM_{1,φ_2}.

COROLLARY 2.2. [1] *Let* $1 \leqslant p < \infty$ *and* (φ_1, φ_2) *satisfies the condition* (2.2). *Then the maximal operator* M *is bounded from* M_{p,φ_1} *to* M_{p,φ_2} *for* $p > 1$ *and from* M_{1,φ_1} *to* WM_{1,φ_2}. *Moreover, for* $p > 1$

$$\|Mf\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}, \qquad \|\mu_j f\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}},$$

and for $p = 1$

$$\|Mf\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}, \qquad \|\mu_j f\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

3. Marcinkiewicz operator μ_j^L in the spaces $M_{p,\varphi}$

In this section, we prove the boundedness of the Marcinkiewicz operator μ_j^L on $M_{p,\varphi}(\mathbb{R}^n)$ spaces.

For $x \in \mathbb{R}^n$, the function $m_V(x)$ is defined by

$$\rho(x) = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V\,(y)dy \leqslant 1 \right\}.$$

LEMMA 3.1. [21] *Let* $V \in B_q$ *with* $q \geqslant n/2$. *Then there exists* $l_0 > 0$ *such that*

$$\frac{l}{C}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-l_0} \leqslant \frac{\rho(y)}{\rho(x)} \leqslant C\left(1 + \frac{|x-y|}{\rho(x)}\right)^{l_0/(l_0+1)}.$$

In particular, $\rho(x) \sim \rho(y)$ *if* $|x-y| < C\rho(x)$.

LEMMA 3.2. [21] *Let* $V \in B_q$ *with* $q \geqslant n/2$. *For any* $l > 0$, *there exists* $C_l > 0$ *such that*

$$\left|K_j^L(x,y)\right| \leqslant \frac{C_l}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^l} \frac{1}{|x-y|^{n-1}},$$

and

$$\left|K_j^L(x,y) - K_j(x,y)\right| \leqslant C \frac{\rho(x)^{-1}}{|x-y|^{n-2}}.$$

The following Theorem has been proved in [10]. For the sake of completeness we give the proof.

THEOREM 3.5. *Let* $V \in B_n$. *Then the operators* μ_j^L, $j = 1, \ldots, n$ *are bounded on* $L_p(\mathbb{R}^n)$ *for* $1 < p < \infty$, *and bounded from* $L_1(\mathbb{R}^n)$ *to weak* $L_1(\mathbb{R}^n)$.

Proof. It suffices to show that

$$\mu_j^L f(x) \leqslant \mu_j f(x) + CMf(x), a.e. \;\; x \in \mathbb{R}^n, \tag{3.1}$$

where M denotes the standard Hardy-Littlewood maximal operator.

Fixing $x \in \mathbb{R}^n$ and let $r = \rho(x)$.

$$\begin{aligned}
\mu_j^L f(x) \leqslant & \left(\int_0^r \left|\int_{|x-y|\leqslant t} K_j^L(x,y) f(y) dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
& + \left(\int_r^\infty \left|\int_{|x-y|\leqslant r} K_j^L(x,y) f(y) dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
& + \left(\int_r^\infty \left|\int_{r<|x-y|\leqslant t} K_j^L(x,y) f(y) dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
\leqslant & \left(\int_0^r \left|\int_{|x-y|\leqslant t} [K_j^L(x,y) - K_j(x,y)] f(y) dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
& + \left(\int_0^r \left|\int_{|x-y|\leqslant t} K_j(x,y) f(y) dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
& + \left(\int_r^\infty \left|\int_{|x-y|\leqslant r} K_j^L(x,y) f(y) dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
& + \left(\int_r^\infty \left|\int_{r<|x-y|\leqslant t} K_j^L(x,y) f(y) dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
:= & E_1 + E_2 + E_3 + E_4.
\end{aligned}$$

For E_1, by Lemma 3.2, we have

$$E_1 \leqslant C\left(\int_0^r \left|\frac{1}{r}\int_{|x-y|\leqslant t} \frac{|f(y)|}{|x-y|^{n-2}} dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \leqslant CMf(x).$$

Obviously,

$$E_2 \leqslant \mu_j f(x).$$

For E_3, using Lemma 3.2 again, we get

$$E_3 \leqslant \left(\int_r^\infty \left|\frac{1}{r}\int_{|x-y|\leqslant r} \frac{|f(y)|}{|x-y|^{n-2}} dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \leqslant CMf(x).$$

It remains to estimate E_4. From Lemma 3.2, we obtain

$$\begin{aligned}
E_4 &\leqslant C\left(\int_r^\infty \left|r\int_{r<|x-y|\leqslant t}\frac{|f(y)|}{|x-y|^n}dy\right|^2\frac{dt}{t^3}\right)^{\frac12}\\
&\leqslant C_r\left(\int_r^\infty \left|\sum_{k=0}^{[\log_2 t/r]+1}(2^k r)^n\int_{|x-y|\leqslant 2^k r}|f(y)|dy\right|^2\frac{dt}{t^3}\right)^{\frac12}\\
&\leqslant C_r\left(\int_r^\infty \left|\left(\left[\log_2\frac{t}{r}\right]+1\right)Mf(x)\right|^2\frac{dt}{t^3}\right)^{\frac12}\\
&\leqslant C_r\left(\int_r^\infty \frac{t}{r}Mf(x)^2\frac{dt}{t^3}\right)^{\frac12}\\
&\leqslant CMf(x).
\end{aligned}$$

Thus, Theorem 3.5 is proved. □

LEMMA 3.3. *Let $V\in B_n$. If $1<p<\infty$, then the inequality*

$$\|\mu_j^L(f)\|_{L_p(B(x_0,r))}\lesssim r^{\frac{n}{p}}\int_{2r}^\infty t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x_0,t))}dt$$

holds for any ball $B(x_0,r)$, and for all $f\in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover, for $p=1$ the inequality

$$\|\mu_j^L(f)\|_{WL_1(B(x_0,r))}\lesssim r^n\int_{2r}^\infty t^{-n-1}\|f\|_{L_1(B(x_0,t))}dt$$

holds for any ball $B(x_0,r)$, and for all $f\in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let $p\in(1,\infty)$. For arbitrary $x_0\in\mathbb{R}^n$, set $B=B(x_0,r)$, for the ball centered at x_0 and of radius r, $2B=B(x_0,2r)$. We represent f as

$$f=f_1+f_2,\quad f_1(y)=f(y)\chi_{2B}(y),\quad f_2(y)=f(y)\chi_{\complement_{(2B)}}(y),\quad r>0,$$

and have

$$\|\mu_j^L(f)\|_{L_p(B)}\leqslant\|\mu_j^L(f_1)\|_{L_p(B)}+\|\mu_j^L(f_2)\|_{L_p(B)}.$$

Since $f_1\in L_p(\mathbb{R}^n)$, $\mu_j^L f_1\in L_p(\mathbb{R}^n)$ and from the boundedness of μ_j^L in $L_p(\mathbb{R}^n)$ (see Theorem 3.5) it follows that:

$$\|\mu_j^L(f_1)\|_{L_p(B)}\leqslant\|\mu_j^L(f_1)\|_{L_p(\mathbb{R}^n)}\lesssim\|f_1\|_{L_p(\mathbb{R}^n)}\approx\|f\|_{L_p(2B)},$$

where constant $C>0$ is independent of f. It's clear that $x\in B,\ y\in {}^{\complement}(2B)$ implies $\frac{1}{2}|x_0-y|\leqslant|x-y|\leqslant\frac{3}{2}|x_0-y|$. We get

$$\begin{aligned}\mu_j^L f_2(x) &\leqslant \int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f_2(y)|\left(\int_{|x-y|}^{\infty}\frac{dt}{t^3}\right)^{\frac{1}{2}}dy\\ &\lesssim \int_{{}^{\complement}(2B)}\frac{|f(y)|}{|x-y|^n}\,dy\\ &\lesssim \int_{{}^{\complement}(2B)}\frac{|f(y)|}{|x_0-y|^n}\,dy.\end{aligned}$$

By Fubini's theorem we have

$$\begin{aligned}\int_{{}^{\complement}(2B)}\frac{|f(y)|}{|x_0-y|^n}dy &\approx \int_{{}^{\complement}(2B)}|f(y)|\int_{|x_0-y|}^{\infty}\frac{dt}{t^{n+1}}dy\\ &\approx \int_{2r}^{\infty}\int_{2r\leqslant|x_0-y|<t}|f(y)|dy\frac{dt}{t^{n+1}}\\ &\lesssim \int_{2r}^{\infty}\int_{B(x_0,t)}|f(y)|\,dy\,\frac{dt}{t^{n+1}}.\end{aligned}$$

By applying Hölder's inequality, we get

$$\int_{{}^{\complement}(2B)}\frac{|f(y)|}{|x_0-y|^n}dy\lesssim\int_{2r}^{\infty}\|f\|_{L_p(B(x_0,t))}\,\frac{dt}{t^{n+1}}.$$

Moreover, for all $p\in[1,\infty)$ the inequatily

$$\|\mu_j^L(f_2)\|_{L_p(B)}\lesssim r^{\frac{n}{p}}\int_{2r}^{\infty}t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x_0,t))}\,dt \tag{3.2}$$

is valid. Thus

$$\|\mu_j^L(f)\|_{L_p(B)}\lesssim\|f\|_{L_p(2B)}+r^{\frac{n}{p}}\int_{2r}^{\infty}t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x_0,t))}\,dt.$$

On the other hand,

$$\begin{aligned}\|f\|_{L_p(2B)} &\approx r^{\frac{n}{p}}\|f\|_{L_p(2B)}\int_{2r}^{\infty}\frac{dt}{t^{\frac{n}{p}+1}}\\ &\lesssim r^{\frac{n}{p}}\int_{2r}^{\infty}t^{-\frac{n}{p}-1}\,\|f\|_{L_p(B(x_0,t))}\,dt.\end{aligned} \tag{3.3}$$

Thus

$$\|\mu_j^L(f)\|_{L_p(B)}\lesssim r^{\frac{n}{p}}\int_{2r}^{\infty}t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x_0,t))}\,dt.$$

If $p = 1$. From the weak $(1,1)$ boundedness of μ_j^L and (3.3) it follows that

$$\begin{aligned}\|\mu_j^L f_1\|_{WL_1(B)} &\leqslant \|\mu_j^L f_1\|_{WL_1(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)} = \|f\|_{L_1(2B)}\\ &\lesssim r^n \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|dy \frac{dt}{t^{n+1}}\\ &\leqslant r^n \int_{2r}^{\infty} t^{-n-1} \|f\|_{L_1(B(x_0,t))}\, dt. \end{aligned} \tag{3.4}$$

Then by (3.2) and (3.4) we get the inequality (3.3).

THEOREM 3.6. *Let $V \in B_n$ and $1 \leqslant p < \infty$ and (φ_1, φ_2) satisfies the condition (2.2). Then the operator μ_j^L is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}. Moreover, for $p > 1$*

$$\|\mu_j^L f\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}},$$

and for $p = 1$

$$\|\mu_j^L f\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

Proof. By Lemma 3.3 and Theorem 1.3 we have for $p > 1$

$$\begin{aligned}\|\mu_j^L f\|_{M_{p,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L_p(B(x,t))} \frac{dt}{t^{\frac{n}{p}+1}}\\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x,r))}\\ &= \|f\|_{M_{p,\varphi_1}}\end{aligned}$$

and for $p = 1$

$$\begin{aligned}\|\mu_j^L f\|_{WM_{1,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L_1(B(x,t))} \frac{dt}{t^{n+1}}\\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} r^{-n} \|f\|_{L_1(B(x,r))}\\ &= \|f\|_{M_{1,\varphi_1}}. \quad \square\end{aligned}$$

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REFERENCES

[1] A. AKBULUT, V. S. GULIYEV AND R. MUSTAFAYEV, *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces*, Math. Bohem. **137** (1) (2012), 27–43.

[2] A. AL-SALMAN, H. AL-QASSEM, L. C. CHENG, Y. PAN, *L_p bounds for the function of Marcinkiewicz*, Math. Res. Lett. **9** (2002) 697–700.

[3] A. BENEDEK, A. P. CALDERON, R. PANZONE, *Convolution operators on Banach value functions*, Proc. Natl. Acad. Sci. USA **48** (1962), 256–265.

[4] V. I. BURENKOV, A. GOGATISHVILI, V. S. GULIYEV, R. CH. MUSTAFAYEV, *Boundedness of the Riesz potential in local Morrey-type spaces*, Potential Analysis, **35** (2011), no. 1, 67–87.

[5] G. DI FAZIO, M. A. RAGUSA, *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. **112** (1993) 241–256.

[6] D. FAN, S. LU AND D. YANG, *Boundedness of operators in Morrey spaces on homogeneous spaces and its applications*, Acta Math. Sinica (N. S.) **14** (1998), suppl., 625–634.

[7] Y. DING, D. YANG, Z. ZHOU, *Boundedness of sublinear operators and commutators on* $L^{p,\omega}(\mathbb{R}^n)$, Yokohama Mathematical, **46** (1998), 15–27.

[8] J. DZIUBAŃSKI, J. ZIENKIEWICZ, *Hardy space* H^1 *associated to Schrödinger operator with potential satisfying reverse Hölder inequality*, Rev. Mat. Iber. **15** (1999), 279–296.

[9] Y. DING, *On Marcinkiewicz integral*, in: Proc. of the Conference "Singular Integrals and Related Topics, III", Osaka, Japan, 2001, pp. 28–38.

[10] W. GAO, L. TANG, *Boundedness for Marcinkiewicz integrals associated with Schrdinger operators*, Indian Academy of Sci., 2012. (to appear).

[11] M. GIAQUINTA, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Princeton Univ. Press, Princeton, NJ, 1983.

[12] V. S. GULIYEV, *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$, Doctoral dissertation, Moscow, Mat. Inst. Steklov, 1994, 329 pp. (in Russian).

[13] V. S. GULIYEV, *Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications*, Baku, Elm. 1999, 332 pp. (Russian).

[14] V. S. GULIYEV, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. 2009, Art. ID 503948, 20 pp.

[15] V. S. GULIYEV, S. S. ALIYEV, T. KARAMAN, *Boundedness of a class of sublinear operators and their commutators on generalized Morrey spaces*, Abstr. Appl. Anal. vol. 2011, Art. ID 356041, 18 pp. doi:10.1155/2011/356041.

[16] V. S. GULIYEV, SEYMUR S. ALIYEV, *Boundedness of parametric Marcinkiewicz integral operator and their commutators on generalized Morrey spaces*, Georgian Math. J. **19** (2012), 195–208.

[17] L. HÖRMANDER, *Translation Invariant Operators*, Acta Math., **104** (1960), 93–139.

[18] A. KUFNER, O. JOHN AND S. FUČIK, *Function Spaces*, Noordhoff International Publishing: Leyden, Publishing House Czechoslovak Academy of Sciences: Prague, 1977.

[19] G. LU, S. LU, D. YANG, *Singular integrals and commutators on homogeneous groups*, Analysis Mathematica, **28** (2002) 103–134.

[20] C. B. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.

[21] Z. SHEN, L^p *estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble) **45** (1995) 513–546.

[22] F. SORIA, G. WEISS, *A remark on singular integrals and power weights*, Indiana Univ. Math. J. **43** (1994) 187–204.

[23] E. M. STEIN, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466.

[24] T. WALSH, *On the function of Marcinkiewicz*, Studia Math. **44** (1972) 203–217.

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