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CERTAIN L^p BOUNDS FOR ROUGH SINGULAR INTEGRALS

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Abstract. In this paper, we prove L^p bounds for singular integrals with rough kernels associated to certain surfaces. Our results extend as well as improve previously obtained results.

1. Introduction and main results

Let $\mathbb{R}^n$, $n \geqslant 2$, be the $n-$dimensional Euclidean space and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with normalized Lebesgue measure $d\sigma$. Let Ω be a homogeneous function of degree zero with $\Omega \in L^1\left(\mathbb{S}^{n-1}\right)$ and

$$\int_{\mathbb{S}^{n-1}} \Omega(y')\, d\sigma(y') = 0 \tag{1}$$

where $y' = \frac{y}{|y|} \in \mathbb{S}^{n-1}$ for $y \neq 0$. The classical Calderón-Zygmund singular integral operator $\mathscr{T}_\Omega$ is defined by

$$(\mathscr{T}_\Omega f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \Omega(y')\, |y|^{-n} f(x-y)\, dy, \tag{2}$$

where $f \in S(\mathbb{R}^n)$, the space of Schwartz functions.

By introducing the "method of rotation", Calderón and Zygmund ([5], [6]) proved that the operator $\mathscr{T}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ provided that Ω is an odd function in $L^1\left(\mathbb{S}^{n-1}\right)$. However, for general functions Ω, Calderón and Zygmund proved that $\mathscr{T}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ provided that $\Omega \in L\log^+ L\left(\mathbb{S}^{n-1}\right)$, i.e.,

$$\int_{\mathbb{S}^{n-1}} |\Omega(y')| \log^+ |\Omega(y')|\, d\sigma(y') < \infty. \tag{3}$$

Moreover, they showed that the L^p bondedness may fail if the condition $\Omega \in L\log^+ L\left(\mathbb{S}^{n-1}\right)$ is replaced by $\Omega \in L\left(\log^+ L\right)^{1-\varepsilon}\left(\mathbb{S}^{n-1}\right)$, for some $\varepsilon > 0$. Subsequently, the condition $\Omega \in L\log^+ L\left(\mathbb{S}^{n-1}\right)$ was independently improved by Connett ([8]) and Ricci-Weiss ([15]) who showed that if $\Omega \in H^1\left(\mathbb{S}^{n-1}\right)$, then $\mathscr{T}_\Omega$ maps $L^p(\mathbb{R}^n)$ into itself for $1 < p < \infty$. Here, $H^1\left(\mathbb{S}^{n-1}\right)$ denotes the Hardy space on the unit sphere which contains the class $L\log^+ L\left(\mathbb{S}^{n-1}\right)$ as a proper subspace.

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On the other hand, by an application of Plancherel's formula, it can be easily seen that $\mathcal{T}_{\Omega}$ maps $L^2(\mathbb{R}^n)$ onto itself boundedly if, and only if the function Ω satisfies the condition

$$\sup_{\xi\in\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}|\Omega(y')|\log^{+}\frac{1}{|\xi\cdot y'|}d\sigma(y')<\infty. \tag{4}$$

However, it is not known weather the Condition 4 alone implies the L^p boundedness of $\mathcal{T}_{\Omega}$ for some $p\neq 2$. For a though discussion of this problem, we advice readers to consult [12], [16], among others.

In [12], Grafakos and Stefanov discussed the L^p bondedness of $\mathcal{T}_{\Omega}$ under conditions related to the Condition 4. In fact, they introduced the following conditions

$$\sup_{\xi\in\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}|\Omega(y')|\left(\log^{+}\frac{1}{|\xi\cdot y'|}\right)^{1+\alpha}d\sigma(y')<\infty, \tag{5}$$

$\alpha>0$. For $\alpha>0$, let $F_{\alpha}(\mathbb{S}^{n-1})$ denote the space of all integrable functions Ω on $\mathbb{S}^{n-1}$ that satisfy (1). It is clear that

$$\bigcup_{q>1}L^{q}(\mathbb{S}^{n-1})\subset\bigcap_{\alpha>0}F_{\alpha}(\mathbb{S}^{n-1}). \tag{6}$$

On the other hand, it is shown in [12] that

$$\bigcap_{\alpha>0}F_{\alpha}(\mathbb{S}^{n-1})\nsubseteq H^{1}(\mathbb{S}^{n-1})\nsubseteq\bigcup_{\alpha>0}F_{\alpha}(\mathbb{S}^{n-1}) \tag{7}$$

and

$$\bigcap_{\alpha>0}F_{\alpha}(\mathbb{S}^{n-1})\nsubseteq L\log^{+}L(\mathbb{S}^{n-1}). \tag{8}$$

The following is the main result in [12]

THEOREM 1.1. ([12]) *Let $\Omega\in F_{\alpha}(\mathbb{S}^{n-1})$ and satisfy (1). Then $\mathcal{T}_{\Omega}$ extends to a bounded operator from $L^p(\mathbb{R}^n)$ into itself for $p\in\left(\frac{2+\alpha}{1+\alpha},2+\alpha\right)$.*

Clearly, if the index α is close to zero, i.e., $\alpha\to0^{+}$, then the interval for p in Theorem 1.1 reduces to $p=2$. In [10], Fan, Guo and Pan were able to improve the range of p. In fact, they showed that the result of Theorem 1.1 still holds for $p\in\left(\frac{2+2\alpha}{1+2\alpha},2+2\alpha\right)$.

The main objective of this paper is to consider the conditions (5) in the context of singular integrals along certain surfaces.

(I) *Singular integrals associated to polynomial mappings*. For a suitable function $\Phi:\mathbb{R}^n\to\mathbb{R}^d$, consider the singular integral operator

$$\mathcal{T}_{\Omega,\Phi}f(x)=\text{p.v.}\int_{\mathbb{R}^n}\Omega(y')|y|^{-n}f(x-\Phi(y))dy. \tag{9}$$

Clearly, if $\Phi(y)=y$, the operator $\mathcal{T}_{\Omega,\Phi}$ reduces to the classical operator $\mathcal{T}_{\Omega}$. In this paper, we are interested in the case when

$$\Phi(y)=P(|y|)\otimes\varphi(y')=\left(P_1(|y|)\varphi_1(y'),P_2(|y|)\varphi_2(y'),\dots,P_d(|y|)\varphi_d(y')\right), \tag{10}$$

where $P_j(t), 1 \leqslant j \leqslant d$ are real valued polynomials on $\mathbb{R}$ and $\varphi_1(y'), \ldots \varphi_d(y')$ are real valued functions that are analytic on $\mathbb{S}^{n-1}$. It can be easily seen that if $d = n$ and $P_1(t) = P_2(t) = \ldots = P_d(t) = P(t) = t$ and $\varphi_1(y') = y'_1, \ldots, \varphi_d(y') = y'_d$, then $\Phi(y) = y$ and hence, as mentioned above the corresponding operator $\mathscr{T}_{\Omega,\Phi}$ reduces to the classical operator $\mathscr{T}_\Omega$. When $d = n$ and $P_1(t) = P_2(t) = \ldots = P_d(t) = P(t)$ and $\varphi_1(y') = y'_1, \ldots, \varphi_d(y') = y'_d$, then the operator $\mathscr{T}_{\Omega,\Phi} = \mathscr{T}_{\Omega,P}$ has been studied by Fan, Guo and Pan in [10]. In [10], Fan, Guo and Pan proved that the operator $\mathscr{T}_{\Omega,P}$ is bounded on L^p for $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$ provided that $\Omega \in F_\alpha(\mathbb{S}^{n-1})$ for some $\alpha > 0$. Furthermore, if $n = 2$, $P_1(t) = P_2(t) = \ldots = P_d(t) = t^m, m \geqslant 1$, and $\varphi(y')$ is homogeneous function of degree zero that is real analytic on $\mathbb{S}^{n-1}$, then Al-Salman showed in [4] that the operator $\mathscr{T}_{\Omega,\Phi}$ is bounded on L^p for $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$ provided that $\Omega \in F_\alpha(\mathbb{S}^{n-1})$ for some $\alpha > 0$. In this paper, we prove the following two results concerning the class of operators $\mathscr{T}_{\Omega,\Phi}$:

THEOREM 1.2. *Suppose that $\Omega \in F_\alpha\left(\mathbb{S}^{n-1}\right)$ for some $\alpha > 0$ and satisfy (1). Suppose also that $\Phi(y) = P(|y|) \otimes y'$ is as in (10). Then $\mathscr{T}_{\Omega,\Phi}$ is bounded in $L^p\left(\mathbb{R}^d\right)$ for $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$. The L^p bounds of $\mathscr{T}_{\Omega,\Phi}$ are independent of the coefficients of the polynomial mappings P_j, $1 \leqslant j \leqslant d$.*

THEOREM 1.3. *Suppose that $\Omega \in F_\alpha\left(\mathbb{S}^1\right)$ for some $\alpha > 0$ and satisfy (1). Suppose also that $P_1(t) = P_2(t) = \ldots = P_d(t) = P(t)$ and $\varphi(y') = (\varphi_1(y'), \ldots \varphi_d(y'))$ is real analytic on $\mathbb{S}^1$. Then $\mathscr{T}_{\Omega,\Phi}$ is bounded on $L^p\left(\mathbb{R}^d\right)$ for $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$.*

It is clear that Theorems 1.2 and 1.3 generalize the corresponding results in [4], [10], and [12]. We turn now to discuss the second class of operators in this paper. Namely, we discuss singular integrals along surfaces of revolution.

(II) *Singular integrals along surfaces of revolution*. We consider singular integral operators along hypersurfaces obtained by rotating one dimensional curves around one of the coordinate axes. For suitable functions $\Phi : \mathbb{R}^n \to \mathbb{R}^d$ and $\phi : [0,\infty) \to \mathbb{R}$, we define the singular integral operator $\mathscr{T}_{\Phi,\phi}$ along the surface

$$\Gamma = \{(\Phi(y), \phi(|y|)) : y \in \mathbb{R}^n\}$$

by

$$\mathscr{T}_{\Phi,\phi} f(x, x_{n+1}) = \text{p.v.} \int_{\mathbb{R}^n} f(x - \Phi(y), x_{n+1} - \phi(|y|)) \Omega(y') |y|^{-n} dy, \tag{11}$$

where $(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$, $n \geqslant 2$. When $\Phi(y) = y$, the operator $\mathscr{T}_\phi = \mathscr{T}_{\Phi,\phi}$ was introduced in 1996 by W. Kim, S. Wainger, J. Wright, and S. Ziesler [13]. It was shown in [13] that $\mathscr{T}_\phi$ is bounded on L^p for every $1 < p < \infty$ provided that ϕ is a convex increasing function with $\phi(0) = 0$ and $\Omega \in \mathscr{C}^\infty(\mathbb{S}^{n-1})$. Subsequently, the condition $\Omega \in \mathscr{C}^\infty(\mathbb{S}^{n-1})$ was relaxed to the weaker condition $L\log^+ L(\mathbb{S}^{n-1})$ by Al-Salman and Pan in [2]. In [7], L. Cheng and Y. Pan studied the classical operator $\mathscr{T}_\phi$ for functions $\Omega \in F_\alpha\left(\mathbb{S}^{n-1}\right)$ and polynomial mappings ϕ. More precisely, L. Cheng and Y. Pan proved the following result:

THEOREM 1.4. ([7]) *Let $\Omega \in F_\alpha\left(\mathbb{S}^{n-1}\right)$ for some $\alpha > 0$ and let ϕ be a polynomial.*

(i) If $n=2$, then $\mathscr{T}_\phi$ is bounded on $L^p\left(\mathbb{R}^3\right)$ for $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$.

(ii) If $n \geqslant 3$ and $\phi'(0)=0$, then $\mathscr{T}_\phi$ is bounded on $L^p\left(\mathbb{R}^{n+1}\right)$ for $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$.

In both (i) and (ii), the bounds on the operator norm are independent of the coefficients of ϕ.

In the following, we consider the general operator $\mathscr{T}_{\Phi,\phi}$ for various functions Φ and ϕ. In order to state our first result concerning the operator $\mathscr{T}_{\Phi,\phi}$, we need to recall the following class of functions introduced by Al-Salman and Pan in [3].

DEFINITION 1.5. ([3]) For $n \geqslant 2$, and $m \geqslant 1$, let $V(n,m)$ be the collection of homogeneous polynomials in $n-$variables of degree m. An integrable function Ω on S^{n-1} is said to be in the space $W(n,m,\alpha)$ if

$$\sup_{\substack{\mathscr{P}\in V(n,m)\\ \|\mathscr{P}\|=1}} \int_{\mathbb{S}^{n-1}} |\Omega(y')| \left(\log \frac{1}{|\mathscr{P}(y)|}\right)^{1+\alpha} d\sigma(y') < \infty. \tag{12}$$

Here, $\|\mathscr{P}\| = \sum\limits_{|\alpha|=m} |a_\alpha|$ where $\mathscr{P}(y) = \sum\limits_{|\alpha|=m} a_\alpha \ y^\alpha$.

It is shown in [3] that $\bigcap\limits_{s=1}^{\infty} W(2,s,\alpha) = F_\alpha\left(\mathbb{S}^1\right)$ while $\bigcap\limits_{s=1}^{\infty} W(n,s,\alpha)$ is a proper subspace of $F_\alpha\left(\mathbb{S}^{n-1}\right)$ for $n \geqslant 3$. Set

$$W(n,\alpha) = \bigcap_{s=1}^{\infty} W(n,s,\alpha).$$

Now, for $\alpha > 0$, $n \geqslant 2$, and $m \geqslant 1$, we let $W^0(n,s,\alpha)$ be the space of all integrable functions Ω on $\mathbb{S}^{n-1}$ that satisfy

$$\sup_{\substack{\mathscr{P}\in V(n,m)\\ \|\mathscr{P}\|=1,\beta\in\mathbb{R}}} \int_{\mathbb{S}^{n-1}} |\Omega(y')| \left(\log \frac{1}{|\mathscr{P}(y)+\beta|}\right)^{1+\alpha} d\sigma(y') < \infty. \tag{13}$$

Also, we set

$$W^0(n,\alpha) = \bigcap_{s=1}^{\infty} W^0(n,s,\alpha).$$

By the argument in [3] and [7], we can easily show that

$$W^0(2,\alpha) = F_\alpha\left(\mathbb{S}^1\right) \tag{14}$$

and

$$W^0(n,\alpha) \subset W(n,\alpha),\ n \geqslant 3. \tag{15}$$

Now we have the following result:

THEOREM 1.6. *Let $n \geqslant 2$, $\Phi = \mathscr{P}$ be a real valued polynomial in $n-$variables, and ϕ be a real valued polynomial on $\mathbb{R}$ with $\phi(0) = 0$. If $\Omega \in W^0(n,\alpha)$ for some $\alpha > 0$, then the operator $\mathscr{T}_{\mathscr{P},\phi}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$. Moreover, the L^p bounds are independent of the coefficients of ϕ and $\mathscr{P}$.*

By (14), it follows that Theorem 1.6 is a substantial improvement of the corresponding result in [7]. Our next result is the following:

THEOREM 1.7. *Suppose that $\Phi(y) = P(|y|) \otimes y'$ where $P(t) = (P_1(t), \ldots, P_d(t))$ is a polynomial mapping. Suppose also that ϕ is a real valued polynomial on $\mathbb{R}$.*

(i) If $n = 2$, and $\Omega \in F_\alpha(S^{n-1})$ for $\alpha > 0$, then $\mathscr{T}_{\Phi,\phi}$ is bounded on $L^p(\mathbb{R}^3)$ for $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$.

(ii) If $n \geqslant 3$, and $\Omega \in W^0(n,1,\alpha)$, then $\mathscr{T}_{\Phi,\phi}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$.

In both (i) and (ii), the L^p bounds on the operator norm are independent of the coefficient of ϕ and P_j.

It should be noticed here that $\Omega \in W^0(n,1,\alpha)$ if

$$\sup_{\substack{\xi \in S^{n-1} \\ \beta \in \mathbb{R}}} \int_{\mathbb{S}^{n-1}} \left|\Omega(y')\right| \log\left(\left|\frac{1}{\langle \xi, y' \rangle + \beta}\right|\right)^{1+\alpha} d\sigma(y') < \infty \tag{16}$$

which is equivalent to (5) in the case $n = 2$.

Now, we move to discuss the third class of operators in this paper.

(III) *Singular integrals along surfaces determined by certain convex functions*. We assume that the surface Γ is in the form

$$\Gamma = \left\{\left(\psi(|y|)y', \phi(|y|)\right) : y \in \mathbb{R}^n\right\}.$$

The singular integral operator associated to the surface Γ is given by

$$\mathscr{T}_{\psi,\phi} f(x, x_{n+1}) = \text{p.v.} \int_{\mathbb{R}^n} f(x - \psi(|y|)y', x_{n+1} - \phi(|y|))\,\Omega(y')\,|y|^{-n}\,dy. \tag{17}$$

Clearly, if ψ and ϕ are polynomials, then the corresponding operator $\mathscr{T}_{\psi,\phi}$ is a special class of the operators discussed in Theorem 1.7 above. However, our aim here is to discuss the L^p boundedness for functions that satisfy certain convexity assumptions. More precisely, our result concerning this class of operators is the following:

THEOREM 1.8. *Let $\psi, \phi \in C^1[0,\infty)$ be convex increasing, $\psi(0) = \phi(0) = \phi'(0) = 0$, and $\psi'(0) \neq 0$. Let $\varphi(t) = \phi(\psi^{-1}(t))$ and assume that φ' is convex and increasing. If $\Omega \in F_\alpha(\mathbb{S}^{n-1})$ for some $\alpha > 0$, and satisfies (1), then $\mathscr{T}_{\psi,\phi}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for all $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$.*

It should be noticed here that by specializing to the case $\psi(t) = t$, one obtain the result in [14].

This paper is organized as follows. In section 2, we shall present the main tools that we shall need to prove our results. In section 3, we shall prove Theorems 1.2 and 1.3. In section 4, we prove Theorems 1.6 and 1.7. The proof of Theorem 1.8 will be presented in Section 5. In Section 6, some further results will be highlighted.

2. Some lemmas

By combining the proofs of Lemma 5.2 in [11] and Lemma 2.1 in [3], we obtain the following version of Lemma 5.2 in [11]

LEMMA 2.1. *Let $\alpha > 0$, m, $d \in \mathbb{N}$ and $\left\{\sigma_{s,k} : 0 \leqslant s \leqslant m \text{ and } k \in \mathbb{Z}\right\}$ be a family of uniformly bounded Borel measure on $\mathbb{R}^d$ with $\sigma_{0,k} = 0$, for every $k \in \mathbb{Z}$. Let $\{\eta_s : 1 \leqslant s \leqslant m\} \subset R^+ \diagdown \{1\}$, $\{\ell_s : 1 \leqslant s \leqslant m\} \subset \mathbb{N}$, and $L_s : \mathbb{R}^d \to \mathbb{R}^{\ell_s}$ be linear transformations for $1 \leqslant s \leqslant m$. Suppose that*

i) $\left|\hat{\sigma}_{s,k}(\xi)\right| \leqslant C\left[\ln\left(\eta_s^k |L_s\xi|\right)\right]^{-(1+\alpha)}$ *for* $\xi \in \mathbb{R}^d$, $k \in \mathbb{Z}$ *and* $1 \leqslant s \leqslant m$,

ii) $\left|\hat{\sigma}_{s,k}(\xi) - \hat{\sigma}_{s-1,k}(\xi)\right| \leqslant C\left|\eta_s^k |L_s\xi|\right|$ *for* $\xi \in \mathbb{R}^d$, $k \in \mathbb{Z}$ *and* $1 \leqslant s \leqslant m$,

iii) For every $q \in (1,\infty)$ *there exist an* $A_q > 0$ *such that*

$$\left\|\sup_{k\in\mathbb{Z}}\left|\sigma_{s,k}\right| * |f|\right\|_q \leqslant A_q \|f\|_q, \tag{18}$$

for all $f \in L^q\left(\mathbb{R}^d\right)$ *and* $1 \leqslant s \leqslant m$. *Then for* $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$, *there exists a positive constant* C_p *such that*

$$\left\|\sum_{k\in\mathbb{Z}} \sigma_{m,k} * f\right\|_p \leqslant C_p \|f\|_p$$

holds for all $f \in L^p\left(\mathbb{R}^d\right)$. *Moreover, the constant* C_p *is independent of the linear transformations* $\{L_s : 1 \leqslant s \leqslant m\}$.

The following lemma can be found in ([16], page 477).

LEMMA 2.2. ([16]) *For every* $1 < p \leqslant \infty$ *there exists a positive constant* C_p *such that the maximal function*

$$(M_P f)(x) = \sup_{r>0} \frac{1}{r}\left|\int_{|t|<r} f(x - P(t))\,dt\right|$$

satisfies

$$\|M_P f\|_p \leqslant C_p \|f\|_p,$$

for $f \in L^p\left(\mathbb{R}^d\right)$. *The constant* C_p *may depend on the degree of the polynomials* $\left\{P_j\right\}$.

In order to handle the oscillatory integrals, we shall need the following lemma due to Van der Corput:

LEMMA 2.3. (Van der Corput [16]) *Let* ϕ *be a real valued and smooth function in* (a,b) *and* $\left|\phi^{(k)}(x)\right| \geqslant 1, \forall x \in (a,b)$. *Then*

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \leqslant c_k |\lambda|^{-\frac{1}{k}}$$

holds when:

(i) $k \geqslant 2$ *or*

(ii) $k = 1$ *and* $\phi'(x)$ *is monotonic. The bound* c_k *is independent of* $a, b,\ \phi$, *and* λ.

The following two lemmas will be useful:

LEMMA 2.4. ([9]) *Let* $\phi : [0,\infty) \to \mathbb{R}$ *be a* C^1 *function such that* ϕ' *is convex, increasing and satisfies* $\phi'(0) = 0$. *Let* (a_k) *be a lacunary sequence with* $\inf \frac{a_{k+1}}{a_k} \geqslant 2$. *Then, there exists a* $c > 0$ *such that*

$$\left| \int_1^b e^{i[a_k at + \eta\phi(a_k t)]} dt \right| \leqslant c\,|a_k a|^{-\frac{1}{2}}$$

holds for all $b \geqslant 1, a, \eta \in \mathbb{R}$, *and* $k \in \mathbb{Z}$.

It should be remarked here that the above lemma, i.e., Lemma 2.4 is proved in [9] for the special case $a_k = 2^k$. However, the proof for the general case follows by minor modifications of the special case.

LEMMA 2.5. ([4]) *Let* $\Phi : \mathbb{S}^1 \to \mathbb{R}^d,\ \Phi = (\Phi_1, \ldots, \Phi_d)$ *be a real analytic function on* $\mathbb{S}^1$. *Suppose also that* $\{\Phi_1, \ldots, \Phi_d\}$ *is linearly independent set. If* $\Omega \in F_\alpha\left(\mathbb{S}^1\right)$, *then*

$$\sup_{\xi \in S^{d-1}} \int_{\mathbb{S}^1} |\Omega(y')| \left(\log^+ \frac{1}{|\xi'.\Phi(y')|}\right)^{1+\alpha} d\sigma(y') < \infty.$$

Now, we prove the following lemma:

LEMMA 2.6. *Let* $\mathscr{P}$ *be a real valued polynomial in* $n-$*variables and* ϕ *be a real valued polynomial on* $\mathbb{R}$. *Let* $\Omega \in L^1\left(\mathbb{S}^{n-1}\right)$ *and let*

$$\mu_{\mathscr{P},\phi}(f)(x, x_{d+1}) = \sup_{j \in \mathbb{Z}} \int\limits_{2^j \leqslant |y| < 2^{j+1}} |f(x - \mathscr{P}(y), x_{d+1} - \phi(|y|))| \frac{|\Omega(y')|}{|y|^n} dy.$$

Then

$$\left\| \mu_{\mathscr{P},\phi}(f) \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_p \|\Omega\|_{L^1(\mathbb{S}^{n-1})} \|f\|_{L^p(\mathbb{R}^{d+1})} \tag{19}$$

for all $1 < p < \infty$ *with constants* C_p *that are independent of the coefficients of the polynomials* $\mathscr{P}$ *and* ϕ.

Proof. The verification of (19) is a straightforward application of Lemma 2.2. In fact, by polar coordinates we have

$$\begin{aligned} &\mu_{\mathscr{P},\phi}(f)(x, x_{d+1}) \\ &\leqslant \int\limits_{\mathbb{S}^{n-1}} |\Omega(y')| \left(\sup_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \left| f(x - \mathscr{P}(ty'), x_{d+1} - \phi(t)) \right| \frac{dt}{t} \right) d\sigma(y'). \end{aligned}$$

By Minkowski's inequality, we get

$$\left\|\mu_{\mathscr{P},\phi}(f)\right\|_{L^p(\mathbb{R}^{d+1})}$$
$$\leqslant \int_{\mathbb{S}^{n-1}} |\Omega(y')| \left\| \left(\sup_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \left| f(x-\mathscr{P}(ty'),x_{d+1}-\phi(t)) \right| \frac{dt}{t} \right) \right\|_p d\sigma(y').$$

Hence, (19) follows by an application of Lemma 2.2. This completes the proof. □

Now, we prove the following result concerning oscillatory integrals:

LEMMA 2.8. *Let ψ,ϕ be as in Theorem 1.8. Then*

$$\left| \int_{2^k}^{2^{k+1}} e^{-i[\lambda_1\psi(t)+\lambda_2\phi(t)]}\frac{dt}{t} \right| \leqslant C\left|\psi(2^k)\lambda_1\right|^{-\frac{1}{2}} \tag{20}$$

with a constant C independent of $\lambda_1,\lambda_2,$ and k.

Proof. By change of variables we have

$$\int_{2^k}^{2^{k+1}} e^{-i[\lambda_1\psi(t)+\lambda_2\phi(t)]}\frac{dt}{t} = \int_{\psi(2^k)}^{\psi(2^{k+1})} e^{-i[\lambda_1 u+\lambda_2\varphi(u))]}\frac{du}{\psi^{-1}(u)\psi'(\psi^{-1}(u))}$$

where $\varphi(t)=\phi(\psi^{-1}(t))$. By Lemma 2.4, we have

$$\left| \int_{\psi(2^k)}^{\psi(2^{k+1})} e^{-i[\lambda_1 u+\lambda_2\varphi(u))]}du \right| \leqslant \psi(2^k)C\left|\psi(2^k)\lambda_1\right|^{-\frac{1}{2}}.$$

Thus, by an integration by parts, we have

$$\begin{aligned}\left| \int_{2^k}^{2^{k+1}} e^{-i[\lambda_1\psi(t)+\lambda_2\phi(t)]}\frac{dt}{t} \right| &\leqslant \psi(2^k)C\left|\psi(2^k)\lambda_1\right|^{-\frac{1}{2}}\left(\frac{1}{2^k\psi'(2^k)}\right)\\ &\leqslant C\left|\psi(2^k)\lambda_1\right|^{-\frac{1}{2}}.\end{aligned}$$

This completes the proof. □

We end this section by the following result concerning maximal functions related to the operators in (17):

LEMMA 2.9. *Let ψ,ϕ be as in Theorem 1.8. Let $y' \in \mathbb{S}^{n-1}$. Then the maximal function*

$$M_{\psi,\phi,y'}(f)(x,x_{n+1}) = \sup_k \left| \int_{2^k}^{2^{k+1}} f(x-\psi(t)y', x_{n+1}-\phi(t))\frac{dt}{t} \right| \tag{21}$$

satisfies

$$\left\| M_{\psi,\phi,y'}(f) \right\|_p \leqslant C_p \|f\|_p \tag{22}$$

for all $1<p<\infty$ with constant C_p independent of y'.

Proof. For $k\in\mathbb{Z}$, define the measure μ_k by

$$\hat{\mu}_k(\xi,\eta) = \int_{2^k}^{2^{k+1}} e^{-i\left(\xi\cdot y'\psi(t)+\eta\phi(t)\right)}\frac{dt}{t}.$$

Then

$$M_{\psi,\phi,y'}(f)(x,x_{n+1}) = \sup_k |\mu_k * f(x,x_{n+1})|.$$

Now, Choose a smooth function θ with the properties $\hat{\theta}(\xi)=1$ for $|\xi|\leqslant \frac{1}{2}$, and $\hat{\theta}(\xi)=0$ for $|\xi|\geqslant 1$. Let $\theta_r(x)=r^{-n}\theta(\frac{x}{r})$ for $r\geqslant 0$. Define the sequence of measures τ_k by

$$\hat{\tau}_k(\xi,\eta) = \hat{\mu}_k(\xi,\eta) - \hat{\theta}_{\psi(2^k)}(\xi)\hat{\mu}_k(0,\eta). \tag{23}$$

By Lemma 2.8, we get

$$|\hat{\tau}_k(\xi,\eta)| \leqslant \left|\psi(2^k)\xi\cdot y'\right|^{-\frac{1}{2}} \tag{24}$$

for $\left|\psi(2^k)\xi\right|\geqslant 1$.

On the other hand, we have

$$|\hat{\tau}_k(\xi,\eta)| \leqslant \left|\psi(2^{k+1})\xi\cdot y'\right|. \tag{25}$$

Now, notice that

$$M_{\psi,\phi,y'}(f)(x,x_{n+1}) \leqslant \Big(\sum_k |\tau_k * f(x,x_{n+1})|^2\Big)^{\frac{1}{2}} + (M_H\otimes M_\phi)f(x,x_{n+1}) \tag{26}$$

where M_H is the Hardy-Littlewood maximal function acting on x-variable and M_ϕ is the maximal function in Lemma 2.2 with P is replaced by ϕ.

Also,

$$\tau^*(f)(x,x_{n+1}) \leqslant M_{\psi,\phi,y'}(f)(x,x_{n+1}) + (M_H\otimes M_\phi)f(x,x_{n+1}) \tag{27}$$

where

$$\tau^*(f)(x,x_{n+1}) = \sup_k |\tau_k * f(x,x_{n+1})|.$$

Thus, by (24)–(27), Lemma 2.2, and a bootstrapping argument as in [2], we obtain (22). This completes the proof. $\square$

3. Proofs of results on singular integrals associated to polynomial mappings

This section is devoted to the proofs of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Assume that $\Omega \in F_\alpha\left(\mathbb{S}^{n-1}\right)$ for some $\alpha > 0$ and satisfy (1). Let

$$M = \deg(P) = \max\{\deg(P_j), 1 \leqslant j \leqslant d\}.$$

For $1 \leqslant l \leqslant n$, let

$$P_l(t) = \sum_{j=1}^{M} a_{j,l} t^j.$$

For $1 \leqslant s \leqslant M$ and $1 \leqslant l \leqslant n$, let

$$P_l^{(s)}(t) = \sum_{j=1}^{s} a_{j,l} t^j,$$

$$P^{(s)}(t) = (P_1^{(s)}(t), \ldots, P_n^{(s)}(t)),$$

and

$$\Phi_s(y) = P^{(s)}(|y|) \otimes y'.$$

For each $k \in \mathbb{Z}$ and $1 \leqslant s \leqslant M$, define the measures $\sigma_{s,k}$ and $\mu_{s,k}$ by

$$\hat{\sigma}_{s,k}(\xi) = \int_{2^k \leqslant |y| < 2^{k+1}} \Omega(y')\, |y|^{-n} e^{-i\Phi_s(y).\xi} dy,$$
$$\hat{\mu}_{s,k}(\xi) = \int_{2^k \leqslant |y| < 2^{k+1}} |\Omega(y')|\, |y|^{-n} e^{-i\Phi_s(y).\xi} dy.$$

Also, we define σ_s^* by

$$\sigma_s^* f(x) = \sup_{k \in \mathbb{Z}} \left| \mu_{s,k} * f(x) \right|.$$

It is clear that $\sigma_{0,k} = 0$ for all $k \in \mathbb{Z}$. Moreover,

$$\mathscr{T}_{\Omega,\Phi} f(x) = \sum_{k \in \mathbb{Z}} \sigma_{M,k} * f. \tag{28}$$

Now,

$$\begin{aligned} \Phi_s(y).\xi &= \sum_{l=1}^{n} \xi_l y_l' P_l^{(s)}(|y|) = \sum_{l=1}^{n} \sum_{j=1}^{s} \xi_l y_l' a_{j,l} |y|^j \\ &= \sum_{j=1}^{s} \left(\sum_{l=1}^{n} \xi_l y_l' a_{j,l} \right) |y|^j = \sum_{j=1}^{s} \left(L_j(\xi) \cdot y' \right) |y|^j \end{aligned}$$

where $L_j : \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation given by

$$L_j(\xi) = (\xi_1 a_{j,1}, \ldots, \xi_n a_{j,n}). \tag{29}$$

Thus,

$$\begin{aligned}&\left|\hat{\sigma}_{s,k}(\xi)-\hat{\sigma}_{s-1,k}(\xi)\right|\\ \leqslant &\int\limits_{2^k\leqslant|y|<2^{k+1}} \left|\exp\left[i\left(\left(L_s(\xi)\cdot y'\right)|y|^s\right)\right]-1\right|\left|\Omega(y')\right||y|^{-n}\,dy\leqslant C\left(2^{sk}\left|L_s\xi\right|\right).\end{aligned}\tag{30}$$

On the other hand,

$$\left|\hat{\sigma}_{s,k}(\xi)\right|\leqslant\int\limits_{\mathbb{S}^{n-1}}\left|\Omega(y')\right|\left|\int_1^2\exp\left[-i\left(\sum_{j=1}^{s}\left(L_j(\xi)\cdot y'\right)2^{kj}\right)\right]\frac{dt}{t}\right|d\sigma(y').\tag{31}$$

By Lemma 2.3, we have

$$\left|\int_1^2\exp\left[-i\left(\sum_{j=1}^{s}\left(L_j(\xi)\cdot y'\right)2^{kj}\right)\right]\frac{dt}{t}\right|\leqslant\left|\left(L_s(\xi)\cdot y'\right)2^{ks}\right|^{\frac{-1}{s}}\tag{32}$$

By (32) and the trivial estimate

$$\left|\int_1^2\exp\left[-i\left(\sum_{j=1}^{s}\left(L_j(\xi)\cdot y'\right)2^{kj}\right)\right]\frac{dt}{t}\right|\leqslant 1,\tag{33}$$

we get

$$\begin{aligned}&\left|\int_1^2\exp\left[-i\left(\sum_{j=1}^{s}\left(L_j(\xi)\cdot y'\right)2^{kj}\right)\right]\frac{dt}{t}\right|\\ &\leqslant C\left(\log^+\left|2^{ks}L_s(\xi)\right|\right)^{-1-\alpha}\left(\log^+\left(\frac{1}{\left|L_s(\xi)/\left|L_s(\xi)\right|\cdot y'\right|}\right)\right)^{1+\alpha}\end{aligned}\tag{34}$$

which when combined with (5) imply that

$$\left|\hat{\sigma}_{s,k}(\xi)\right|\leqslant C\left(\log^+\left|2^{ks}L_s(\xi)\right|\right)^{-1-\alpha}.\tag{35}$$

Now, as in the proof of Lemma 2.6, we get

$$\left\|\sigma_s^*f\right\|_p\leqslant\int\limits_{\mathbb{S}^{n-1}}\left|\Omega(y')\right|\left\|\left(\sup_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}\left|f(x-P^{(s)}(t)\otimes y')\right|\frac{dt}{t}\right)\right\|_p d\sigma(y').$$

Thus, by Lemma 2.2, we get

$$\left\|\sigma_s^*f\right\|_p\leqslant C_p\left\|\Omega\right\|_1\left\|f\right\|_p\tag{36}$$

for all $1<p<\infty$. Hence, by (30), (35), (36), and Lemma 2.1, the proof is complete. □

Proof of Theorem 1.3. Assume that $\Omega \in F_\alpha\left(\mathbb{S}^1\right)$ for some $\alpha > 0$ and satisfy (1). Assume also that $\varphi(y') = (\varphi_1(y'), \ldots \varphi_d(y'))$ is real analytic on $\mathbb{S}^1$. Let $P(t) = P_1(t) = P_2(t) = \ldots = P_d(t)$ and that $P(t) = \sum_{m=1}^{M} a_m t^m$. For $1 \leqslant s \leqslant M$, let $P_s(t) = \sum_{m=1}^{s} a_m t^m$. For each $k \in \mathbb{Z}$, we define

$$\hat{\sigma}_{s,k}(\xi) = \int_{2^k \leqslant |y| < 2^{k+1}} \Omega(y')\,|y|^{-n}\, e^{-iP_s(|y|)\varphi(y')\cdot\xi}\,dy \tag{37}$$

$$\hat{\mu}_{s,k}(\xi) = \int_{2^k \leqslant |y| < 2^{k+1}} |\Omega(y')|\,|y|^{-n}\, e^{-iP_s(|y|)\varphi(y')\cdot\xi}\,dy. \tag{38}$$

Let $\left\{\varphi_{i_1}, \ldots \varphi_{i_\ell}\right\}$ be a maximal linearly independent subset of $\{\varphi_1, \ldots \varphi_d\}$, where $1 \leqslant \ell \leqslant d$, $1 \leqslant i_j \leqslant d$ and $j = 1, \ldots, \ell$. Thus, for $j \notin \{i_1, \ldots, i_l\}$, there exist $a^{(j)} = (a_{j,1}, \ldots, a_{j,l})$ $\in \mathbb{R}^\ell$ such that

$$\varphi_j(y') = a^{(j)} \cdot (\varphi_{i_1}(y'), \ldots \varphi_{i_\ell}(y')) = \sum_{o=1}^{l} a_{j,o}\varphi_{i_o}(y').$$

This implies that there exists a linear transformation $L : \mathbb{R}^d \to \mathbb{R}^\ell$ such that

$$\varphi\left(y'\right) \cdot \xi = L(\xi) \cdot \tilde{\varphi}\left(y'\right), \quad \xi \in \mathbb{R}^d$$

where $\tilde{\varphi}(y') = \left(\varphi_{i_1}(y'), \ldots \varphi_{i_\ell}(y')\right)$.

Now,

$$\left|\hat{\sigma}_{s,k}(\xi)\right| \leqslant \int_{\mathbb{S}^1} |\Omega(y')| \left|I_{s,k}(\xi)\right| d\sigma(y'),$$

where

$$I_{s,k}(\xi) = \int_1^2 \exp\left[-iP_s\left(2^k t\right)\left(L(\xi)\,.\tilde{\varphi}(y')\right)\right] \tfrac{dt}{t}.$$

By Lemma 2.3, we get

$$\left|I_{s,k}(\xi)\right| \leqslant \left|2^{ks} a_s L(\xi) \cdot \tilde{\varphi}\left(y'\right)\right|^{-\frac{1}{s}}. \tag{39}$$

By (39) and the estimate $\left|I_{s,k}(\xi)\right| \leqslant 1$, we get

$$\left|I_{s,k}(\xi)\right| \leqslant \frac{\left(\log^+ \left|L(\xi)'\,.\tilde{\varphi}(y')\right|^{-1}\right)^{1+\alpha}}{\left(\log^+ |2^{sk} a_s|\,|L(\xi)|\right)^{1+\alpha}} \tag{40}$$

where $L(\xi)' = \frac{L(\xi)}{|L(\xi)|}$. By combining inequality (40) and Lemma 2.5, we get

$$\left|\hat{\sigma}_{s,k}(\xi)\right| \leqslant C\left(\log^+ \left|2^{sk} a_s\right| |L(\xi)|\right)^{-1-\alpha}. \tag{41}$$

On the other hand, by using the vanishing property of Ω, we have

$$\begin{aligned}
&\left|\hat{\sigma}_{s,k}(\xi)-\hat{\sigma}_{s-1,k}(\xi)\right| \\
&\leqslant \int\limits_{2^k\leqslant|y|<2^{k+1}} \left|\exp\left[i\left(L(\xi).\tilde{\varphi}(y')\right)a_s|y|^s\right]-1\right|\left|\Omega(y')\right||y|^{-n}\,dy \\
&\leqslant C\left|a_s 2^{sk}L(\xi)\right|\int\limits_{\mathbb{S}^1}\left|\Omega(y')\right|\left\|\tilde{\varphi}(y')\right\|d\sigma(y') \\
&\leqslant C\left|a_s 2^{sk}L(\xi)\right|\|\Omega\|_1\sup_{y'\in\mathbb{S}^1}\left\|\tilde{\varphi}\left(y'\right)\right\|\leqslant C\left|a_s 2^{sk}L(\xi)\right|. \qquad (42)
\end{aligned}$$

In order to conclude the proof, we only need to prove the boundedness on L^p for all $p>1$ of the maximal operator

$$\sigma_s^* f(x)=\sup_{k\in\mathbb{Z}}\left|\mu_{s,k}*f(x)\right|.$$

Notice that

$$\begin{aligned}
\sigma_s^* f(x) &\leqslant \sup_k\int\limits_{\mathbb{S}^1}\left|\Omega(y')\right|\left|\int\limits_{2^k}^{2^{k+1}} f\left(x-P_s(t)\varphi(y')\right)\tfrac{dt}{t}\right|d\sigma(y') \\
&\leqslant \int\limits_{\mathbb{S}^1}\left|\Omega(y')\right|M^{(s)}_{\varphi(y')}f(x)\,d\sigma(y'),
\end{aligned}$$

where

$$M^{(s)}_{\varphi(y')}(f)(x)=\sup_k\left|\int\limits_{2^k}^{2^{k+1}} f\left(x-P_s(t)\varphi(y')\right)\tfrac{dt}{t}\right|.$$

By Lemma 2.2, we have

$$\left\|M^{(s)}_{\varphi(y')}(f)\right\|_p\leqslant C_p\|f\|_p \qquad (43)$$

for all $1<p<\infty$ with constant C_p independent of $\varphi(y')$. By (43) and Minkowski's inequality, we get

$$\left\|\sigma_s^* f\right\|_p\leqslant C_p\|f\|_p \qquad (44)$$

for all $1<p<\infty$. Hence, by (41), (42), (44), and Lemma 2.1, the proof is complete. □

4. Proofs of results on singular integrals along surfaces of revolution

This section is devoted to the proofs of Theorems 1.6 and 1.7. We shall start by the proof of Theorem 1.6.

Proof of Theorem 1.6. Let $n\geqslant 2$, $\mathscr{P}=(P_1,P_2,\ldots,P_d)$ where $P_j:\mathbb{R}^n\longrightarrow\mathbb{R}$ is a polynomial mapping for $1\leqslant j\leqslant d$. Let ϕ be a polynomial on $\mathbb{R}$ and that $\Omega\in W^0(n,\alpha)$ for some $\alpha>0$. Let

$$M=\max\{\deg(\phi),\deg(P_1),\deg(P_1),\ldots,\deg(P_d)\}.$$

Let

$$\phi(t) = \sum_{j=1}^{M} b_j t^j \quad \text{and} \quad P_j(y) = \sum_{|\beta| \leqslant M} a_{j\beta} y^\beta$$

for $j = 1, 2, ..., d$. For $1 \leqslant s \leqslant M$, we let $\mathscr{P}^{(s)}(y) = \left(P_{1,s}, P_{2,s}, ..., P_{d,s}\right)$ and $\phi_s(t) = \sum_{j=1}^{s} b_j t^j$ where

$$P_{j,s}(y) = \sum_{|\beta| \leqslant s} a_{j\beta} y^\beta, \quad j = 1, 2, ..., d.$$

We shall set $\mathscr{P}^{(0)} = 0$ and $\phi_0 = 0$.

For $k \in \mathbb{Z}$ and $0 \leqslant s \leqslant M$, we define the measure $\sigma_{s,k}$ on $\mathbb{R}^{n+1}$ by

$$\int_{\mathbb{R}^{n+1}} f d\sigma_{s,k} = \int_{2^k \leqslant |y| < 2^{k+1}} f\left(\mathscr{P}^{(s)}(y), \phi_s(|y|)\right) \Omega(y') |y|^{-n} dy. \tag{45}$$

We let σ_s^* be the maximal function

$$\sigma_s^* f(x) = \sup_{k \in \mathbb{Z}} \left| \left| \sigma_{s,k} \right| * f(x) \right|. \tag{46}$$

For $1 \leqslant s \leqslant M$, let ℓ_s denote the number of multi indices $\beta = (\beta_1, \beta_2, ..., \beta_n)$ satisfying $|\beta| = s$, and define the linear transformation $L_s : \mathbb{R}^d \to \mathbb{R}^{\ell_s}$ by

$$L_s \xi = \left(\sum_{j=1}^{d} a_{j\beta} \xi_j \right)_{|\beta| = s}. \tag{47}$$

For $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}$, $y \in \mathbb{S}^{n-1}$ and $k \in \mathbb{Z}$, we have

$$\begin{aligned}
\left| \hat{\sigma}_{s,k}(\xi, \eta) \right| &= \left| \int_{2^k \leqslant |y| < 2^{k+1}} \Omega(y') |y|^{-n} e^{-i\left[\xi . \mathscr{P}^{(s)}(y) + \eta \phi_s(|y|)\right]} dy \right| \\
&\leqslant \int_{\mathbb{S}^{n-1}} |\Omega(y')| \left| \left\{ \int_1^2 \exp\left[-i \left(\xi . \mathscr{P}^{(s)}\left(2^k y' t\right) + \eta \phi_s\left(2^k t\right) \right) \right] \tfrac{dt}{t} \right\} \right| d\sigma(y')
\end{aligned}$$

Let $I_{k,s}(\xi, \eta)$ denote the integral inside the brackets, i.e.,

$$I_{k,s}(\xi, \eta) = \int_1^2 \exp\left[-i \left(\xi . \mathscr{P}^{(s)}\left(2^k y' t\right) + \eta \phi_s\left(2^k t\right) \right) \right] \tfrac{dt}{t}.$$

Notice that

$$\xi.\mathscr{P}^{(s)}\left(2^k y' t\right)+\eta\phi_s\left(2^k t\right)=\sum_{j=1}^{d}\xi_j P_{j,s}\left(2^k y' t\right)+\eta\phi_s\left(2^k t\right)$$

$$=\sum_{j=1}^{d}\sum_{|\beta|\leqslant s}\xi_j a_{j\beta}2^{|\beta|k}t^{|\beta|}y'^{\beta}+\eta\phi_s\left(2^k t\right)$$

$$=\left(\eta b_s+\sum_{|\beta|=s}\left(\sum_{j=1}^{d}\xi_j a_{j\beta}\right)y'^{\beta}\right)2^{sk}t^s+(\text{lower powers in } t)$$

$$=\left(\eta b_s+\sum_{|\beta|=s}\left(\sum_{j=1}^{d}\xi_j a_{j\beta}\right)y'^{\beta}\right)2^{sk}t^s+(\text{lower powers in } t)$$

$$=\left(\eta b_s+L_s\xi\cdot\left(y'^{\beta}\right)_{|\beta|=s}\right)2^{sk}t^s+(\text{lower powers in } t).$$

Let

$$Q_{s\xi}\left(y'\right)=\left(L_s\xi\right)'\cdot\left(y'^{\beta}\right)_{|\beta|=s}$$

where $(L_s\xi)'=L_s\xi/|L_s\xi|$. By Lemma 2.3, we get

$$\left|I_{k,s}(\xi,\eta)\right|\leqslant C\left[2^{ks}\left|L_s\xi\right|\left(\left|Q_{s\xi}\left(y'\right)+\rho(\xi,\eta)\right|\right)\right]^{-\frac{1}{s}},\tag{48}$$

where $\rho(\xi,\eta)=\frac{b_s\eta}{|L_s\xi|}$. Thus, by (48) and the estimate $|I_k(\xi,\eta)|\leqslant 1$, we have

$$\left|I_{k,s}(\xi,\eta)\right|\leqslant C\left(\log^{+}\left(2^{ks}\left|L_s\xi\right|\right)\right)^{-(1+\alpha)}\left(s+\alpha+\log^{+}\frac{1}{\left|Q_{s\xi}\left(y'\right)+\rho(\xi,\eta)\right|}\right)^{1+\alpha}.\tag{49}$$

Since $\Omega\in W^0(n,\alpha)$, $Q_{s\xi}\in V(n,m)$ and $\left\|Q_{s\xi}\right\|=1$, we immediately obtain

$$\left|\hat{\sigma}_{s,k}(\xi,\eta)\right|\leqslant C\left(\log^{+}\left(2^{ks}\left|L_s\xi\right|\right)\right)^{-(1+\alpha)}\tag{50}$$

for $0\leqslant s\leqslant m$ and $k\in\mathbb{Z}$ and $(\xi,\eta)\in\mathbb{R}^{n+1}$.

On the other hand, we have

$$\begin{aligned}&\left|\hat{\sigma}_{s,k}(\xi,\eta)-\hat{\sigma}_{s-1,k}(\xi,\eta)\right|\\ \leqslant&\int_{2^k\leqslant|y|<2^{k+1}}\left|\exp\left[-i\left(\sum_{j=1}^{d}\sum_{|\beta|=s}a_{j\beta}\xi_j y^{\beta}+b_s\eta|y|^s\right)\right]-1\right|\left|\Omega\left(y'\right)\right||y|^{-n}dy\\ \leqslant&\, C\left(2^{ks}\left|L_s\xi\right|\right).\end{aligned}$$

Finally, by an argument similar to that led to (44) and an application of Lemma 2.6, we get

$$\left\|\sigma_s^*(f)\right\|_q\leqslant C_q\|f\|_q\tag{51}$$

for all $1 < q < \infty$. Hence, by (50)–(51) and Lemma 2.1, we get $T_{\mathscr{P},\phi}$ is bounded on $L^p\left(\mathbb{R}^{n+1}\right)$ for $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$ with abound on $\left\|T_{\mathscr{P},\phi}\right\|_{p,p}$ independent of the coefficients of the P_j's and ϕ. This completes the proof. $\square$

Proof of Theorem 1.7. We start by the case $n \geqslant 3$. Let

$$M = \max\{\phi, \deg(P_1), ..., \deg(P_d)\}.$$

Let $P_l, P_l^{(s)}, P^{(s)}$, and $\Phi_s, 1 \leqslant s \leqslant M$ and $1 \leqslant l \leqslant d$, be as in the proof of Theorem 1.2. We set $P_l^{(0)} = P^{(0)} = \Phi_0 = 0$. Set $\phi_s(t) = \sum_{j=0}^{s} b_j t^j$ with $b_0 = 0$. For $0 \leqslant s \leqslant M$ and $k \in \mathbb{Z}$, we define the measure $\sigma_{s,k}$ on $\mathbb{R}^{n+1}$ by

$$\int_{\mathbb{R}^{n+1}} F d\sigma_{s,k} = \int_{2^k \leqslant |y| < 2^{k+1}} F\left(\Phi_s(y), \phi_s(|y|)\right) \Omega(y') |y|^{-n} dy. \tag{52}$$

Set

$$\sigma_s^*(f) = \sup_{k \in \mathbb{Z}} \left| |\sigma_{s,k}| * f \right|. \tag{53}$$

Then it follows that

$$\mathscr{T}_{\Phi,\phi} f(x) = \sum_{k \in \mathbb{Z}} \sigma_{s,k} * f. \tag{54}$$

In order to conclude the proof, we only need to show that the measures $\sigma_{s,k}$ satisfy the assumptions (i) and (ii) in Lemma 2.1.

Notice that for $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$, we have

$$\Phi_s(y).\xi + \eta\phi_s(|y|) = \sum_{j=1}^{s} \left(\left(L_j(\xi) \cdot y'\right) + b_s\eta\right) |y|^j$$

where L_j is the linear transformation given in (29). Thus,

$$\left|\hat{\sigma}_{s,k}(\xi, \eta)\right| \leqslant \int_{\mathbb{S}^{n-1}} \left|I_{s,k}(\xi, \eta)\right| |\Omega(y')| d\sigma(y'),$$

where

$$I_{s,k}(\xi, \eta) = \int_1^2 \exp\left[-i\left(\Phi_s(2^k t y').\xi + \eta\phi_s\left(2^k t\right)\right)\right] \frac{dt}{t}. \tag{55}$$

By Lemma 2.3, we get

$$\left|I_{s,k}(\xi, \eta)\right| \leqslant C\left[2^{sk} |L_s(\xi)| \rho.y' + \delta\right]^{-\frac{1}{s}} \tag{56}$$

where

$$\rho = |L_s(\xi)|^{-1} L_s(\xi)$$

and

$$\delta = \min\left\{\left(|L_s(\xi)|^{-1} |b_s\eta|, 2\right) sgn(b_s\eta)\right\}.$$

By (56) and the observation $\left|I_{s,k}(\xi,\eta)\right| \leqslant 1$, we obtain

$$\left|I_{s,k}(\xi,\eta)\right| \leqslant C \frac{\left[\ln\left(4\left|\langle\rho.y'\rangle+\delta\right|^{-1}\right)\right]^{1+\alpha}}{\left[\ln\left(2^{sk}\left|L_s(\xi)\right|\right)\right]^{1+\alpha}},$$

whenever $2^{sk}\left|L_s(\xi)\right| \geqslant 2$. Therefore, by the fact that $\Omega \in W^0(n,1,\alpha)$, we immediately obtain

$$\begin{aligned}
&\left|\hat{\sigma}_{s,k}(\xi,\eta)\right| \\
&\leqslant C\left[\ln\left(2^{sk}\left|L_s(\xi)\right|\right)\right]^{-1-\alpha} \int_{\mathbb{S}^{n-1}} \left|\Omega(y')\right| \left[\ln\left(4\left|\langle\rho.y'\rangle+\delta\right|^{-1}\right)\right]^{1+\alpha} d\sigma(y') \\
&\leqslant C\left[\ln\left(2^{sk}\left|L_s(\xi)\right|\right)\right]^{-1-\alpha} \qquad (57)
\end{aligned}$$

provided that $2^{sk}\left|L_s(\xi)\right| \geqslant 2$.

On the other hand, it can be easily seen that

$$\left|\hat{\sigma}_{s,k}(\xi,\eta) - \hat{\sigma}_{s-1,k}(\xi,\eta)\right| \leqslant C\left|2^{sk}\left|L_s(\xi)\right|\right|. \qquad (58)$$

By (57), (58), the boundedness of the maximal functions σ_s^* on L^p for all $1<p<\infty$ (which follows by Lemma 2.6) and Lemma 2.1, the theorem is proved for the case $n \geqslant 3$.

Now, the proof of the case $n=2$ follows by minor modification of the corresponding proof of the case $n \geqslant 3$. In fact, by adapting the argument in ([7], p. 167,168), we can show that the estimate (57) holds provided that $\Omega \in F_\alpha\left(\mathbb{S}^{n-1}\right)$. □

5. Proofs of the result concerning convex functions

In the following, we give a sketch of the proof of Theorem 1.8.

Proof of Theorem 1.8. Let $\psi, \phi, \varphi(t)$, and Ω be as in the statement 1.8. For $k \in \mathbb{Z}$, define the measure σ_k, using Fourier transform, by

$$\hat{\sigma}_k(\xi,\eta) = \int_{2^k \leqslant |y| < 2^{k+1}} \Omega(y')\,|y|^{-n} e^{-i\left[\xi\cdot\psi(|y|)y' + \eta\phi(|y|)\right]} dy.$$

Then it follows that

$$\mathscr{T}_{\psi,\phi}(f)(x,x_{n+1}) = \sum_{k\in\mathbb{Z}} \sigma_k * f(x,x_{n+1}). \qquad (59)$$

By Lemma 2.8, we have

$$\left|\hat{\sigma}_k(\xi,\eta)\right| \leqslant \left[\log^+\left(\left|\psi(2^k)\xi\right|\right)\right]^{-1-\alpha}. \qquad (60)$$

On the other hand, by making use of the cancellation property of Ω on $\mathbb{S}^{n-1}$, we can show that

$$\left|\hat{\sigma}_k(\xi,\eta)\right| \leqslant \log^+\left|\psi(2^{k+1})\xi\right|. \tag{61}$$

Now, let σ^* be the corresponding maximal function

$$\sigma^* f(x,x_{n+1}) = \sup_{k\in\mathbb{Z}} \left|\left|\sigma_k\right| * f(x)\right|.$$

It can be easily seen that

$$\sigma^* f(x,x_{n+1}) \leqslant \int_{\mathbb{S}^{n-1}} \left|\Omega(y')\right| M_{\psi,\phi,y'}(f)(x,x_{n+1})\, d\sigma(y') \tag{62}$$

where

$$M_{\psi,\phi.y'}(f)(x,x_{n+1}) = \sup_k \left|\int_{2^k}^{2^{k+1}} f\left(x-\psi(t)y', x_{n+1}-\phi(t)\right)\frac{dt}{t}\right|. \tag{63}$$

In order to finish the proof, we only need to prove that σ^* is bounded on L^p for all $1<p<\infty$. In fact, by Lemma 2.9, we have

$$\left\|M_{\psi,\phi,y'}(f)\right\|_p \leqslant C_p \left\|f\right\|_p \tag{64}$$

for all $1<p<\infty$ with constant independent of y'. Thus, by (62), (64), and Minkowski's inequality, we get

$$\left\|\sigma^* f\right\|_p \leqslant C_p \left\|f\right\|_p \tag{65}$$

for all $1<p<\infty$. This completes the proof of the boundedness of σ^* and hence the proof of the theorem. □

6. Further results

In this section we highlight some of the results that can be obtained using the estimates obtained in the previous sections. Namely, we consider the truncated maximal operators corresponding to the operators stated in the paragraphs (I), (II), and (III) in the introduction section. The truncated maximal operators corresponding to the operators in (9), (11), and (17) are given, respectively by

$$(\mathscr{T}_{\Omega,\Phi})^*(f)(x) = \sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} \Omega\left(y'\right)|y|^{-n} f\left(x-\Phi(y)\right)dy\right|,$$

$$(\mathscr{T}_{\Phi,\phi} f)^*(f)(x,x_{n+1}) = \sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} f\left(x-\Phi(y), x_{n+1}-\phi(|y|)\right)\Omega\left(y'\right)|y|^{-n}dy\right|,$$

and

$$(\mathscr{T}_{\psi,\phi} f)^*(f)(x,x_{n+1}) = \sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} f\left(x-\psi(|y|)y', x_{n+1}-\phi(|y|)\right)\Omega\left(y'\right)|y|^{-n}dy\right|.$$

Our results concerning these operators are the following:

THEOREM 6.1. *Suppose that* $\Omega \in F_\alpha\left(\mathbb{S}^{n-1}\right)$ *for some* $\alpha > \frac{1}{2}$ *and satisfy (1). Suppose also that* $\Phi(y) = P(|y|) \otimes y'$ *is as in (10). Then* $(\mathscr{T}_{\Omega,\Phi})^*$ *is bounded in* $L^p\left(\mathbb{R}^d\right)$ *for* $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$. *The* L^p *bounds of* $(\mathscr{T}_{\Omega,\Phi})^*$ *are independent of the coefficients of the polynomial mappings* P_j, $1 \leqslant j \leqslant d$.

THEOREM 6.2. *Suppose that* $\Omega \in F_\alpha\left(\mathbb{S}^1\right)$ *for some* $\alpha > \frac{1}{2}$ *and satisfy (1). Suppose also that* $P_1(t) = P_2(t) = \ldots = P_d(t) = P(t)$ *and* $\varphi(y') = (\varphi_1(y'), \ldots \varphi_d(y'))$ *is real analytic on* $\mathbb{S}^1$. *Then* $(\mathscr{T}_{\Omega,\Phi})^*$ *is bounded on* $L^p\left(\mathbb{R}^d\right)$ *for* $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$.

THEOREM 6.3. *Let* $n \geqslant 2$, $\Phi = \mathscr{P}$ *be a real valued polynomial in* $n-$*variables, and* ϕ *be a real valued polynomial on* $\mathbb{R}$ *with* $\phi(0) = 0$. *If* $\Omega \in W^0(n, \alpha)$ *for some* $\alpha > \frac{1}{2}$, *then the operator* $(\mathscr{T}_{\Phi,\phi} f)^*$ *is bounded on* $L^p\left(\mathbb{R}^{n+1}\right)$ *for* $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$. *Moreover, the* L^p *bounds are independent of the coefficients of* ϕ *and* $\mathscr{P}$.

THEOREM 6.4. *Suppose that* $\Phi(y) = P(|y|) \otimes y'$ *where* $P(t) = (P_1(t), \ldots, P_d(t))$ *is a polynomial mapping. Suppose also that* ϕ *is a real valued polynomial on* $\mathbb{R}$.

(i) If $n = 2$, *and* $\Omega \in F_\alpha\left(S^{n-1}\right)$ *for* $\alpha > \frac{1}{2}$, *then* $(\mathscr{T}_{\Phi,\phi} f)^*$ *is bounded on* $L^p\left(\mathbb{R}^3\right)$ *for* $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$.

(ii) If $n \geqslant 3$, *and* $\Omega \in W^0(n, 1, \alpha)$, *then* $(\mathscr{T}_{\Phi,\phi} f)^*$ *is bounded on* $L^p\left(\mathbb{R}^{n+1}\right)$ *for* $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$.

In both (i) and (ii), the L^p *bounds on the operator norm are independent of the coefficient of* ϕ *and* P_j.

THEOREM 6.5. *Let* $\psi, \phi \in C^1[0, \infty)$ *be convex increasing,* $\psi(0) = \phi(0) = \phi'(0) = 0$, *and* $\psi'(0) \neq 0$. *Let* $\varphi(t) = \phi(\psi^{-1}(t))$ *and assume that* φ' *is convex and increasing. If* $\Omega \in F_\alpha\left(\mathbb{S}^{n-1}\right)$ *for some* $\alpha > \frac{1}{2}$ *and satisfies (1), then* $(\mathscr{T}_{\psi,\phi} f)^*$ *is bounded on* $L^p\left(\mathbb{R}^{n+1}\right)$ *for all* $p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$.

It should be noticed here that proofs of the above results can be constructed by using the same estimates obtained in the proofs of the corresponding results in Sections 3, 4, and 5, and adapting a similar argument as in [2] (see also [12]). We omit the details.

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