Journal of Mathematical Inequalities
Volume 8, Number 4 (2014), 823–837 doi:10.7153/jmi-08-62

NONLINEAR DIFFERENCE INEQUALITIES WITH AN INFINITE SUMMATION AND THEIR APPLICATIONS

YING LIANG, XIAOPEI LI, HAISHAN DONG AND SHUISHENG CHEN

(*Communicated by A. Gilányi*)

Abstract. Some new infinite difference inequalities involving two independent variables with more than one nonlinear terms are established. These inequalities provide a handy tool in deriving the boundedness and uniqueness of solutions of certain nonlinear infinite difference equations.

1. Introduction

In this paper, we study a class of nonlinear infinite difference inequalities with the following form

$$u(m,n) \leqslant a(m,n) + \sum_{i=1}^{k} \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} f_i(m,n,s,t)\upsilon_i(u(s,t)), \quad m,n \in \mathbf{N}_0, \tag{1.1}$$

where $a(m,n)$, $f_i(m,n,s,t)$, $\upsilon_i(u(s,t))$ are real functions and $m,n,s,t \in N_0 = \{0,1,2,...\}$.

In the research of solutions of certain difference equations, if the solutions are unknown, then it is necessary to study their qualitative and quantitative properties such as existence, uniqueness, boundedness, stability and continuous dependence on initial data and so on. The Gronwall-Bellman inequality and its various generalizations that provide explicit bounds play a fundamental role in the research of this domain. Many such generalized inequalities have been established in the literature (for example, see [1, 4, 7, 8, 9, 11, 12, 15, 18] for continuous cases, and [3, 5, 6, 17] for discrete cases, also see the books [2, 14]). As we know, the theory of difference equations is very important to the study of dynamics of physical systems. In the study of many finite difference and sum-difference equations, finite difference inequalities which provide explicit estimates on unknown functions have become very effective and powerful tools for studying the qualitative behaviors of their solutions. In the past few years a large number of new finite difference inequalities have been discovered.

Mathematics subject classification (2010): 26D15, 26D20, 39A10.
Keywords and phrases: Infinite difference inequality, nonlinear, boundedness, uniqueness.
This work was supported by National Natural Science Foundation of China (No. 11371314), Guangdong Natural Science Foundation (No. S2013010015957), and the Natural Science Funds of Zhanjiang Normal University (No. LZL1101).

Paper JMI-08-62

For Eq. (1.1), if we take $a(m,n) = c$ ($c > 0$ is a constant), $i = 1$, $f_i(m,n,s,t) = f(t)$, $\upsilon_i(u(s,t)) = \upsilon_i(u(t))$, then Eq. (1.1) becomes the variation of [10] considered by Mate and Nevai

$$u(n) \leqslant c + \sum_{t=n+1}^{\infty} p(t)u(t). \tag{1.2}$$

If we take $i = 1$, $f_i(m,n,s,t) = f(s,t)$, $\upsilon_i(u(s,t)) = \upsilon(s,t)$, then Eq. (1.1) is reduced to the following inequalities [13]

$$u(m,n) \leqslant a(m,n) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} b(s,t)u(s,t), \quad m,n \in \mathbf{N}_0. \tag{1.3}$$

If we take k nonlinear finite terms of Eq. (1.1), it is as follows

$$u(m,n) \leqslant a(m,n) + \sum_{i=1}^{k} \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} f_i(m,n,s,t)\upsilon_i(u(s,t)), \quad m,n \in \mathbf{N}_0, \tag{1.4}$$

which was discussed by Deng [6] under the condition that $\frac{\upsilon_{i+1}}{\upsilon_i}$ is increasing.

In order to fulfill the analysis of qualitative and quantitative properties of the solutions of difference equations, the results provided by the earlier inequalities are inadequate. Thus, it is necessary to seek some new discrete inequalities so as to obtain desired results.

In this paper, we present a more general discrete inequality (1.1), which has two independent variables with more than one nonlinear terms. Moreover, we do not require that $\frac{\upsilon_{i+1}}{\upsilon_i}$ is increasing.

This paper is organized as follows. In section 2, we give some assumptions and lemmas. In section 3, using the mathematical induction, we discuss the upper bound of the unknown function $u(m,n)$ of (1.1). In section 4, we present some corollaries. In section 5, we give an example to show the boundedness and uniqueness of solutions of a certain nonlinear infinite difference equation.

2. Preliminaries

In this section we make some assumptions and lemmas which will be used later. Assume that

(A_1) $a(m,n)$ is nonnegative for $m,n \in \mathbf{N}_0$ and $a(0,\infty) > 0$; $\Delta_1 \tilde{a}(m,n)$ is nonnegative and decreasing in n where $\tilde{a}(m,n) = \max_{0 \leqslant \tau \leqslant m, \eta \geqslant n, \tau,\eta \in \mathbf{N}_0} a(\tau,\eta)$ and $\Delta_1 \tilde{a}(m,n) = \tilde{a}(m+1,n) - \tilde{a}(m,n)$;

(A_2) $f_i(m,n,s,t)$ $(i = 1,\cdots,k)$ is nonnegative for $m,n,s,t \in \mathbf{N}_0$;

(A_3) υ_i $(i = 1,\cdots,k)$ is continuous on $[0,\infty)$ and positive on $(0,\infty)$.

Define that

$$\begin{aligned}
&\tilde{f}_i(m,n,s,t) \triangleq \max_{0\leqslant\tau\leqslant m,\eta\geqslant n,\tau,\eta\in\mathbf{N}_0} f_i(\tau,\eta,s,t),\\
&w_1(s) \triangleq \max_{0\leqslant\tau\leqslant s} \upsilon_1(\tau), \quad w_i(s) \triangleq \max_{0\leqslant\tau\leqslant s}\{\frac{\upsilon_i(\tau)}{w_{i-1}(\tau)}\}w_{i-1}(s),\\
&W_i(u) \triangleq \int_{u_i}^{u}\frac{dz}{w_i(z)}, \quad \phi_i(u) \triangleq \frac{w_i(u)}{w_{i-1}(u)}, \qquad i=1,2,\cdots,k-1, \qquad (2.5)
\end{aligned}$$

where u_i is a given positive constant, $\phi_1(u)=w_1(u)$, $W_0=I$ (Identity), and $r_k(m,n,s,t)$ are defined as

$$\begin{aligned}
r_1(m,n,s,t) &\triangleq \tilde{a}(s,t),\\
r_{i+1}(m,n,s,t) &\triangleq W_i(r_1(m,n,0,t)) + \sum_{\tau=0}^{s-1}\sum_{\eta=t+1}^{\infty}\tilde{f}_i(m,n,\tau,\eta)\\
&\quad + \sum_{\tau=0}^{s-1}\frac{\Delta_3 r_i(m,n,\tau,t)}{\phi_i(W_{i-1}^{-1}(r_i(0,\infty,\tau,\infty)))}, \quad i=1,\cdots,k-1,\\
\Delta_3 r_i(m,n,s,t) &\triangleq r_i(m,n,s+1,t) - r_i(m,n,s,t)\\
&\triangleq \sum_{\eta=t+1}^{\infty}\tilde{f}_i(m,n,s,\eta) + \frac{\Delta_3 r_i(m,n,s,t)}{\phi_i(W_{i-1}^{-1}(r_i(0,\infty,s,\infty)))}, \quad i=2,\cdots,k. \qquad (2.6)
\end{aligned}$$

LEMMA 2.1. *w_i $(i=1,\cdots,k)$ is increasing and satisfies the relationship $w_1 \propto w_2 \propto \cdots \propto w_k$ (See [15]) where $w_i \propto w_{i+1}$ means that $\frac{w_{i+1}}{w_i}$ is increasing on $(0,\infty)$; then, $\phi_i(u)$ is continuous and increasing in its corresponding domain and is positive; W_i is strictly increasing so its inverse W_i^{-1} is well defined, continuous and increasing in its corresponding domain; $\tilde{a}(m,n)$ and $\tilde{f}_i(m,n,s,t)$ are nonnegative, increasing in m and decreasing in n, $\tilde{a}(m,n)\geqslant a(m,n)$ and $\tilde{f}_i(m,n,s,t)\geqslant f_i(m,n,s,t)$ where $i=1,\cdots,k$; $a(0,\infty)>0$ in (A_1) implies that $\tilde{a}(m,n)>0$ for all $m,n\in\mathbf{N}_0$.*

LEMMA 2.2. *$\Delta_3 r_i(m,n,s,t)$ is nonnegative, increasing in m, decreasing in n and t, and $r_i(m,n,s,t)$ is nonnegative, increasing in m, decreasing in n and t where $i=1,\cdots,k$.*

Proof. By the definitions of $\tilde{a}(m,n)$ and $\tilde{f}_i(m,n,s,t)$ and the fact that $r_1(0,\infty,s,\infty) = \tilde{a}(s,\infty)>0$, we have

$$\begin{aligned}
&\Delta_3 r_1(m+1,n,s,t) - \Delta_3 r_1(m,n,s,t) = 0,\\
&\Delta_3 r_2(m+1,n,s,t) - \Delta_3 r_2(m,n,s,t) = \sum_{j=t+1}^{\infty}\tilde{f}_1(m+1,n,s,j) - \sum_{j=t+1}^{\infty}\tilde{f}_1(m,n,s,j)\\
&\qquad\qquad + \frac{\Delta_3 r_1(m+1,n,s,t) - \Delta_3 r_1(m,n,s,t)}{w_1(r_1(0,\infty,s,\infty))} \geqslant 0,
\end{aligned}$$

so we know that $\Delta_3 r_1(m,n,s,t)$ and $\Delta_3 r_2(m,n,s,t)$ are increasing in m. Assume that $\Delta_3 r_l(m,n,s,t)$ is increasing in m. Then we have

$$\begin{aligned}
&\Delta_3 r_{l+1}(m+1,n,s,t)-\Delta_3 r_{l+1}(m,n,s,t)\\
&=\sum_{j=t+1}^{\infty}\tilde{f}_l(m+1,n,s,j)-\sum_{j=t+1}^{\infty}\tilde{f}_l(m,n,s,j)+\frac{\Delta_3 r_l(m+1,n,s,t)-\Delta_3 r_l(m,n,s,t)}{\phi_l(W_{l-1}^{-1}(r_l(0,\infty,s,\infty)))}\geqslant 0,
\end{aligned}$$

which means that $\Delta_3 r_{l+1}(m,n,s,t)$ is increasing in m. By the mathematical induction, $\Delta_3 r_i(m,n,s,t)$ is increasing in m.

Similarly, since $\tilde{f}_1(m,n,s,t)$ is decreasing in n, we can have

$$\begin{aligned}
&\Delta_3 r_1(m,n+1,s,t)-\Delta_3 r_1(m,n,s,t)=0,\\
&\Delta_3 r_2(m,n+1,s,t)-\Delta_3 r_2(m,n,s,t)=\sum_{j=t+1}^{\infty}\left(\tilde{f}_1(m,n+1,s,j)-\tilde{f}_1(m,n,s,j)\right)\\
&\qquad\qquad+\frac{\Delta_3 r_1(m,n+1,s,t)-\Delta_3 r_1(m,n,s,t)}{w_1(r_1(0,\infty,s,\infty))}\leqslant 0.
\end{aligned}$$

Thus, $\Delta_3 r_1(m,n,s,t)$ and $\Delta_3 r_2(m,n,s,t)$ are decreasing in n. Assume that $\Delta_3 r_l(m,n,s,t)$ is decreasing in n. Then

$$\begin{aligned}
\Delta_3 r_{l+1}(m,n+1,s,t)-\Delta_3 r_{l+1}(m,n,s,t)&=\sum_{j=t+1}^{\infty}\left(\tilde{f}_l(m,n+1,s,j)-\tilde{f}_l(m,n,s,j)\right)\\
&\quad+\frac{\Delta_3 r_l(m,n+1,s,t)-\Delta_3 r_l(m,n,s,t)}{\phi_l(W_{l-1}^{-1}(r_l(0,\infty,s,\infty)))}\leqslant 0,
\end{aligned}$$

which implies that $\Delta_3 r_i(m,n,s,t)$ is decreasing in n. It is easy to check that $\Delta_3 r_i(m,n,s,t)$ is nonnegative and decreasing in t by the mathematical induction again. Thus, $r_i(m,n,s,t)$ is nonnegative, increasing in m, and decreasing in n and t. □

3. Main results

THEOREM 3.1. *Suppose that* (A_1)–(A_3) *hold and* $u(m,n)$ *is a nonnegative function for* $m,n\in\mathbf{N}_0$ *satisfying* (1.1). *Then*

$$\begin{aligned}
u(m,n)\leqslant W_k^{-1}\Big[&W_k(\tilde{a}(0,n))+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\tilde{f}_k(m,n,s,t)\\
&+\sum_{s=0}^{m-1}\frac{\Delta_3 r_k(m,n,s,n)}{\phi_k(W_{k-1}^{-1}(r_k(0,\infty,s,\infty)))}\Big],\quad 0\leqslant m\leqslant M_1,\quad n\geqslant N_1,
\end{aligned}\tag{3.1}$$

where M_1 *and* N_1 *are positive integers satisfying*

$$\begin{aligned}
W_i(\tilde{a}(0,N_1))+\sum_{s=0}^{M_1-1}\sum_{t=N_1+1}^{\infty}\tilde{f}_i(M_1,N_1,s,t)+\sum_{s=0}^{M_1-1}\frac{\Delta_3 r_i(M_1,N_1,s,N_1)}{\phi_i(W_{i-1}^{-1}(r_i(0,\infty,s,\infty)))}\\
\leqslant\int_{u_i}^{\infty}\frac{dz}{w_i(z)},\quad i=1,\cdots,k.
\end{aligned}\tag{3.2}$$

Proof. Consider an auxiliary inequality

$$u(m,n) \leqslant r_1(\tilde{m},\tilde{n},m,n) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_i(\tilde{m},\tilde{n},s,t)w_i(u(s,t)) \tag{3.3}$$

for $0 \leqslant m \leqslant \tilde{m}$ and $n \geqslant \tilde{n}$ where the arbitrary positive integers $\tilde{m}$ and $\tilde{n}$ satisfy $\tilde{m} \leqslant M_1$ and $\tilde{n} \geqslant N_1$. Claim that $u(m,n)$ in (3.3) satisfies

$$\begin{aligned} u(m,n) \leqslant W_k^{-1}\Big[& W_k(r_1(\tilde{m},\tilde{n},0,n)) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_k(\tilde{m},\tilde{n},s,t) \\ & + \sum_{s=0}^{m-1} \frac{\Delta_3 r_k(\tilde{m},\tilde{n},s,n)}{\phi_k(W_{k-1}^{-1}(r_k(0,\infty,s,\infty)))}\Big] \end{aligned} \tag{3.4}$$

for $0 \leqslant m \leqslant \min\{\tilde{m}, M_2\}$ and $n \geqslant \max\{\tilde{n}, N_2\}$ where M_2 and N_2 are positive integers satisfying

$$\begin{aligned} & W_i(r_1(\tilde{m},\tilde{n},0,N_2)) + \sum_{s=0}^{M_2-1}\sum_{t=N_2+1}^{\infty} \tilde{f}_i(\tilde{m},\tilde{n},s,t) + \sum_{s=0}^{M_2-1} \frac{\Delta_3 r_i(\tilde{m},\tilde{n},s,N_2)}{\phi_i(W_{i-1}^{-1}(r_i(0,\infty,s,\infty)))} \\ & \quad \leqslant \int_{u_i}^{\infty} \frac{dz}{w_i(z)}, \quad i = 1,\cdots,k. \end{aligned} \tag{3.5}$$

Since $r_i(\tilde{m},\tilde{n},m,n)$, $\Delta_3 r_i(\tilde{m},\tilde{n},m,n)$ and $\tilde{f}_i(\tilde{m},\tilde{n},m,n)$ are increasing in $\tilde{m}$ and decreasing in $\tilde{n}$ by Lemma 2.1 and Lemma 2.2, M_2 gets smaller and N_2 gets bigger as $\tilde{m}$ is chosen bigger and $\tilde{n}$ is chosen smaller. In particular, M_2 and N_2 satisfy the same (3.2) as M_1 and N_1 for $\tilde{m} = M_1$ and $\tilde{n} = N_1$. Thus, we may choose $M_1 \leqslant M_2$ and $N_1 \geqslant N_2$ so that $0 \leqslant m \leqslant \min\{\tilde{m}, M_2\}$ and $n \geqslant \max\{\tilde{n}, N_2\}$ are reduced to $0 \leqslant m \leqslant \tilde{m}$ and $n \geqslant \tilde{n}$.

Now we prove (3.4) into two steps by using the mathematical induction.

Step 1. $k = 1$.

For $k = 1$, we have

$$u(m,n) \leqslant r_1(\tilde{m},\tilde{n},m,n) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},s,t)w_1(u(s,t)) \tag{3.6}$$

and let $z(m,n) = \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},s,t)w_1(u(s,t))$ and $z(0,n) = 0$. Thus, $z(m,n)$ is nonnegative, increasing in m and decreasing in n. Hence (3.6) is equivalent to $u(m,n) \leqslant r_1(\tilde{m},\tilde{n},m,n) + z(m,n)$ for $0 \leqslant m \leqslant \tilde{m}, n \geqslant \tilde{n}$ and

$$\begin{aligned} \Delta_1 z(m,n) &= \sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},m,t)w_1(u(m,t)) \\ &\leqslant \sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},m,t)w_1(r_1(\tilde{m},\tilde{n},m,t) + z(m,t)) \\ &\leqslant w_1(r_1(\tilde{m},\tilde{n},m,n) + z(m,n)) \sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},m,t). \end{aligned}$$

Since w_1 is increasing and $r_1(\tilde{m},\tilde{n},m,n) > 0$, we have

$$\begin{aligned}\frac{\Delta_1 z(m,n)+\Delta_3 r_1(\tilde{m},\tilde{n},m,n)}{w_1(z(m,n)+r_1(\tilde{m},\tilde{n},m,n))} &\leqslant \sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},m,t)+\frac{\Delta_3 r_1(\tilde{m},\tilde{n},m,n)}{w_1(z(m,n)+r_1(\tilde{m},\tilde{n},m,n))}\\ &\leqslant \sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},m,t)+\frac{\Delta_3 r_1(\tilde{m},\tilde{n},m,n)}{w_1(r_1(0,\infty,m,\infty))}. \end{aligned} \tag{3.7}$$

Note that

$$\begin{aligned}\int_{z(m,n)+r_1(\tilde{m},\tilde{n},m,n)}^{z(m+1,n)+r_1(\tilde{m},\tilde{n},m+1,n)} \frac{d\tau}{w_1(\tau)} &\leqslant \int_{z(m,n)+r_1(\tilde{m},\tilde{n},m,n)}^{z(m+1,n)+r_1(\tilde{m},\tilde{n},m+1,n)} \frac{d\tau}{w_1(z(m,n)+r_1(\tilde{m},\tilde{n},m,n))}\\ &\leqslant \frac{\Delta_1 z(m,n)+\Delta_3 r_1(\tilde{m},\tilde{n},m,n)}{w_1(z(m,n)+r_1(\tilde{m},\tilde{n},m,n))},\end{aligned}$$

which implies together with (3.7)

$$\int_{z(m,n)+r_1(\tilde{m},\tilde{n},m,n)}^{z(m+1,n)+r_1(\tilde{m},\tilde{n},m+1,n)} \frac{d\tau}{w_1(\tau)} \leqslant \sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},m,t)+\frac{\Delta_3 r_1(\tilde{m},\tilde{n},m,n)}{w_1(r_1(0,\infty,m,\infty))}.$$

Therefore,

$$\begin{aligned}\int_{z(0,n)+r_1(\tilde{m},\tilde{n},0,n)}^{z(m,n)+r_1(\tilde{m},\tilde{n},m,n)} \frac{d\tau}{w_1(\tau)} &= \sum_{s=0}^{m-1}\int_{z(s,n)+r_1(\tilde{m},\tilde{n},s,n)}^{z(s+1,n)+r_1(\tilde{m},\tilde{n},s+1,n)} \frac{d\tau}{w_1(\tau)}\\ &\leqslant \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},s,t)+\sum_{s=0}^{m-1}\frac{\Delta_3 r_1(\tilde{m},\tilde{n},s,n)}{w_1(r_1(0,\infty,s,\infty))}.\end{aligned}$$

The definition of W_1 in (2.5) and $z(0,n)=0$ yield

$$\begin{aligned}W_1(z(m,n)+r_1(\tilde{m},\tilde{n},m,n)) \leqslant W_1(r_1(\tilde{m},\tilde{n},0,n)) &+ \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},s,t)\\ &+\sum_{s=0}^{m-1}\frac{\Delta_3 r_1(\tilde{m},\tilde{n},s,n)}{w_1(r_1(0,\infty,s,\infty))}, \quad 0\leqslant m\leqslant \tilde{m},\ n\geqslant \tilde{n}. \end{aligned}\tag{3.8}$$

(3.5) shows that the right side of (3.8) is in the domain of W_1^{-1} for all $0\leqslant m\leqslant \tilde{m}$ and $n\geqslant \tilde{n}$. Thus,

$$\begin{aligned}u(m,n) &\leqslant z(m,n)+r_1(\tilde{m},\tilde{n},m,n)\\ &\leqslant W_1^{-1}\Big[W_1(r_1(\tilde{m},\tilde{n},0,n))+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},s,t)+\sum_{s=0}^{m-1}\frac{\Delta_3 r_1(\tilde{m},\tilde{n},s,n)}{w_1(r_1(0,\infty,s,\infty))}\Big]\end{aligned}\tag{3.9}$$

for $0\leqslant m\leqslant \tilde{m}$ and $n\geqslant \tilde{n}$. Hence (3.4) holds for $k=1$.

Step 2. $k+1$.

Suppose that (3.4) is true for k. Consider

$$u(m,n) \leqslant r_1(\tilde{m},\tilde{n},m,n) + \sum_{i=1}^{k+1}\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_i(\tilde{m},\tilde{n},s,t)w_i(u(s,t))$$

for $0 \leqslant m \leqslant \tilde{m}$ and $n \geqslant \tilde{n}$. Let $z(m,n) = \sum_{i=1}^{k+1}\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_i(\tilde{m},\tilde{n},s,t)w_i(u(s,t))$ and $z(0,n) = 0$. It is clear that $z(m,n)$ is nonnegative, increasing in m, decreasing in n and satisfies $u(m,n) \leqslant r_1(\tilde{m},\tilde{n},m,n) + z(m,n)$ for $0 \leqslant m \leqslant \tilde{m}$ and $n \geqslant \tilde{n}$. Obviously,

$$\begin{aligned}
\Delta_1 z(m,n) &= \sum_{i=1}^{k+1}\sum_{t=n+1}^{\infty} \tilde{f}_i(\tilde{m},\tilde{n},m,t)w_i(u(m,t)) \\
&\leqslant \sum_{i=1}^{k+1}\sum_{t=n+1}^{\infty} \tilde{f}_i(\tilde{m},\tilde{n},m,t)w_i(r_1(\tilde{m},\tilde{n},m,t) + z(m,t)).
\end{aligned}$$

We have

$$\begin{aligned}
\frac{\Delta_1 z(m,n) + \Delta_3 r_1(\tilde{m},\tilde{n},m,n)}{w_1(z(m,n) + r_1(\tilde{m},\tilde{n},m,n))} &\leqslant \frac{\sum_{i=1}^{k+1}\sum_{t=n+1}^{\infty} \tilde{f}_i(\tilde{m},\tilde{n},m,t)w_i(z(m,t) + r_1(\tilde{m},\tilde{n},m,t))}{w_1(z(m,n) + r_1(\tilde{m},\tilde{n},m,n))} \\
&\quad + \frac{\Delta_3 r_1(\tilde{m},\tilde{n},m,n)}{w_1(r_1(\tilde{m},\tilde{n},m,n))} \\
&\leqslant \sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},m,t) + \frac{\Delta_3 r_1(\tilde{m},\tilde{n},m,n)}{w_1(r_1(0,\infty,m,\infty))} \\
&\quad + \sum_{i=2}^{k+1}\sum_{t=n+1}^{\infty} \tilde{f}_i(\tilde{m},\tilde{n},m,t)\frac{w_i(z(m,t) + r_1(\tilde{m},\tilde{n},m,t))}{w_1(z(m,t) + r_1(\tilde{m},\tilde{n},m,t))} \\
&\leqslant \sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},m,t) + \frac{\Delta_3 r_1(\tilde{m},\tilde{n},m,n)}{w_1(r_1(0,\infty,m,\infty))} \\
&\quad + \sum_{i=1}^{k}\sum_{t=n+1}^{\infty} \tilde{f}_{i+1}(\tilde{m},\tilde{n},m,t)\tilde{\phi}_{i+1}(z(m,t) + r_1(\tilde{m},\tilde{n},m,t))
\end{aligned}$$

for $0 \leqslant m \leqslant \tilde{m}$ and $n \geqslant \tilde{n}$ where $\tilde{\phi}_{i+1}(u) = \frac{w_{i+1}(u)}{w_1(u)}$ for $i = 1,\cdots,k$, which implies

$$\begin{aligned}
\int_{z(m,n)+r_1(\tilde{m},\tilde{n},m,n)}^{z(m+1,n)+r_1(\tilde{m},\tilde{n},m+1,n)} \frac{d\tau}{w_1(\tau)} &\leqslant \sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},m,t) + \frac{\Delta_3 r_1(\tilde{m},\tilde{n},m,n)}{w_1(r_1(0,\infty,m,\infty))} \\
&\quad + \sum_{i=1}^{k}\sum_{t=n+1}^{\infty} \tilde{f}_{i+1}(\tilde{m},\tilde{n},m,t)\tilde{\phi}_{i+1}(z(m,t) + r_1(\tilde{m},\tilde{n},m,t)).
\end{aligned}$$

Thus, we obtain

$$\int_{z(0,n)+r_1(\tilde{m},\tilde{n},0,n)}^{z(m,n)+r_1(\tilde{m},\tilde{n},m,n)} \frac{d\tau}{w_1(\tau)} \leqslant \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},s,t) + \sum_{s=0}^{m-1} \frac{\Delta_3 r_1(\tilde{m},\tilde{n},s,n)}{w_1(r_1(0,\infty,s,\infty))}$$
$$+ \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_{i+1}(\tilde{m},\tilde{n},s,t)\tilde{\phi}_{i+1}(z(s,t)+r_1(\tilde{m},\tilde{n},s,t)),$$

which yields

$$W_1(z(m,n)+r_1(\tilde{m},\tilde{n},m,n)) \leqslant W_1(r_1(\tilde{m},\tilde{n},0,n)) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},s,t)$$
$$+ \sum_{s=0}^{m-1} \frac{\Delta_3 r_1(\tilde{m},\tilde{n},s,n)}{w_1(r_1(0,\infty,s,\infty))}$$
$$+ \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_{i+1}(\tilde{m},\tilde{n},s,t)\tilde{\phi}_{i+1}(z(s,t)+r_1(\tilde{m},\tilde{n},s,t)),$$

or equivalently

$$\xi(m,n) \leqslant c_1(\tilde{m},\tilde{n},m,n) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_{i+1}(\tilde{m},\tilde{n},s,t)\tilde{\phi}_{i+1}(W_1^{-1}(\xi(s,t)))$$

for $0 \leqslant m \leqslant \tilde{m}$ and $n \geqslant \tilde{n}$ the same as (3.3) for k where

$$\xi(m,n) = W_1(z(m,n)+r_1(\tilde{m},\tilde{n},m,n)),$$
$$c_1(\tilde{m},\tilde{n},m,n) = W_1(r_1(\tilde{m},\tilde{n},0,n)) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_1(\tilde{m},\tilde{n},s,t) + \sum_{s=0}^{m-1} \frac{\Delta_3 r_1(\tilde{m},\tilde{n},s,n)}{w_1(r_1(0,\infty,s,\infty))}.$$

From the assumption (C_3) and the definition of w_i, $\tilde{\phi}_{i+1}(W_1^{-1})$ $(i=1,\cdots,k)$ is continuous and increasing on $[0,\infty)$ and is positive on $(0,\infty)$ since W_1^{-1} is continuous and increasing on $[0,\infty)$. Moreover, $\tilde{\phi}_2(W_1^{-1}) \propto \tilde{\phi}_3(W_1^{-1}) \propto \cdots \propto \tilde{\phi}_{k+1}(W_1^{-1})$. By the inductive assumption, we have

$$\xi(m,n) \leqslant \Phi_{k+1}^{-1}\Big[\Phi_{k+1}(c_1(\tilde{m},\tilde{n},0,n)) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_{k+1}(\tilde{m},\tilde{n},s,t)$$
$$+ \sum_{s=0}^{m-1} \frac{\Delta_3 c_k(\tilde{m},\tilde{n},s,n)}{\psi_{k+1}(\Phi_k^{-1}(c_k(0,\infty,s,\infty)))}\Big] \tag{3.10}$$

for $0 \leqslant m \leqslant \min\{\tilde{n}, M_3\}$ and $n \geqslant \max\{\tilde{n}, N_3\}$ where $\Phi_{i+1}(u) = \int_{\tilde{u}_{i+1}}^{u} \frac{dz}{\tilde{\phi}_{i+1}(W_1^{-1}(z))}$, $u>0$, $\Phi_1 = I$ (Identity), $\tilde{u}_{i+1} = W_1(u_{i+1})$, Φ_{i+1}^{-1} is the inverse of Φ_{i+1}, $\psi_{i+1}(u) = \frac{\tilde{\phi}_{i+1}(W_1^{-1}(u))}{\tilde{\phi}_i(W_1^{-1}(u))} = \frac{w_{i+1}(W_1^{-1}(u))}{w_i(W_1^{-1}(u))}$, $i=1,\cdots,k$,

$$c_{i+1}(\tilde{m},\tilde{n},m,n) = \Phi_{i+1}(c_1(\tilde{m},\tilde{n},0,n)) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \tilde{f}_{i+1}(\tilde{m},\tilde{n},s,t)$$

$$+\sum_{s=0}^{m-1}\frac{\Delta_3 c_i(\tilde{m},\tilde{n},s,n)}{\psi_{i+1}(\Phi_i^{-1}(c_i(0,\infty,s,\infty)))},\quad i=1,\cdots,k-1,$$

and M_3 and N_3 are positive integers satisfying

$$\Phi_{i+1}(c_1(\tilde{m},\tilde{n},0,N_3))+\sum_{s=0}^{M_3-1}\sum_{t=N_3+1}^{\infty}\tilde{f}_{i+1}(\tilde{m},\tilde{n},s,t)+\sum_{s=0}^{M_3-1}\frac{\Delta_3 c_i(\tilde{m},\tilde{n},s,N_3)}{\psi_{i+1}(\Phi_i^{-1}(c_i(0,\infty,s,\infty)))}$$
$$\leqslant\int_{\tilde{u}_{i+1}}^{W_1(\infty)}\frac{dz}{\tilde{\phi}_{i+1}(W_1^{-1}(z))},\quad i=1,\cdots,k. \tag{3.11}$$

Note that $c_1(\tilde{m},\tilde{n},0,n)=W_1(r_1(\tilde{m},\tilde{n},0,n))$,

$$\begin{aligned}\Phi_i(u)&=\int_{\tilde{u}_i}^{u}\frac{dz}{\tilde{\phi}_i(W_1^{-1}(z))}=\int_{W_1(u_i)}^{u}\frac{w_1(W_1^{-1}(z))dz}{w_i(W_1^{-1}(z))}=\int_{u_i}^{W_1^{-1}(u)}\frac{dz}{w_i(z)}\\&=W_i\circ W_1^{-1}(u),\qquad i=2,\cdots,k+1\end{aligned}$$

and

$$\begin{aligned}\psi_{i+1}(\Phi_i^{-1}(u))&=\frac{w_{i+1}(W_1^{-1}(\Phi_i^{-1}(u)))}{w_i(W_1^{-1}(\Phi_i^{-1}(u)))}=\frac{w_{i+1}(W_1^{-1}(W_1(W_i^{-1}(u))))}{w_i(W_1^{-1}(W_1(W_i^{-1}(u))))}\\&=\frac{w_{i+1}(W_i^{-1}(u))}{w_i(W_i^{-1}(u))}=\phi_{i+1}(W_i^{-1}(u)),\qquad i=1,\cdots,k.\end{aligned}$$

We have from (3.10) that

$$\begin{aligned}u(m,n)&\leqslant r_1(\tilde{m},\tilde{n},m,n)+z(m,n)=W_1^{-1}(\xi(m,n))\\&\leqslant W_{k+1}^{-1}\Big[W_{k+1}(W_1^{-1}(c_1(\tilde{m},\tilde{n},0,n)))+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\tilde{f}_{k+1}(\tilde{m},\tilde{n},s,t)\\&\quad+\sum_{s=0}^{m-1}\frac{\Delta_3 c_k(\tilde{m},\tilde{n},s,n)}{\phi_{k+1}(W_k^{-1}(c_k(0,\infty,s,\infty)))}\Big]\\&\leqslant W_{k+1}^{-1}\Big[W_{k+1}(r_1(\tilde{m},\tilde{n},0,n))+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\tilde{f}_{k+1}(\tilde{m},\tilde{n},s,t)\\&\quad+\sum_{s=0}^{m-1}\frac{\Delta_3 c_k(\tilde{m},\tilde{n},s,n)}{\phi_{k+1}(W_k^{-1}(c_k(0,\infty,s,\infty)))}\Big]\end{aligned} \tag{3.12}$$

for $0\leqslant m\leqslant\min\{\tilde{n},M_3\}$ and $n\geqslant\max\{\tilde{n},N_3\}$.

Now we prove that $c_i(\tilde{m},\tilde{n},m,n)=r_{i+1}(\tilde{m},\tilde{n},m,n)$ by the mathematical induction again.

It is clear that $c_1(\tilde{m},\tilde{n},m,n)=r_2(\tilde{m},\tilde{n},m,n)$. Suppose that it is true for $i=l$. For $i=l+1$, we have

$$\begin{aligned}&c_{l+1}(\tilde{m},\tilde{n},m,n)\\&=\Phi_{l+1}(c_1(\tilde{m},\tilde{n},0,n))+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\tilde{f}_{l+1}(\tilde{m},\tilde{n},s,t)+\sum_{s=0}^{m-1}\frac{\Delta_3 c_l(\tilde{m},\tilde{n},s,n)}{\psi_{l+1}(\Phi_l^{-1}(c_l(0,\infty,s,\infty)))}\end{aligned}$$

$$= W_{l+1}(r_1(\tilde m,\tilde n,0,n)) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\tilde f_{l+1}(\tilde m,\tilde n,s,t) + \sum_{s=0}^{m-1}\frac{\Delta_3 r_{l+1}(\tilde m,\tilde n,s,n)}{\phi_{l+1}(W_l^{-1}(r_{l+1}(0,\infty,s,\infty)))}$$
$$= r_{l+2}(\tilde m,\tilde n,m,n),$$

which implies that it is true for $i=l+1$. Therefore, $c_i(\tilde m,\tilde n,m,n)=r_{i+1}(\tilde m,\tilde n,m,n)$ for $i=1,\cdots,k$.

Thus, (3.11) becomes

$$W_{i+1}(r_1(\tilde m,\tilde n,0,N_3)) + \sum_{s=0}^{M_3-1}\sum_{t=N_3+1}^{\infty}\tilde f_{i+1}(\tilde m,\tilde n,s,t) + \sum_{s=0}^{M_3-1}\frac{\Delta_3 r_{i+1}(\tilde m,\tilde n,s,N_3)}{\phi_{i+1}(W_i^{-1}(r_{i+1}(0,\infty,s,\infty)))}$$
$$\leqslant \int_{\tilde u_{i+1}}^{W_1(\infty)}\frac{dz}{\tilde\phi_{i+1}(W_1^{-1}(z))} = \int_{\tilde u_{i+1}}^{W_1(\infty)}\frac{w_1(W_1^{-1}(z))}{w_{i+1}(W_1^{-1}(z))}dz = \int_{u_{i+1}}^{\infty}\frac{dz}{w_{i+1}(z)} \tag{3.13}$$

for $i=1,\cdots,k$. Again we may choose $M_3=M_2$ and $N_3=N_2$. Thus, (3.12) becomes

$$u(m,n) \leqslant W_{k+1}^{-1}\Big[W_{k+1}(r_1(\tilde m,\tilde n,0,n)) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\tilde f_{k+1}(\tilde m,\tilde n,s,t)$$
$$+ \sum_{s=0}^{m-1}\frac{\Delta_3 r_{k+1}(\tilde m,\tilde n,s,n)}{\phi_{k+1}(W_k^{-1}(r_{k+1}(0,\infty,s,\infty)))}\Big]$$

for $0\leqslant m\leqslant\tilde m$ and $n\geqslant\tilde n$. It shows that (3.4) is true for $k+1$. Therefore, the claim is proved.

Replacing m and n by $\tilde m$ and $\tilde n$ in (3.4) respectively, we have

$$u(\tilde m,\tilde n) \leqslant W_k^{-1}\Big[W_k(r_1(\tilde m,\tilde n,0,\tilde n)) + \sum_{s=0}^{\tilde m-1}\sum_{t=\tilde n+1}^{\infty}\tilde f_k(\tilde m,\tilde n,s,t)$$
$$+ \sum_{s=0}^{\tilde m-1}\frac{\Delta_3 r_k(\tilde m,\tilde n,s,\tilde n)}{\phi_k(W_{k-1}^{-1}(r_k(0,\infty,s,\infty)))}\Big].$$

Since (3.4) is true for any $\tilde m\leqslant M_1$ and $\tilde n\geqslant N_1$, we replace $\tilde m$ and $\tilde n$ by m and n and get

$$u(m,n) \leqslant W_k^{-1}\Big[W_k(r_1(m,n,0,n)) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\tilde f_k(m,n,s,t)$$
$$+ \sum_{s=0}^{m-1}\frac{\Delta_3 r_k(m,n,s,n)}{\phi_k(W_{k-1}^{-1}(r_k(0,\infty,s,\infty)))}\Big].$$

Note that $r_1(m,n,0,n)=\tilde a(0,n)$ so we have

$$u(m,n) \leqslant W_k^{-1}\Big[W_k(\tilde a(0,n)) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\tilde f_k(m,n,s,t)$$
$$+ \sum_{s=0}^{m-1}\frac{\Delta_3 r_k(m,n,s,n)}{\phi_k(W_{k-1}^{-1}(r_k(0,\infty,s,\infty)))}\Big], \quad 0\leqslant m\leqslant M_1,\ \ n\geqslant N_1.$$

This proves Theorem 3.1. □

REMARK 3.1.

(1) In [6], υ_i is increasing for $i = 1, \cdots, k$. Here we delete this condition by using the method in [16].

(2) If $a(m,n) = 0$ for all $m,n \in N_0$, then $a(0,\infty) = 0$ so $\tilde{a}(m,n) = 0$. Define $W_i(0) = 0$, for $i = 1, \cdots, k$, and $\frac{\Delta_3 r_l(m,n,\tau,t)}{\phi_l(W_{l-1}^{-1}(r_l(0,\infty,\tau,\infty)))} = 0$ if $\Delta_3 r_l(m,n,\tau,t) = 0$, where $l = 1, \cdots, k$. Then (3.1) is still true.

4. Corollaries

Assume that $\chi \in C(\mathbf{R}_+, \mathbf{R}_+)$ is a strictly increasing function with $\chi(\infty) = \infty$ where $\mathbf{R}_+ = [\mathbf{0}, \infty)$. Consider the following inequality

$$\chi(u(m,n)) \leqslant a(m,n) + \sum_{i=1}^{k} \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} f_i(m,n,s,t)\upsilon_i(u(s,t)), \quad m,n \in \mathbf{N}_0. \tag{4.14}$$

COROLLARY 4.1. *Suppose that* (A_1)–(A_3) *hold and* $u(m,n)$ *is a nonnegative function for* $m,n \in \mathbf{N}_0$ *satisfying* (4.14). *Then*

$$\begin{aligned} u(m,n) \leqslant \chi^{-1}\Big[\tilde{W}_k^{-1}[\tilde{W}_k(\tilde{a}(0,n)) &+ \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} \tilde{f}_k(m,n,s,t) \\ &+ \sum_{s=0}^{m-1} \frac{\Delta_3 r_k(m,n,s,n)}{\hat{\phi}_k(\tilde{W}_{k-1}^{-1}(r_k(0,\infty,s,\infty)))}]\Big], \quad 0 \leqslant m \leqslant M_1,\ n \geqslant N_1, \end{aligned} \tag{4.15}$$

where $\tilde{W}_i(u) = \int_{u_i}^{u} \frac{dz}{w_i(\chi^{-1}(z))}$, $\tilde{W}_i^{-1}$ *is the inverse of* $\tilde{W}_i$, $\tilde{W}_0 = I$ *(Identity)*, $\hat{\phi}_i(u) = \frac{w_i(\chi^{-1}(u))}{w_{i-1}(\chi^{-1}(u))}$, $\hat{\phi}_1(u) = w_1(\chi^{-1}(u))$, *and other related functions are given in Theorem* 3.1 *by replacing* $w_i(u)$ *with* $w_i(\chi^{-1}(u))$.

Proof. Let $\xi(m,n) = \chi(u(m,n))$. Then (4.14) becomes

$$\xi(m,n) \leqslant a(m,n) + \sum_{i=1}^{k} \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} f_i(m,n,s,t)\upsilon_i(\chi^{-1}(\xi(s,t))), \quad m,n \in \mathbf{N}_0. \tag{4.16}$$

Note that $w_i(\chi^{-1}(u))$ satisfies the condition (A_3). Using Theorem 3.1, we obtain the estimate about $\xi(m,n)$ by replacing $w_i(u)$ with $w_i(\chi^{-1}(u))$. Then use the fact that $u(m,n) = \chi^{-1}(\xi(m,n))$ and we get Corollary 4.1.

If $\chi(u) = u^p$ where $p > 0$, then (4.14) reads

$$u^p(m,n) \leqslant a(m,n) + \sum_{i=1}^{k} \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} f_i(m,n,s,t)\upsilon_i(u(s,t)), \quad m,n \in \mathbf{N}_0. \tag{4.17}$$

Directly using Corollary 4.1, we have the following result. □

COROLLARY 4.2. *Suppose that (A_1)–(A_3) hold and $u(m,n)$ is a nonnegative function for $m,n\in\mathbf{N}_0$ satisfying (4.17). Then*

$$u(m,n)\leqslant [\tilde{W}_k^{-1}\Big[\tilde{W}_k(\tilde{a}(0,\infty))+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\tilde{f}_k(m,n,s,t)$$
$$+\sum_{s=0}^{m-1}\frac{\Delta_3 r_k(m,n,s,n)}{\hat{\phi}_k(\tilde{W}_{k-1}^{-1}(r_k(0,\infty,s,\infty)))}\Big]^{\frac{1}{p}},\quad 0\leqslant m\leqslant M_1, n\geqslant N_1, \tag{4.18}$$

where $\tilde{W}_i(u)=\int_{u_i}^{u}\frac{dz}{w_i(z^{\frac{1}{p}})}$, $\tilde{W}_i^{-1}$ is the inverse of $\tilde{W}_i$, $\tilde{W}_0=I$ (Identity), $\hat{\phi}_i(u)=\frac{w_i(u^{\frac{1}{p}})}{w_{i-1}(u^{\frac{1}{p}})}$, $\hat{\phi}_1(u)=w_1(u^{\frac{1}{p}})$, and other related functions are in Theorem 3.1 by replacing $w_i(u)$ with $w_i(u^{\frac{1}{p}})$.

5. Applications to a difference equation

In this section, we will apply our results to study the boundedness and uniqueness of solutions of a nonlinear difference equation.

EXAMPLE 5.1. Consider the following nonlinear difference equation

$$b(m,n)=\beta(m,n)+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}G(m,n,s,t,b(s,t))+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}P(m,n,s,t,b(s,t)) \tag{5.1}$$

for $m,n\in\mathbf{N}_0$ where $b:\mathbf{N}_0^2\to\mathbf{R}$ is an unknown function, β maps from $\mathbf{N}_0^2$ to $\mathbf{R}$, and G and P map from $\mathbf{N}_0^4\times\mathbf{R}$ to $\mathbf{R}$.

Let $a(m,n)=|\beta(m,n)|$ and $\tilde{a}(m,n)=\max_{0\leqslant\tau\leqslant m,\eta\geqslant n,\tau,\eta\in\mathbf{N}_0}a(\tau,\eta)$. We have the following theorem.

THEOREM 5.1. *Suppose that the condition (A_1) is valid, and the functions G and P in (5.1) satisfy the conditions*

$$\begin{aligned}|G(m,n,s,t,b)|&\leqslant f_1(m,n,s,t)\sqrt{|b|},\\ |P(m,n,s,t,b)|&\leqslant f_2(m,n,s,t)|b|,\end{aligned}\tag{5.2}$$

where $f_1,f_2:\mathbf{N}_0^4\to[0,\infty)$. If $b(m,n)$ is a solution of (5.1) on $\mathbf{N}_0^2$, then

$$|b(m,n)|\leqslant\tilde{a}(0,\infty)\exp\Big[\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\tilde{f}_2(m,n,s,t)+\sum_{s=0}^{m-1}\frac{\sum_{t=n+1}^{\infty}\tilde{f}_1(m,n,s,t)+\frac{\Delta_1\tilde{a}(s,n)}{\sqrt{\tilde{a}(s,\infty)}}}{p(s)}\Big], \tag{5.3}$$

where

$$\tilde{f}_1(m,n,s,t)=\max_{0\leqslant\tau\leqslant m,\eta\geqslant n,\tau,\eta\in\mathbf{N}_0}f_1(\tau,\eta,s,t),$$

$$\tilde{f}_2(m,n,s,t) = \max_{0\leqslant \tau \leqslant m, \eta \geqslant n, \tau,\eta \in \mathbf{N}_0} f_2(\tau,\eta,s,t),$$
$$p(s) = \sqrt{\tilde{a}(0,\infty)} + \frac{1}{2}\sum_{\tau=0}^{s-1} \frac{\Delta_1 \tilde{a}(\tau,\infty)}{\sqrt{\tilde{a}(\tau,\infty)}}.$$

Proof. Using (5.1) and (5.2), the solution $b(m,n)$ satisfies

$$u(m,n) \leqslant a(m,n) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} f_1(m,n,s,t)\upsilon_1(u(s,t)) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} f_2(m,n,s,t)\upsilon_2(u(s,t)), \quad m,n \in \mathbf{N}_0, \tag{5.4}$$

where

$$u(m,n) = |b(m,n)|, \quad \upsilon_1(u) = \sqrt{u}, \quad \upsilon_2(u) = u.$$

Clearly, $\tilde{a}(m,n) > 0$ for all $m,n \in \mathbf{N}_0$ since the condition (A_1) holds. For positive constants u_1, u_2 by (5.1), we have

$$w_1(u) = \sqrt{u}, \qquad w_2(u) = u,$$
$$W_1(u) = \int_{u_1}^{u} \frac{dz}{w_1(z)} = 2(\sqrt{u} - \sqrt{u_1}), \qquad W_1^{-1}(u) = \Big(\frac{u}{2} + \sqrt{u_1}\Big)^2,$$
$$W_2(u) = \int_{u_2}^{u} \frac{dz}{w_2(z)} = \ln\frac{u}{u_2}, \qquad W_2^{-1}(u) = u_2\exp(u),$$
$$r_1(m,n,s,t) = \tilde{a}(s,t) > 0, \qquad r_1(m,n,0,t) = \tilde{a}(0,t),$$
$$r_2(m,n,s,t) = 2(\sqrt{\tilde{a}(0,t)} - \sqrt{u_1}) + \sum_{\tau=0}^{s-1}\sum_{\eta=t+1}^{\infty} \tilde{f}_1(m,n,\tau,\eta) + \sum_{\tau=0}^{s-1} \frac{\Delta_1\tilde{a}(\tau,t)}{\sqrt{\tilde{a}(\tau,\infty)}},$$
$$\Delta_3 r_2(m,n,s,t) = \sum_{\eta=t+1}^{\infty} \tilde{f}_1(m,n,s,\eta) + \frac{\Delta_1\tilde{a}(s,t)}{\sqrt{\tilde{a}(s,\infty)}}, \qquad \phi_2(u) = \frac{w_2(u)}{w_1(u)} = \sqrt{u}.$$

It is obvious that w_1 and w_2 satisfy the condition (A_3). Applying Theorem 3.1 gives

$$u(m,n) \leqslant \tilde{a}(0,\infty)\exp\Big[\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\tilde{f}_2(m,n,s,t) + \sum_{s=0}^{m-1} \frac{\sum_{\eta=n+1}^{\infty}\tilde{f}_1(m,n,s,\eta) + \frac{\Delta_1\tilde{a}(s,n)}{\sqrt{\tilde{a}(s,\infty)}}}{p(s)}\Big]$$

which implies (5.3). □

THEOREM 5.2. *Suppose that the functions G and P in (5.1) satisfy the conditions*

$$\begin{aligned} |G(m,n,s,t,b_1) - P(m,n,s,t,b_2)| &\leqslant f_1(m,n,s,t)\sqrt{|b_1-b_2|}, \\ |P(m,n,s,t,b_1) - P(m,n,s,t,b_2)| &\leqslant f_2(m,n,s,t)|b_1-b_2|. \end{aligned} \tag{5.5}$$

where $f_1, f_2 : \mathbf{N}_0^4 \to [0,\infty)$. Then (5.1) has at most one solution on $\mathbf{N}_0^4$.

Proof. Let $b_1(m,n)$ and $b_1(m,n)$ be two solutions of (5.1) on $\mathbf{N}_0^4$. From (4.18), we have

$$\begin{aligned} |b_1(m,n)-b_2(m,n)| &\leqslant \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}|G(m,n,s,t,b_1)-G(m,n,s,t,b_2)| \\ &\quad+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}|P(m,n,s,t,b_1)-P(m,n,s,t,b_2)| \\ &\leqslant \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} f_1(m,n,s,t)\sqrt{|b_1-b_2|} \\ &\quad+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} f_2(m,n,s,t)|b_1-b_2|, \quad m,n\in\mathbf{N}_0. \end{aligned} \tag{5.6}$$

Rewriting the above, we have

$$\begin{aligned} u(m,n,s,t) &\leqslant \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} f_1(m,n,s,t)\upsilon_1(u(s,t)) \\ &\quad+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} f_2(m,n,s,t)\upsilon_2(u(s,t)), \end{aligned} \tag{5.7}$$

where $u(m,n)=|b_1(m,n)-b_2(m,n)|$, $a(m,n)=0$, $\upsilon_2(u)=\sqrt{u}$, $\upsilon_1(u)=u$, so $w_2(u)=\sqrt{u}$, $w_1(u)=u$. Apply Theorem 3.1 and (2) of Remark 3.1, the inequality (5.7) leads us to the equality $u(m,n)=0$ which implies that the solution is unique. $\square$

REFERENCES

[1] R. P. AGARWAL, S. DENG AND W. ZHANG, *Generalization of a retarded Gronwall-like inequality and its applications*, Applied Mathematics and Computation **165**, (2005), 599–612.

[2] D. BAINOV AND P. SIMEONOV, *Integral Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, 1992.

[3] W. CHEUNG, *Some discrete nonlinear inequalities and applications to boundary value problems for difference equations*, Journal of Difference Equations and Applications **10**, (2004), 213–223.

[4] W. S. CHEUNG AND Q.-H. MA, *On certain new Gronwall-Ou-Iang type integral inequalities in two variables and their applications*, Journal of Inequalities and Applications **6**, (2005), 347–361.

[5] W. CHEUNG AND J. REN, *Discrete non-linear inequalities and applications to boundary value problems*, Journal of Mathematical Analysis and Applications **319**, (2006), 708–724.

[6] S. DENG, *Nonlinear discrete inequalities with two variables and their applications*, Applied Mathematics and Computation **217**, (2010): 2217–2225.

[7] S. DENG AND C. PRATHER, *Generalization of an impulsive nonlinear singular Gronwall-Bihari inequality with delay*, Journal of Inequalities in Pure and Applied Mathematics **9** (2), (2008), Art. 34, 11 pp.

[8] S. DENG AND W. ZHANG, *Integral Gronwall inequalities*, Review and Research of Mathematics **22**, (2002), 307–313.

[9] S. S. DRAGOMIR AND Y.-H. KIM, *Some integral inequalities for functions of two variables*, Electronic Journal of Differential Equations **2003**, (2003), 1–13.

[10] A. MATE AND P. NEVAI, *Sublinear perturbations of the differential equation $y^{(n)}=0$ and of the analogous difference equation*, Journal of Differential Equations **53**, (1984), 234–257.

[11] F. W. MENG AND W. N. LI, *On some new integral inequalities and their applications*, Applied Mathematics and Computation **148**, (2004), 381–392.

[12] B. G. PACHPATTE, *On some new inequalities related to certain inequalities in the theory of differential equations*, Journal of Mathematical Analysis and Applications **189**, (1995), 128–144.

[13] B. G. PACHPATTE, *On some fundamental integral inequalities and their discrete analogues*, Journal of Inequalities in Pure and Applied Mathematics **2**, (2001) 1–13.

[14] B. G. PACHPATTE, *Integral and Finite Difference Inequalities and Applications*, Elsevier Science B. V., Amsterdam, 2006.

[15] M. PINTO, *Integral inequalities of Bihari-type and applications*, Funkcialaj Ekvacioj **33**, (1990), 387–403.

[16] W. WANG AND C. SHEN, *On a generalized retarded integral inequality with two variables*, Journal of Inequalities and Applications, 2008, Article ID 518646.

[17] Y. WU, X. LI AND S. DENG, *Nonlinear delay discrete inequalities and their applications to Volterra type difference equations*, Advances in Difference Equations, 2010, Article ID 795145.

[18] W. ZHANG AND S. DENG, *Projected Gronwall-Bellman's inequality for integrable functions*, Mathematical and Computer Modelling **34**, (2001), 393–402.

(Received July 6, 2013)

Ying Liang
Mathematics and Computational School
Zhanjiang Normal University
Zhanjiang, Guangdong 524048, P. R. China
e-mail: lydd2013@163.com

Xiaopei Li
Mathematics and Computational School
Zhanjiang Normal University
Zhanjiang, Guangdong 524048, P. R. China
e-mail: lxpzhanjiang2013@163.com

Haishan Dong
Mathematics and Computational School
Zhanjiang Normal University
Zhanjiang, Guangdong 524048, P. R. China
e-mail: donghaishan@126.com

Shuisheng Chen
Mathematics and Computational School
Zhanjiang Normal University
Zhanjiang, Guangdong 524048, P. R. China
e-mail: chenshuishengzj@yeah.net