Journal of
Mathematical
Inequalities
Volume 8, Number 4 (2014), 855–862 doi:10.7153/jmi-08-64

THE SCHUR CONVEXITY FOR THE GENERALIZED MUIRHEAD MEAN

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(*Communicated by N. Elezović*)

Abstract. For $x,y>0$, $a,b\in\mathbb{R}$ with $a+b\neq 0$, the generalized Muirhead mean is defined by $M(a,b;x,y)=\left(\frac{x^a y^b+x^b y^a}{2}\right)^{\frac{1}{a+b}}$. In this paper, we prove that $M(a,b;x,y)$ is Schur convex with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ if and only if $(a,b)\in\{(a,b)\in\mathbb{R}^2:(a-b)^2\geqslant a+b>0\ \&\ ab\leqslant 0\}$ and Schur concave with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ if and only if $(a,b)\in\{(a,b)\in\mathbb{R}_+^2:(a-b)^2\leqslant a+b\ \&\ (a,b)\neq(0,0)\}\cup\{(a,b)\in\mathbb{R}^2:a+b<0\}$, where $\mathbb{R}_+:=[0,\infty)$.

1. Introduction

Recall that the following notion of Schur convexity (see [1]).

DEFINITION 1.1. Let $E\subseteq\mathbb{R}^n (n\geqslant 2)$ be a set with nonempty interior, a real-valued function $f:E\to\mathbb{R}$ is said to be Schur convex on E if $f(x)\leqslant f(y)$ for each pair of n-tuples $x=(x_1,\cdots,x_n)$ and $y=(y_1,\cdots,y_n)$ in E with $x\prec y$, i.e.

$$\sum_{i=1}^{k} x_{[i]}\leqslant\sum_{i=1}^{k} y_{[i]},\quad k=1,2,\cdots,n-1$$

and

$$\sum_{i=1}^{n} x_{[i]}=\sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the ith largest component in x. f is called Schur concave if $-f$ is Schur convex.

The notion of Schur convexity was introduced by I. Schur in 1923 [2]. In the recent past, the Schur convexity has been the subject of intensive research, many remarkable inequalities have been established by using the Schur convexity theory [3–9]. In particular, the Schur convexity for means and symmetric functions has attracted the attention of many researchers [10–24].

Mathematics subject classification (2010): Primary 26B25; Secondary 26E60.
Keywords and phrases: Generalized Muirhead mean, Schur convexity, Schur concavity.
This research was supported by the Natural Science Foundation of China under Grants 61374086 and 11171307, and the Natural Science Foundation of the Department of Education of Hunan Province under Grant 13C127.

For $x, y > 0$, and $a, b \in \mathbb{R}$ with $a+b \neq 0$, the generalized Muirhead mean $M(a,b;x,y)$ was introduced by T. Trif [25] as follows:

$$M(a,b;x,y) = \left(\frac{x^a y^b + x^b y^a}{2}\right)^{\frac{1}{a+b}}. \tag{1.1}$$

It is easy to see that the generalized Muirhead mean $M(a,b;x,y)$ is continuous on the domain $\{(a,b;x,y) : a+b \neq 0; x,y > 0\}$ and differentiable with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for fixed $a, b \in \mathbb{R}$ with $a+b \neq 0$. It is of symmetry between a and b and between x and y. Many means are the special case of the generalized Muirhead mean, for example

$$\begin{aligned} M_p(x,y) &= M(p,0;x,y) \quad \text{is the power or Hölder mean,} \\ A(x,y) &= M(0,1;x,y) \quad \text{is the arithmetic mean,} \\ G(x,y) &= M(a,a;x,y) \quad \text{is the geometric mean} \end{aligned}$$

and

$$H(x,y) = M(0,-1;x,y) \quad \text{is the harmonic mean.}$$

In paper [25], the monotonicity of $M(a,b;x,y)$ with respect to a or b was discussed, and a comparison theorem and a Minkowski-type inequality involving the generalized Muirhead mean $M(a,b;x,y)$ were established.

The well-known Muirhead's inequality (see [26]) implies that if $x, y > 0$ are fixed then $M(a,b;x,y)$ is Schur convex on the domain $\{(a,b) \in \mathbb{R}^2 : a+b > 0\}$ and Schur concave on the domain $\{(a,b) \in \mathbb{R}^2 : a+b < 0\}$. But no one has ever researched the Schur convexity or Schur concavity of $M(a,b;x,y)$ with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for fixed $a, b \in \mathbb{R}$ with $a+b \neq 0$.

Our purpose is to discuss the Schur convexity and Schur concavity of $M(a,b;x,y)$ with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for fixed $a, b \in \mathbb{R}$ with $a+b \neq 0$. Our main result is the following theorem.

THEOREM 1.1. *The generalized Muirhead mean $M(a,b;x,y)$ is Schur convex with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ if and only if $(a,b) \in \{(a,b) \in \mathbb{R}^2 : (a-b)^2 \geqslant a+b > 0 \ \& \ ab \leqslant 0\}$ and Schur concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ if and only if $(a,b) \in \{(a,b) \in \mathbb{R}_+^2 : (a-b)^2 \leqslant a+b \ \& \ (a,b) \neq (0,0)\} \cup \{(a,b) \in \mathbb{R}^2 : a+b < 0\}$, where $\mathbb{R} := [0,\infty)$.*

2. Preliminary Results

In this section we introduce and establish two lemmas, which will be used in the proof of Theorem 1.1.

LEMMA 2.1. (see [1]) *Let $E \subset \mathbb{R}^2$ be a symmetric convex set with nonempty interior $\mathrm{int}E$ and $\varphi : E \to \mathbb{R}$ be a continuous and symmetric function on E. If φ is* *differentiable on* $\operatorname{int}E$*, then* φ *is Schur convex (or Schur concave, respectively) on* E *if and only if*

$$(y-x)\left(\frac{\partial\varphi}{\partial y}-\frac{\partial\varphi}{\partial x}\right)\geqslant 0 \quad (or \leqslant 0,\ respectively)$$

for all $(x,y)\in \operatorname{int}E$.

LEMMA 2.2. *Let* a *and* b *be two real numbers such that* $a>b$ *and* $a+b\neq 0$*. Let us define the function* $f:[1,\infty)\to\mathbb{R}$,

$$f(t)=\frac{1}{a+b}(at^{a-b}-bt^{a-b+1}-at+b),\ \ for\ \ t\geqslant 1.$$

Then the following statements hold:

(1) If $b>0$ *and* $(a-b)^2<a+b$*, then* $f(t)\leqslant 0$ *for all* $t\geqslant 1$*;*

(2) If $b>0$ *and* $(a-b)^2>a+b$*, then there exist* $t_1,t_2\in(1,\infty)$ *such that* $f(t_1)<0$ *and* $f(t_2)>0$*;*

(3) If $b<0$ *and* $(a-b)^2<a+b$*, then there exist* $t_3,t_4\in(1,\infty)$ *such that* $f(t_3)<0$ *and* $f(t_4)>0$*;*

(4) If $a+b>0$, $b<0$ *and* $(a-b)^2>a+b$*, then* $f(t)\geqslant 0$ *for all* $t\geqslant 1$*;*

(5) If $a+b<0$*, then* $f(t)\leqslant 0$ *for all* $t\geqslant 1$.

Proof. Let $g(t)=\frac{a-b}{a+b}(a(a-b-1)-b(a-b+1)t)$. Then simple computation leads to

$$f(1)=0, \tag{2.1}$$

$$f'(t)=\frac{1}{a+b}\Big(a(a-b)t^{a-b-1}-b(a-b+1)t^{a-b}-a\Big),$$

$$f'(1)=\frac{(a-b)^2-(a+b)}{a+b}, \tag{2.2}$$

$$f''(t)=t^{a-b-2}g(t),$$

$$g(1)=\frac{a-b}{a+b}\left((a-b)^2-(a+b)\right) \tag{2.3}$$

and

$$g'(t)=-\frac{b(a-b)(a-b+1)}{a+b}. \tag{2.4}$$

(1) If $b>0$ and $(a-b)^2<a+b$, then (2.2)–(2.4) imply

$$f'(1)<0, \tag{2.5}$$

$$g(1)<0 \tag{2.6}$$

and

$$g'(t)<0. \tag{2.7}$$

Therefore, Lemma 2.2 (1) follows from (2.1) and (2.5)–(2.7).

(2) If $b>0$ and $(a-b)^2>a+b$, then from (2.2) we clearly see that $f'(1)>0$. Then the continuity of $f'(t)$ implies that there exists $\delta_1>0$ such that

$$f'(t)>0 \tag{2.8}$$

for $t\in[1,1+\delta_1)$. From (2.1) and (2.8) we know that $f(t)>0$ for $t\in(1,1+\delta_1)$.

On the other hand, it is easy to see that $\lim\limits_{t\to+\infty} f(t)=-\infty$. Hence Lemma 2.2 (2) is true.

(3) If $b<0$ and $(a-b)^2<a+b$, then (2.5) again holds. Then the continuity of $f'(t)$ implies that there exists $\delta_2>0$ such that

$$f'(t)<0 \tag{2.9}$$

for $t\in[1,1+\delta_2)$. From (2.1) and (2.9) we know that $f(t)<0$ for $t\in(1,1+\delta_2)$.

On the other hand, we clearly see that $\lim\limits_{t\to+\infty} f(t)=+\infty$. Hence Lemma 2.2 (3) is true.

(4) If $a+b>0$, $b<0$ and $(a-b)^2>a+b$, then (2.2)–(2.4) lead to

$$f'(1)>0, \tag{2.10}$$

$$g(1)>0 \tag{2.11}$$

and

$$g'(t)>0. \tag{2.12}$$

Therefore, Lemma 2.2(4) follows from (2.1) and (2.10)–(2.12).

(5) If $a+b<0$, then inequalities (2.5)–(2.7) again hold. Therefore, Lemma 2.2 (5) follows from (2.1) and (2.5)–(2.7).

3. Proof of Theorem 1.1

Proof of Theorem 1.1. We use Lemma 2.1 to discuss the nonegativity and nonpositivity of $(y-x)(\frac{\partial M}{\partial y}-\frac{\partial M}{\partial x})$ for all $(x,y)\in(0,\infty)\times(0,\infty)$ and fixed $(a,b)\in\mathbb{R}^2$ with $a+b\neq 0$. Since $(y-x)(\frac{\partial M}{\partial y}-\frac{\partial M}{\partial x})=0$ for $x=y$ and it is symmetric with respect to x and y, hence we assume $y>x$ in the following discussion.

Let

$$E_1=\{(a,b)\in\mathbb{R}^2:(a-b)^2\geqslant a+b>0\ \&\ ab\leqslant 0\},$$

$$E_2=\{(a,b)\in\mathbb{R}^2_+:(a-b)^2\leqslant a+b\ \&\ (a,b)\neq(0,0)\}\cup\{(a,b)\in\mathbb{R}^2:a+b<0\}$$

and

$$\begin{aligned} E_3=&\{(a,b)\in\mathbb{R}^2:b<0\ \&\ (a-b)^2<a+b\}\\ &\cup\{(a,b)\in\mathbb{R}^2:a<0\ \&\ (a-b)^2<a+b\}\\ &\cup\{(a,b)\in\mathbb{R}^2:a>b>0\ \&\ (a-b)^2>a+b\}\\ &\cup\{(a,b)\in\mathbb{R}^2:b>a>0\ \&\ (a-b)^2>a+b\}. \end{aligned}$$

Then $E_1 \cup E_2 \cup E_3 = \{(a,b) \in \mathbb{R}^2 : a+b \neq 0\}$. It is obvious that Theorem 1.1 is true if once we prove that $M(a,b;x,y)$ is Schur convex, Schur concave, and neither Schur convex nor Schur concave with respect to $(x,y) \in (0,\infty)^2$ for $(a,b) \in E_1, E_2$ and E_3, respectively.

From (1.1) we get the following identity

$$(y-x)\left(\frac{\partial M}{\partial y} - \frac{\partial M}{\partial x}\right) = 2^{-\frac{1}{a+b}} x^a y^{b-1} (y-x)\left(x^a y^b + x^b y^a\right)^{\frac{1}{a+b}-1} f\left(\frac{y}{x}\right) \tag{3.1}$$

for $y > x > 0$, where f is defined as in Lemma 2.2. Further, it is sufficed to study the sign of $f\left(\frac{y}{x}\right)$, we divide our discussion into three cases.

Case 1. $(a,b) \in E_1$. Let $E_{11} = \{(a,b) \in \mathbb{R}^2 : a+b > 0, b < 0 \ \& \ (a-b)^2 > a+b\}$ and $E_{12} = \{(a,b) \in \mathbb{R}^2 : a+b > 0, a < 0 \ \& \ (a-b)^2 > a+b\}$.

From Lemma 2.2 (4) and the assumption $y > x$ we know that $f\left(\frac{y}{x}\right) \geqslant 0$ for $(a,b) \in E_{11}$, then (3.1) and Lemma 2.1 together with the continuity and symmetry of $M(a,b;x,y)$ with respect to (a,b) imply that $M(a,b;x,y)$ is Schur convex with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for $(a,b) \in E_1$.

Case 2. $(a,b) \in E_2$. Let $E_{21} = \{(a,b) \in \mathbb{R}^2 : a > b > 0 \ \& \ (a-b)^2 < a+b\}$, $E_{22} = \{(a,b) \in \mathbb{R}^2 : b > a > 0 \ \& \ (a-b)^2 < a+b\}$, $E_{23} = \{(a,b) \in \mathbb{R}^2 : a > b \ \& \ a+b < 0\}$ and $E_{24} = \{(a,b) \in \mathbb{R}^2 : b > a \ \& \ a+b < 0\}$.

From Lemmas 2.2 (1) and 2.2 (5) together with the assumption $y > x$ we clearly see that $f\left(\frac{y}{x}\right) \leqslant 0$ for $(a,b) \in E_{21} \cup E_{23}$, then (3.1) and Lemma 2.1 together with the continuity and symmetry of $M(a,b;x,y)$ with respect to (a,b) lead to the conclusion that $M(a,b;x,y)$ is Schur concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for $(a,b) \in E_2$.

Case 3. Let $E_{31} = \{(a,b) \in \mathbb{R}^2 : b < 0 \ \& \ (a-b)^2 < a+b\}$, $E_{32} = \{(a,b) \in \mathbb{R}^2 : a < 0 \ \& \ (a-b)^2 < a+b\}$, $E_{33} = \{(a,b) \in \mathbb{R}^2 : a > b > 0 \ \& \ (a-b)^2 > a+b\}$ and $E_{34} = \{(a,b) \in \mathbb{R}^2 : b > a > 0 \ \& \ (a-b)^2 > a+b\}$.

From Lemmas 2.2 (3) and 2.2 (2) together with assumption $y > x$ we clearly see that $f\left(\frac{y}{x}\right)$ is neither nonpositivity nor nonnegativity for $(a,b) \in E_{31}$ or E_{33}, then (3.1) and Lemma 2.1 together with the symmetry of $M(a,b;x,y)$ with respect to (a,b) show that $M(a,b;x,y)$ is neither Schur convex nor Schur concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for $(a,b) \in E_3$. □

4. Applications

In this section, we establish some inequalities by use of Theorem 1.1 and the theory of majorization.

The following Corollary 4.1 follows from Theorem 1.1 immediately.

COROLLARY 4.1. *The power mean* $M_p(x,y) = \begin{cases} \left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}}, & p \neq 0, \\ \sqrt{xy}, & p = 0 \end{cases}$ *of order* p *is Schur convex with respect to* $(x,y) \in (0,\infty)^2$ *if and only if* $p \geqslant 1$ *and Schur concave with respect to* $(x,y) \in (0,\infty)^2$ *if and only if* $p \leqslant 1$.

COROLLARY 4.2. *Let us consider two real numbers a and b, such that $a+b\neq 0$. Then*

(1) $M(a,b;(1-\alpha)x+\alpha y,\alpha x+(1-\alpha)y)\leqslant M(a,b;x,y)$ for all $(x,y)\in(0,\infty)^2$ and $\alpha\in[0,1]$ if and only if $(a,b)\in E_1$.

In particular, taking $\alpha=1/2$, we have

$$A(x,y)\leqslant M(a,b;x,y),\ \textit{for}\ x,y>0\ \textit{and}\ (a,b)\in E_1;$$

(2) $M(a,b;(1-\alpha)x+\alpha y,\alpha x+(1-\alpha)y)\geqslant M(a,b;x,y)$ for all $(x,y)\in(0,\infty)^2$ and $\alpha\in[0,1]$ if and only if $(a,b)\in E_2$.

In particular, taking $\alpha=1/2$, we have

$$A(x,y)\geqslant M(a,b;x,y),\ \textit{for}\ x,y>0\ \textit{and}\ (a,b)\in E_2.$$

Proof. Corollary 4.2 follows from Theorem 1.1 and the fact that

$$\big((1-\alpha)x+\alpha y,\alpha x+(1-\alpha)y\big)\prec(x,y)$$

for all $(x,y)\in(0,\infty)\times(0,\infty)$ and $\alpha\in[0,1]$.

As a geometric application of Theorem 1.1 we have the following Corollary 4.3. □

COROLLARY 4.3. *Let $a,b\in\mathbb{R}$ with $a+b\neq 0$, $A=A_1A_2A_3$ be a triangle in $\mathbb{R}^2$ with vertexes A_1,A_2,A_3, and P be an arbitrary point in the interior of A. If B_1,B_2 and B_3 are the intersection points of the straight line A_1P with segment A_2A_3, straight line A_2P with segment A_1A_3 and straight line A_3P with segment A_1A_2, respectively, then*

(1) $M(a,b;\frac{PB_1}{A_1B_1},\frac{PB_2}{A_2B_2})\geqslant\frac{PA_3}{2A_3B_3}$ and $M(a,b;\frac{PA_1}{A_1B_1},\frac{PA_2}{A_2B_2})\geqslant 1-\frac{PA_3}{2A_3B_3}$ for $(a,b)\in E_1$;

(2) $M(a,b;\frac{PB_1}{A_1B_1},\frac{PB_2}{A_2B_2})\leqslant\frac{PA_3}{2A_3B_3}$ and $M(a,b;\frac{PA_1}{A_1B_1},\frac{PA_2}{A_2B_2})\leqslant 1-\frac{PA_3}{2A_3B_3}$ for $(a,b)\in E_2$.

Proof. We clearly see that

$$\frac{PB_1}{A_1B_1}+\frac{PB_2}{A_2B_2}+\frac{PB_3}{A_3B_3}=1. \tag{4.1}$$

Equation (4.1) implies

$$\left(\frac{PA_3}{2A_3B_3},\frac{PA_3}{2A_3B_3}\right)\prec\left(\frac{PB_1}{A_1B_1},\frac{PB_2}{A_2B_2}\right) \tag{4.2}$$

and

$$\left(1-\frac{PA_3}{2A_3B_3},1-\frac{PA_3}{2A_3B_3}\right)\prec\left(\frac{PA_1}{A_1B_1},\frac{PA_2}{A_2B_2}\right) \tag{4.3}$$

Therefore, Corollary 4.3 follows from (1.1) and Theorem 1.1 together with (4.2) and (4.3). □

REFERENCES

[1] A. W. MARSHALL & I. OLKIN, *Inequalities: theory of majorization and its applications*, Academic Press, New York, 1979.

[2] I. SCHUR, *Über eine Klasse Von Mittelbildungen mit Anwendungen auf die Determinantentheorie*, Sitzunsber. Belin. Math. Ges., **22** (1923), 9–20.

[3] F. K. HWANG & U. G. ROTHBLUM, *Partition-optimization with Schur convex sum objective functions*, SIAM J. Discrete Math., **18** (2004/2005), no. 3, 512–524.

[4] X.-M. ZHANG, *Schur-convex functions and isoperimetric inequalities*, Proc. Amer. Math. Soc., **126** (1998), no. 2, 461–470.

[5] C. STEPNIAK, *Stochastic ordering and Schur-convex functions in comparison of linear experiments*, Metrika, **36** (1989), no. 5, 291–298.

[6] G. M. CONSTANTINE, *Schur-convex functions on the spectra of graphs*, Discrete Math., **45** (1983), no. 2–3, 181–188.

[7] M. MERKLE, *Convexity, Schur-convexity and bounds for the gamma function involving the digamma function*, Rocky Mountain J. Math., **28** (1998), no. 3, 1053–1066.

[8] F. K. HWANG, U. G. ROTHBLUM & L. SHEEP, *Monotone optimal multipartitions using Schur convexity with respect to partial orders*, SIAM J. Discrete Math., **6** (1993), no. 4, 533–547.

[9] N. N. CHAN, *Schur-convexity for A-optimal designs*, J. Math. Anal. Appl., **122** (1987), no. 1, 1–6.

[10] Y.-M. CHU & Y.-P. LV, *The Schur harmonic convexity of the Hamy symmetric function and its applications*, J. Inequal. Appl., 2009, Art. ID 838529, 1–10.

[11] Y.-M. CHU & X.-M. ZHANG, *Necessary and sufficient conditions such that extended mean values are Schur-convex or Schur-concave*, J. Math. Kyoto Univ., **48** (2008), no. 1, 229–238.

[12] Y.-M. CHU, X.-M. ZHANG & G.-D. WANG, *The Schur geometrical convexity of the extended mean values*, J. Convex Anal., **15** (2008), no. 4, 707–718.

[13] Y.-M. CHU & W.-F. XIA, *Solution of an open problem for Schur convexity or concavity of the Gini mean values*, Sci. China Ser. A, **52** (2009), no. 10, 2099–2106.

[14] F. QI, *A note on Schur-convexity of extended mean values*, Rocky Mountain J. Math., **35** (2005), no. 5, 1787–1793.

[15] F. QI, J. SÁNDOR, S. S. DRAGOMIR & A. SOFO, *Notes on the Schur-convexity of the extended mean values*, Taiwanese J. Math., **9** (2005), no. 3, 411–420.

[16] J. SÁNDOR, *The Schur-convexity of Stolarsky and Gini means*, Bannch J. Math. Anal., **1** (2007), no. 2, 212–215.

[17] H.-N. SHI, M. BENCZE, S.-H. WU & D.-M. LI, *Schur convexity of generalized Heronian means involving two parameters*, J. Inequal. Appl., 2008, Art. ID 879273, 1–9.

[18] H.-N. SHI, S.-H. WU & F. QI, *An alternative note on the Schur-convexity of the extended mean values*, Math. Inequal. Appl., **9** (2006), no. 2, 219–224.

[19] W.-D. JIANG, *Some properties of dual form of the Hamy's symmetric function*, J. Math. Inequal., **1** (2007), no. 1, 117–125.

[20] K.-Z. GUAN, *Some properties of a class of symmetric functions*, J. Math. Anal. Appl., **336** (2007), no. 1, 70–80.

[21] K.-Z. GUAN, *The Hamy symmetric function and its generalization*, Math. Inequal. Appl., **9** (2006), no. 4, 797–805.

[22] K.-Z. GUAN, *Schur-convexity of the complete symmetric function*, Math. Inequal. Appl., **9** (2006), no. 4, 567–576.

[23] K.-Z. GUAN & J.-H. SHEN, *Schur-convexity for a class of symmetric function and its applications*, Math. Inequal. Appl., **9** (2006), no. 2, 199–210.

[24] K.-Z. GUAN, *Schur-convexity of the complete elementary symmetric function*, J. Inequal. Appl., 2006, Art. ID 67624, 1–9.

[25] T. TRIF, *Monotonicity, comparison and Minkowski's inequality for generalized Muirhead means in two variables*, Mathematica, **48** (71) (2006), no. 1, 99–110.

[26] G. H. HARDY, J. E. LITTLEWOOD & G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, 1934.

(Received July 18, 2013)

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