\textbf{L}_p-\textit{mixed projection bodies and L}_p-\textit{mixed quermassintegrals}

\textbf{Weidong Wang and Xiaoyan Wan}

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Abstract. In this paper, we research the \textit{L}_p-\textit{mixed projection bodies by the L}_p-\textit{mixed quermassintegrals. First, we give an equivalent conclusion of L}_p-\textit{mixed projection bodies. Further, the Shephard type problem for the L}_p-\textit{mixed projection bodies are shown.}

1. Introduction

Let \( K \subset \mathbb{R}^n \) denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space \( \mathbb{R}^n \). For the set of convex bodies containing the origin in their interiors and the class of origin-symmetric convex bodies, write \( K_o^n \) and \( K_s^n \), respectively. Let \( S^{n-1} \) denote the unit sphere in \( \mathbb{R}^n \), denote by \( V(K) \) the \( n \)-dimensional volume of body \( K \). For the standard unit ball \( B \) in \( \mathbb{R}^n \), denote \( \omega_n = V(B) \).

If \( K \in \mathbb{K}^n \), then its support function, \( h_K(h(K, \cdot)) \), is defined by (see [5])

\[
    h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,
\]

where \( x \cdot y \) denotes the standard inner product of \( x \) and \( y \).

For each \( K \in \mathbb{K}^n \), the projection body, \( \Pi K \), of \( K \) is an origin-symmetric convex body whose support function is defined by (see [5, 27])

\[
    h_{\Pi K}(u) = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| \, dS(K, v)
\]

for all \( u \in S^{n-1} \), where \( S(K, \cdot) \) is the surface area measure of \( K \) on \( S^{n-1} \). The projection body is a very important object in the Brunn-Minkowski theory. During past four decades, a number of important results regarding classical projection bodies were obtained (see [1, 2, 3, 5, 6, 9, 10, 12, 14, 15, 16, 21, 23, 24, 26, 27, 34]).

The notion of the projection body was extended to mixed projection body by Lutwak (see [12, 14]). For each \( K \in \mathbb{K}^n \), the mixed projection body, \( \Pi_i K \) \( (i = 0, 1, \ldots, n - 1) \), of \( K \) is origin-symmetric convex body whose support function is defined by

\[
    h_{\Pi_i K}(u) = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| \, dS_i(K, v)
\]


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for all $u \in S^{n-1}$, where $S_i(K, \cdot)$ $(i = 0, 1, \cdots, n - 1)$ is the mixed surface area measure of $K$ on $S^{n-1}$. Obviously, $\Pi_0 K = \Pi K$.

The projection bodies were extended to $L_p$-space by Lutwak, Yang and Zhang. They (see [18]) introduced the notion of $L_p$ -projection body as follows: For $K \in \mathcal{K}_o^n$, and real number $p \geq 1$, the $L_p$ -projection body, $\Pi_p K$, of $K$ is origin-symmetric convex body whose support function is given by

$$h_{\Pi_p K}(u) = \frac{1}{(n+p)c_{n,p} \omega_n} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v) \quad (1.2)$$

for all $u \in S^{n-1}$, where

$$c_{n,p} = \omega_{n+p} / \omega_2 \omega_{n-1} \omega_{p-1}.$$  

The positive Borel measure $S_p(K, \cdot)$ on $S^{n-1}$ is called the $L_p$ -surface area measure of $K$, and has Radon-Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h(K, \cdot)^{1-p}.$$  

The unusual normalization of definition (1.2) is chosen so that for the unit ball $B$, we have $\Pi_p B = B$. In particular, for $p = 1$, $\Pi_1 K$ is the classical projection body $\Pi K$ of $K$ under the normalization of (1.2), and $\Pi B = B$, rather than the $\omega_{n-1} B$ (see [18]).

$L_p$ -projection bodies extended the classical projection bodies from the Brunn-Minkowski theory to the $L_p$ -Brunn-Minkowski theory. The studies of $L_p$ -projection bodies have received considerable attention, except see [18], for example also see [7, 8, 11, 19, 20, 25, 28, 29, 30, 31, 32, 33].

Similar to the definition of $L_p$ -projection body, Wang and Leng in [29] gave the definition of $L_p$ -mixed projection body as follows: For each $K \in \mathcal{K}_o^n$, real $p \geq 1$ and $i = 0, 1, \cdots, n - 1$, the $L_p$ -mixed projection body, $\Pi_{p,i} K$, of $K$ is origin-symmetric convex body whose support function is defined by

$$h_{\Pi_{p,i} K}(u) = \frac{1}{(n+p)c_{n,p} \omega_n} \int_{S^{n-1}} |u \cdot v|^p dS_{p,i}(K, v) \quad (1.3)$$

for all $u \in S^{n-1}$. Here the positive Borel measure $S_{p,i}(K, \cdot)$ $(i = 0, 1, \cdots, n - 1)$ on $S^{n-1}$ is called the $L_p$ -mixed surface area measure of $K$ which was introduced by Lutwak (see [17]). It turns out that the measure $S_{p,i}(K, \cdot)$ is absolutely continuous with respect to $S_i(K, \cdot)$, and has Radon-Nikodym derivative

$$\frac{dS_{p,i}(K, \cdot)}{dS_i(K, \cdot)} = h^{1-p}(K, \cdot). \quad (1.4)$$

The case $i = 0$, $S_{p,0}(K, \cdot)$ is just $L_p$ -surface area measure $S_p(K, \cdot)$. The unusual normalization of definition (1.3) is chosen so that for the unit ball $B$, we have $\Pi_{p,i} B = B$. Note that for $p = 1$, $\Pi_{1,i} K$ is the classical mixed projection body $\Pi_i K$ of $K$ under the normalization of (1.3).

From (1.3), if $i = 0$, then $\Pi_{p,0} K = \Pi_p K$. This means that $L_p$ -mixed projection body is an extension of $L_p$ -projection body in the $L_p$ -Brunn-Minkowski theory.
According to (1.3) and (1.4), we easily know that for \( \lambda > 0 \) and \( n - i \neq p \geq 1 \),
\[
\Pi_{p,i} \lambda K = \lambda^{\frac{n-i-p}{p}} \Pi_{p,i} K.
\] (1.5)

In this paper, we continuously research the \( L_p \)-mixed projection bodies. First, associated with \( L_p \)-mixed quermassintegrals (see [17]), we give an equivalent conclusion of the \( L_p \)-mixed projection bodies as follows:

**THEOREM 1.1.** If \( K, L \in \mathcal{K}_n \), \( p \geq 1 \) and \( i = 0, 1, \cdots, n - 1 \), then
\[
\Pi_{p,i} K = \Pi_{p,i} L \iff W_{p,i}(K, Q) = W_{p,i}(L, Q),
\] (1.6)
for any \( Q \in \mathcal{K}_n \).

Here \( W_{p,i}(M, N) \) \( (i = 0, 1, \cdots, n - 1) \) denotes the \( L_p \)-mixed quermassintegrals of \( M \) and \( N \), \( W_{p,0}(M, N) \) is just the \( L_p \)-mixed volume \( V_p(M, N) \) (see [17]). Let \( i = 0 \) in Theorem 1.1, we immediately obtain the following equivalent conclusion of the \( L_p \)-projection bodies.

**COROLLARY 1.1.** If \( K, L \in \mathcal{K}_n \), \( p \geq 1 \), then
\[
\Pi_{p} K = \Pi_{p} L \iff V_p(K, Q) = V_p(L, Q),
\] for any \( Q \in \mathcal{K}_n \).

Further, we study the Shephard type problems for the \( L_p \)-mixed projection bodies. Recall that Wang and Leng (see [29]) gave an affirmation of the Shephard type problems for the \( L_p \)-mixed projection bodies as follows:

**THEOREM 1.A.** Let \( K, L \in \mathcal{K}_n \), \( i = 0, 1, \cdots, n - 1 \) and \( n - i \neq p \geq 1 \). If \( L \) is an \( L_p \)-mixed projection body and \( \Pi_{p,i} K \subseteq \Pi_{p,i} L \), then for \( 0 \leq i < n - p \),
\[
W_i(K) \leq W_i(L);
\]
for \( n - p < i < n \),
\[
W_i(K) \geq W_i(L);
\]
with equality if and only if \( K = L \).

Here \( W_i(Q) \) denotes the quermassintegrals of \( Q \in \mathcal{K}_n \).

Using the \( L_p \)-mixed quermassintegrals, we give a general form of Theorem 1.A as follows:

**THEOREM 1.2.** Let \( K, L \in \mathcal{K}_n \), \( i = 0, 1, \cdots, n - 1 \) and \( n - i \neq p \geq 1 \). If
\[
\Pi_{p,i} K \subseteq \Pi_{p,i} L,
\]
then for any \( L_p \)-mixed projection body \( Q \),
\[
W_{p,i}(K, Q) \leq W_{p,i}(L, Q),
\] (1.7)
with equality if and only if \( K = L \).

Moreover, as the application of Theorem 1.1, we also obtain an improved version of Theorem 1.A.
THEOREM 1.3. Let $K \in \mathcal{K}_o^n$, $L \in \mathcal{K}_o^n$, $i = 0, 1, \cdots, n-1$ and $n - i \neq p > 1$. If $\Pi_{p,i}K = \Pi_{p,i}L$, then for $0 \leq i < n - p$,

$$W_i(K) \leq W_i(L);$$

(1.8)

for $n - p < i < n$,

$$W_i(K) \geq W_i(L).$$

(1.9)

Equality hold in (1.8) and (1.9) if and only if $K = L$.

In this paper, the proof of Theorem 1.1 is given in the Section 3; Theorems 1.2–1.3 are proven in the Section 4.

2. $L_p$-mixed quermassintegrals

For $K, L \in \mathcal{K}_o^n$ and $\varepsilon > 0$, the Minkowski combination, $K + \varepsilon L \in \mathcal{K}_o^n$, of $K$ and $L$ is defined by (see [5])

$$h(K + \varepsilon L, \cdot) = h(K, \cdot) + \varepsilon h(L, \cdot).$$

For $p \geq 1$, $K, L \in \mathcal{K}_o^n$ and $\varepsilon > 0$, the Firey $L_p$-combination (also called the $L_p$-Minkowski combination), $K +_{p \varepsilon} L \in \mathcal{K}_o^n$, of $K$ and $L$ is defined by (see [4, 17])

$$h(K +_{p \varepsilon} L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p,$$

(2.1)

where “$\cdot$” in $\varepsilon \cdot L$ denotes the Firey scalar multiplication.

If $K \in \mathcal{K}_o^n$, the quermassintegrals, $W_i(K)$ ($i = 0, 1, \cdots, n$), of $K$ is defined by (see [13])

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} h_K(v) dS_i(K, v).$$

(2.2)

Here $S_i(K, \cdot)$ is the mixed surface area measure of $K$, if $i = 0$, then $S_0(K, \cdot)$ is the surface area measure $S(K, \cdot)$ of $K$ (see [13]).

From definition (2.2), we easily see that

$$W_0(K) = \frac{1}{n} \int_{S^{n-1}} h_K(v) dS(K, v) = V(K).$$

(2.3)

Associated with the Firey $L_p$-combination, Lutwak (see [17]) defined the $L_p$-mixed quermassintegrals (who are called mixed $p$-quermassintegrals) as follows: For $K, L \in \mathcal{K}_o^n$, and real $p \geq 1$, the $L_p$-mixed quermassintegral $W_{p,i}(K, L)$ ($i = 0, 1, \cdots, n-1$) is defined by

$$\frac{n - i}{p} W_{p,i}(K, L) = \lim_{\varepsilon \rightarrow 0^+} \frac{W_i(K +_{p \varepsilon} L) - W_i(K)}{\varepsilon}.$$

Obviously, for $p = 1$, $W_{1,i}(K, L) = W_i(K, L)$ (see [17]). If $i = 0$, by (2.3) then the $L_p$-mixed quermassintegrals $W_{p,0}(K, L)$ is just the $L_p$-mixed volume $V_p(K, L)$, namely

$$W_{p,0}(K, L) = V_p(K, L).$$
In [17], Lutwak showed that for each $K \in \mathcal{K}_o^n$, $p \geq 1$, $i = 0, 1, \cdots, n-1$, there exist positive Borel measures $S_{p,i}(K, \cdot)$ on $\mathbb{S}^{n-1}$, such that the $L_p$-mixed quermassintegrals $W_{p,i}(K,L)$ has the following integral representation

$$W_{p,i}(K,L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L^p(v) dSp_i(K,v)$$

(2.4)

for all $L \in \mathcal{K}_o^n$. Here $S_{p,i}(K, \cdot)$ is the $L_p$-mixed surface area measure of $K$. From (2.4), the integral representation of $L_p$-mixed volume $V_p(K,L)$ is given by

$$V_p(K,L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L^p(v) dSp(K,v).$$

(2.5)

From (2.2) and (2.4), we immediately have that for each $K \in \mathcal{K}_o^n$ and $p \geq 1$,

$$W_{p,i}(K,K) = W_i(K).$$

(2.6)

The Minkowski inequality for the $L_p$-mixed quermassintegrals $W_{p,i}$ can be stated that (see [17]):

**Theorem 2.A.** For $K, L \in \mathcal{K}_o^n$, and $p > 1$, $i = 0, 1, \cdots, n-1$, then

$$W_{p,i}(K,L)^{n-i} \geq W_i(K)^{n-i} - p W_i(L)^p,$$

(2.7)

with equality if and only if $K$ and $L$ are dilates.

An immediate consequence of inequality (2.7) is that (see [17])

**Theorem 2.B.** For $K, L \in \mathcal{K}_o^n$, $n-i \neq 0$ and $i = 0, 1, \cdots, n-1$, if for any $Q \in \mathcal{K}_o^n$,

$$W_{p,i}(K,Q) = W_{p,i}(L,Q) \quad \text{or} \quad W_{p,i}(Q,K) = W_{p,i}(Q,L),$$

then $K = L$.

### 3. An equivalent conclusion of $L_p$-mixed projection bodies

In this section, we will give an equivalent conclusion of $L_p$-mixed projection bodies, i.e., we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** From (1.3), we have for all $u \in \mathbb{S}^{n-1}$,

$$h_{\Pi_{p,i}K}^p(u) = \frac{1}{(n+p)c_{n,p}\omega_n} \int_{\mathbb{S}^{n-1}} |u \cdot v|^p dS_{p,i}(K,v)$$

$$= \frac{1}{(n+p)c_{n,p}\omega_n} \int_{\mathbb{S}^{n-1}} |u \cdot (-v)|^p dS_{p,i}(K,-v)$$

$$= \frac{1}{(n+p)c_{n,p}\omega_n} \int_{\mathbb{S}^{n-1}} |u \cdot v|^p dS_{p,i}(-K,v) = h_{\Pi_{p,i}(-K)}^p(u).$$

This yields

$$\Pi_{p,i}K = \Pi_{p,i}(-K).$$

(3.1)
Using (3.1), we know for all $u \in S^{n-1}$,
\[
h_{\Pi_{p,i}K}^p(u) = \frac{1}{2} h_{\Pi_{p,i}K}^p(u) + \frac{1}{2} h_{\Pi_{p,i}(-K)}^p(u)
= \frac{1}{2(n+p)c_{n,p} \omega_n} \int_{S^{n-1}} |u \cdot v|^p |dS_{p,i}(K,v) + dS_{p,i}(-K,v)|
= \frac{1}{2(n+p)c_{n,p} \omega_n} \int_{S^{n-1}} |u \cdot v|^p |dS_{p,i}(K,v) + dS_{p,i}(K,-v)|.
\]
(3.2)

Thus, if $\Pi_{p,i}K = \Pi_{p,i}L$, by (3.2) then for all $u \in S^{n-1}$,
\[
\int_{S^{n-1}} |u \cdot v|^p [dS_{p,i}(K,v) + dS_{p,i}(K,-v) - dS_{p,i}(L,v) - dS_{p,i}(L,-v)] = 0.
\]

Let
\[
\mu(v) = S_{p,i}(K,v) + S_{p,i}(K,-v) - S_{p,i}(L,v) - S_{p,i}(L,-v),
\]
we see that $\mu(v)$ is finite even Borel measure and
\[
\int_{S^{n-1}} |u \cdot v|^p d\mu(v) = 0
\]
for all $u \in S^{n-1}$. Hence $\mu(v) = 0$, i.e.
\[
S_{p,i}(K,v) + S_{p,i}(K,-v) = S_{p,i}(L,v) + S_{p,i}(L,-v)
\]
(3.3)
for all $v \in S^{n-1}$.

But $Q \in \mathcal{K}_s^n$ gives $h_Q(v) = h_{-Q}(v) = h_{-Q}(-v)$ for all $v \in S^{n-1}$, thus by (2.4) we get
\[
W_{p,i}(K,Q) = \frac{1}{n} \int_{S^{n-1}} h_Q^p(v) dS_{p,i}(K,v)
\]
and
\[
W_{p,i}(K,Q) = \frac{1}{n} \int_{S^{n-1}} h_Q^p(-v) dS_{p,i}(K,-v)
= \frac{1}{n} \int_{S^{n-1}} h_Q^p(v) dS_{p,i}(K,-v).
\]
Therefore, combining with (3.3), we have that for $Q \in \mathcal{K}_s^n$,
\[
W_{p,i}(K,Q) = \frac{1}{2n} \int_{S^{n-1}} h_Q^p(v)[dS_{p,i}(K,v) + dS_{p,i}(K,-v)]
= \frac{1}{2n} \int_{S^{n-1}} h_Q^p(v)[dS_{p,i}(L,v) + dS_{p,i}(L,-v)] = W_{p,i}(L,Q).
\]

Conversely, for $Q \in \mathcal{K}_s^n$, let $Q = [-u,u]$ for all $u \in S^{n-1}$, then for all $v \in S^{n-1}$, $h_Q(v) = |u \cdot v|$. Thus
\[
W_{p,i}(K,Q) = \frac{1}{n} \int_{S^{n-1}} h_Q^p(v) dS_{p,i}(K,v)
= \frac{1}{n} \int_{S^{n-1}} |u \cdot v|^p dS_{p,i}(K,v)
= \frac{1}{n} (n+p)c_{n,p} \omega_n h_{\Pi_{p,i}K}^p(u).
\]
From this, if for any \( Q \in \mathcal{K}_s^n \),

\[
W_{p,i}(K, Q) = W_{p,i}(L, Q),
\]
then for all \( u \in S^{n-1} \),

\[
h_{\Pi_{p,i}K}^p(u) = h_{\Pi_{p,i}L}^p(u).
\]
This gives \( \Pi_{p,i}K = \Pi_{p,i}L \). \( \square \)

As an application of Theorems 1.1, we get the following interesting fact.

**THEOREM 3.1.** Let \( i = 0,1,\ldots,n-1 \) and \( n-i \neq p > 1 \). If \( K,L \in \mathcal{K}_s^n \) and \( \Pi_{p,i}K = \Pi_{p,i}L \), then \( K = L \).

**Proof.** Using Theorem 1.1, if \( \Pi_{p,i}K = \Pi_{p,i}L \), then for any \( Q \in \mathcal{K}_s^n \),

\[
W_{p,i}(K, Q) = W_{p,i}(L, Q).
\]
Since \( K,L \in \mathcal{K}_s^n \), thus using Theorem 2.B, we obtain \( K = L \). \( \square \)

4. The Shephard type problems

The Shephard problems for projection bodies were shown in [5]. Ryabogin and Zvavitch in [25] gave the Shephard type problems of \( L_p \)-projection bodies. Recently, Wang and Wan in [33] researched the Shephard type problems for general \( L_p \)-projection bodies. Here we will give the Shephard type problems for the \( L_p \)-mixed projection bodies which are stated by Theorems 1.2–1.3.

**LEMMA 4.1.** If \( K,L \in \mathcal{K}_o^n \), \( p \geq 1 \) and \( i,j = 0,1,\ldots,n-1 \), then

\[
W_{p,i}(K, \Pi_{p,j}L) = W_{p,j}(L, \Pi_{p,i}K). \tag{4.1}
\]

**Proof.** Using formula (2.4) and definition (1.3), we have that

\[
W_{p,i}(K, \Pi_{p,j}L) = \frac{1}{n} \int_{S^{n-1}} h_{\Pi_{p,j}L}^p(u) dS_{p,i}(K, u)
\]
\[
= \frac{1}{n} \int_{S^{n-1}} \frac{1}{(n+p)c_{n,p} \omega_n} \int_{S^{n-1}} |u \cdot v|^p dS_{p,j}(L, v) dS_{p,i}(K, u)
\]
\[
= \frac{1}{n} \int_{S^{n-1}} h_{\Pi_{p,i}K}^p(v) dS_{p,j}(L, v)
\]
\[
= W_{p,j}(L, \Pi_{p,i}K). \quad \square
\]

**Proof of Theorem 1.2.** Since \( \Pi_{p,j}K \subseteq \Pi_{p,i}L \), thus by (2.4) we know for any \( M \in \mathcal{K}_o^n \),

\[
W_{p,j}(M, \Pi_{p,i}K) \leq W_{p,j}(M, \Pi_{p,i}L),
\]
this together with (4.1), then

\[ W_{p,i}(K, \Pi_{p,j}M) \leq W_{p,i}(L, \Pi_{p,j}M). \]

Hence, for any \( L_p \)-mixed projection body \( Q \), taking \( Q = \Pi_{p,j}M \), we get

\[ W_{p,i}(K, Q) \leq W_{p,i}(L, Q), \]

this is (1.7). According to Theorem 2.B, we see that equality holds in (1.7) if and only if \( K = L \) for \( n - i \neq p \), this equality condition implies \( \Pi_{p,i}K = \Pi_{p,i}L \). □

Let \( Q = L \) in Theorem 1.2, and together with the Minkowski’s inequality (2.7) of the \( L_p \)-mixed quermassintegrals, we easily get Theorem 1.A.

Using (4.1), we can prove a reversed form of Theorem 1.2 as follows:

**Theorem 4.1.** Let \( K, L \in \mathcal{K}_o^n \), \( i, j = 0, 1, \ldots, n-1 \) and \( n - i \neq p > 1 \). If for any \( L_p \)-mixed projection body \( Q \),

\[ W_{p,i}(K, Q) \leq W_{p,i}(L, Q), \]  

(4.2)

then

\[ W_j(\Pi_{p,i}K) \leq W_j(\Pi_{p,i}L). \]  

(4.3)

Equality hold in (4.2) and (4.3) if and only if \( K = L \).

**Proof.** Since for any \( L_p \)-mixed projection body \( Q \),

\[ W_{p,i}(K, Q) \leq W_{p,i}(L, Q), \]

thus let \( Q = \Pi_{p,j}M \) (\( j = 0, 1, \ldots, n-1 \)) for any \( M \in \mathcal{K}_o^n \), we have

\[ W_{p,i}(K, \Pi_{p,j}M) \leq W_{p,i}(L, \Pi_{p,j}M), \]

this together with (4.1), then

\[ W_{p,j}(M, \Pi_{p,i}K) \leq W_{p,j}(M, \Pi_{p,i}L). \]

Taking \( M = \Pi_{p,i}L \) in above inequality and using inequality (2.7), we get

\[ W_j(\Pi_{p,i}L) \geq W_{p,j}(\Pi_{p,i}L, \Pi_{p,i}K) \geq W_j(\Pi_{p,i}L)^{\frac{n-p-j}{n-j}} W_j(\Pi_{p,i}K)^{\frac{p}{n-j}}. \]  

(4.4)

According to the equality condition of inequality (2.7), we see that equality holds in second inequality of (4.4) if and only if \( \Pi_{p,j}K \) and \( \Pi_{p,i}L \) are dilates. From (4.4), we give (4.3).

By Theorem 2.B we know that equality holds in (4.2) if and only if \( K = L \) for \( n - i \neq p \), this means that equality holds in first inequality of (4.4) if and only if \( K = L \). But \( K = L \) implies \( \Pi_{p,i}K \) and \( \Pi_{p,i}L \) are dilates, hence equality hold in (4.2) and (4.3) if and only if \( K = L \). □
Proof of Theorem 1.3. Since \( \Pi_{p,i}K = \Pi_{p,i}L \), thus, by Theorem 1.1 we know for any \( Q \in \mathcal{K}_s^n \),

\[
W_{p,i}(K, Q) = W_{p,i}(L, Q).
\] (4.5)

But \( L \in \mathcal{K}_s^n \), then let \( Q = L \) in (4.5), and use (2.6) and inequality (2.7), we have

\[
W_i(L) = W_{p,i}(K, L) \geq W_i(K)^{\frac{n-i-p}{n-i}} W_i(L)^{\frac{p}{n-i}},
\] (4.6)

i.e.,

\[
W_i(K)^{\frac{n-i-p}{n-i}} \leq W_i(L)^{\frac{n-i-p}{n-i}}.
\]

Thus for \( 0 \leq i < n - p \),

\[
W_i(K) \leq W_i(L);
\]

for \( n - p < i < n \),

\[
W_i(K) \geq W_i(L).
\]

This give (1.8) and (1.9).

According to the equality condition of inequality (2.7), we see that equality holds in (4.6) if and only if \( K \) and \( L \) are dilates. Therefore, let \( K = \lambda L \), by \( \Pi_{p,i}K = \Pi_{p,i}L \) and (1.5) we see \( \lambda = 1 \), i.e., \( K = L \). Hence, equality hold in (1.8) and (1.9) if and only if \( K = L \). \( \square \)

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Weidong Wang
Department of Mathematics
China Three Gorges University
Yichang, 443002, China
e-mail: wdwxh722@163.com

Xiaoyan Wan
Department of Mathematics
China Three Gorges University
Yichang, 443002, China
e-mail: wdwxh722@163.com