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L_p–MIXED PROJECTION BODIES AND L_p–MIXED QUERMASSINTEGRALS

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Abstract. In this paper, we research the L_p-mixed projection bodies by the L_p-mixed quermassintegrals. First, we give an equivalent conclusion of L_p-mixed projection bodies. Further, the Shephard type problem for the L_p-mixed projection bodies are shown.

1. Introduction

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the class of origin-symmetric convex bodies, write $\mathscr{K}_o^n$ and $\mathscr{K}_s^n$, respectively. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, denote by $V(K)$ the n-dimensional volume of body K. For the standard unit ball B in $\mathbb{R}^n$, denote $\omega_n = V(B)$.

If $K \in \mathscr{K}^n$, then its support function, $h_K = h(K,\cdot)$, is defined by (see [5])

$$h(K,x) = \max\{x \cdot y : y \in K\}, \qquad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

For each $K \in \mathscr{K}^n$, the projection body, ΠK, of K is an origin-symmetric convex body whose support function is defined by (see [5, 27])

$$h_{\Pi K}(u) = \frac{1}{2}\int_{S^{n-1}} |\, u \cdot v \,|\, dS(K,v)$$

for all $u \in S^{n-1}$, where $S(K,\cdot)$ is the surface area measure of K on S^{n-1}. The projection body is a very important object in the Brunn-Minkowski theory. During past four decades, a number of important results regarding classical projection bodies were obtained (see [1, 2, 3, 5, 6, 9, 10, 12, 14, 15, 16, 21, 23, 24, 26, 27, 34]).

The notion of the projection body was extended to mixed projection body by Lutwak (see [12, 14]). For each $K \in \mathscr{K}^n$, the mixed projection body, $\Pi_i K$ $(i = 0, 1, \cdots, n-1)$, of K is origin-symmetric convex body whose support function is defined by

$$h_{\Pi_i K}(u) = \frac{1}{2}\int_{S^{n-1}} |\, u \cdot v \,|\, dS_i(K,v) \tag{1.1}$$

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for all $u \in S^{n-1}$, where $S_i(K,\cdot)$ $(i=0,1,\cdots,n-1)$ is the mixed surface area measure of K on S^{n-1}. Obviously, $\Pi_0 K = \Pi K$.

The projection bodies were extended to L_p-space by Lutwak, Yang and Zhang. They (see [18]) introduced the notion of L_p-projection body as follows: For $K \in \mathscr{K}_o^n$ and real number $p \geqslant 1$, the L_p-projection body, $\Pi_p K$, of K is origin-symmetric convex body whose support function is given by

$$h^p_{\Pi_p K}(u) = \frac{1}{(n+p)c_{n,p}\omega_n} \int_{S^{n-1}} |u\cdot v|^p \, dS_p(K,v) \tag{1.2}$$

for all $u \in S^{n-1}$, where

$$c_{n,p} = \omega_{n+p}/\omega_2\omega_n\omega_{p-1}.$$

The positive Borel measure $S_p(K,\cdot)$ on S^{n-1} is called the L_p-surface area measure of K, and has Radon-Nikodym derivative

$$\frac{dS_p(K,\cdot)}{dS(K,\cdot)} = h(K,\cdot)^{1-p}.$$

The unusual normalization of definition (1.2) is chosen so that for the unit ball B, we have $\Pi_p B = B$. In particular, for $p=1$, $\Pi_1 K$ is the classical projection body ΠK of K under the normalization of (1.2), and $\Pi B = B$, rather than the $\omega_{n-1}B$ (see [18]).

L_p-projection bodies extended the classical projection bodies from the Brunn-Minkowski theory to the L_p-Brunn-Minkowski theory. The studies of L_p-projection bodies have received considerable attention, except see [18], for example also see [7, 8, 11, 19, 20, 25, 28, 29, 30, 31, 32, 33].

Similar to the definition of L_p-projection body, Wang and Leng in [29] gave the definition of L_p-mixed projection body as follows: For each $K \in \mathscr{K}_o^n$, real $p \geqslant 1$ and $i = 0,1,\cdots,n-1$, the L_p-mixed projection body, $\Pi_{p,i}K$, of K is origin-symmetric convex body whose support function is defined by

$$h^p_{\Pi_{p,i} K}(u) = \frac{1}{(n+p)c_{n,p}\omega_n} \int_{S^{n-1}} |u\cdot v|^p dS_{p,i}(K,v) \tag{1.3}$$

for all $u \in S^{n-1}$. Here the positive Borel measure $S_{p,i}(K,\cdot)$ $(i=0,1,\cdots,n-1)$ on S^{n-1} is called the L_p-mixed surface area measure of K which was introduced by Lutwak (see [17]). It turns out that the measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to $S_i(K,\cdot)$, and has Radon-Nikodym derivative

$$\frac{dS_{p,i}(K,\cdot)}{dS_i(K,\cdot)} = h^{1-p}(K,\cdot). \tag{1.4}$$

The case $i=0$, $S_{p,0}(K,\cdot)$ is just L_p-surface area measure $S_p(K,\cdot)$. The unusual normalization of definition (1.3) is chosen so that for the unit ball B, we have $\Pi_{p,i}B = B$. Note that for $p=1$, $\Pi_{1,i}K$ is the classical mixed projection body $\Pi_i K$ of K under the normalization of (1.3).

From (1.3), if $i=0$, then $\Pi_{p,0}K = \Pi_p K$. This means that L_p-mixed projection body is an extension of L_p-projection body in the L_p-Brunn-Minkowski theory.

According to (1.3) and (1.4), we easily know that for $\lambda > 0$ and $n-i \neq p \geqslant 1$,

$$\Pi_{p,i}\lambda K = \lambda^{\frac{n-i-p}{p}}\Pi_{p,i}K. \tag{1.5}$$

In this paper, we continuously research the L_p-mixed projection bodies. First, associated with L_p-mixed quermassintegrals (see [17]), we give an equivalent conclusion of the L_p-mixed projection bodies as follows:

THEOREM 1.1. *If $K,L \in \mathscr{K}_o^n$, $p \geqslant 1$ and $i = 0,1,\cdots,n-1$, then*

$$\Pi_{p,i}K = \Pi_{p,i}L \iff W_{p,i}(K,Q) = W_{p,i}(L,Q), \tag{1.6}$$

for any $Q \in \mathscr{K}_s^n$.

Here $W_{p,i}(M,N)$ $(i = 0,1,\cdots,n-1)$ denotes the L_p-mixed quermassintegrals of M and N, $W_{p,0}(M,N)$ is just the L_p-mixed volume $V_p(M,N)$ (see [17]). Let $i = 0$ in Theorem 1.1, we immediately obtain the following equivalent conclusion of the L_p-projection bodies.

COROLLARY 1.1. *If $K,L \in \mathscr{K}_o^n$, $p \geqslant 1$, then*

$$\Pi_p K = \Pi_p L \iff V_p(K,Q) = V_p(L,Q),$$

for any $Q \in \mathscr{K}_s^n$.

Further, we study the Shephard type problems for the L_p-mixed projection bodies. Recall that Wang and Leng (see [29]) gave an affirmation of the Shephard type problems for the L_p-mixed projection bodies as follows:

THEOREM 1.A. *Let $K,L \in \mathscr{K}_o^n$, $i = 0,1,\cdots,n-1$ and $n-i \neq p > 1$. If L is an L_p-mixed projection body and $\Pi_{p,i}K \subseteq \Pi_{p,i}L$, then for $0 \leqslant i < n-p$,*

$$W_i(K) \leqslant W_i(L);$$

for $n-p < i < n$,

$$W_i(K) \geqslant W_i(L);$$

with equality if and only if $K = L$.

Here $W_i(Q)$ denotes the quermassintegrals of $Q \in \mathscr{K}^n$.

Using the L_p-mixed quermassintegrals, we give a general form of Theorem 1.A as follows:

THEOREM 1.2. *Let $K,L \in \mathscr{K}_o^n$, $i = 0,1,\cdots,n-1$ and $n-i \neq p > 1$. If*

$$\Pi_{p,i}K \subseteq \Pi_{p,i}L,$$

then for any L_p-mixed projection body Q,

$$W_{p,i}(K,Q) \leqslant W_{p,i}(L,Q), \tag{1.7}$$

with equality if and only if $K = L$.

Moreover, as the application of Theorem 1.1, we also obtain an improved version of Theorem 1.A.

THEOREM 1.3. *Let $K \in \mathscr{K}_o^n$, $L \in \mathscr{K}_s^n$, $i = 0,1,\cdots,n-1$ and $n-i \neq p > 1$. If $\Pi_{p,i}K = \Pi_{p,i}L$, then for $0 \leqslant i < n-p$,*

$$W_i(K) \leqslant W_i(L); \tag{1.8}$$

for $n-p < i < n$,

$$W_i(K) \geqslant W_i(L). \tag{1.9}$$

Equality hold in (1.8) and (1.9) if and only if $K = L$.

In this paper, the proof of Theorem 1.1 is given in the Section 3; Theorems 1.2–1.3 are proven in the Section 4.

2. L_p-mixed quermassintegrals

For $K,L \in \mathscr{K}^n$ and $\varepsilon > 0$, the Minkowski combination, $K + \varepsilon L \in \mathscr{K}^n$, of K and L is defined by (see [5])

$$h(K+\varepsilon L,\cdot) = h(K,\cdot) + \varepsilon h(L,\cdot).$$

For $p \geqslant 1$, $K,L \in \mathscr{K}_o^n$ and $\varepsilon > 0$, the Firey L_p-combination (also called the L_p-Minkowski combination), $K +_p \varepsilon \cdot L \in \mathscr{K}_o^n$, of K and L is defined by (see [4, 17])

$$h(K +_p \varepsilon \cdot L,\cdot)^p = h(K,\cdot)^p + \varepsilon h(L,\cdot)^p, \tag{2.1}$$

where "$\cdot$" in $\varepsilon \cdot L$ denotes the Firey scalar multiplication.

If $K \in \mathscr{K}^n$, the quermassintegrals, $W_i(K)$ $(i = 0,1,\cdots,n)$, of K is defined by (see [13])

$$W_i(K) = \frac{1}{n}\int_{S^{n-1}} h_K(v)dS_i(K,v). \tag{2.2}$$

Here $S_i(K,\cdot)$ is the mixed surface area measure of K, if $i = 0$, then $S_0(K,\cdot)$ is the surface area measure $S(K,\cdot)$ of K (see [13]).

From definition (2.2), we easily see that

$$W_0(K) = \frac{1}{n}\int_{S^{n-1}} h_K(v)dS(K,v) = V(K). \tag{2.3}$$

Associated with the Firey L_p-combination, Lutwak (see [17]) defined the L_p-mixed quermassintegrals (who are called mixed p-quermassintegrals) as follows: For K,L
$\in \mathscr{K}_o^n$, and real $p \geqslant 1$, the L_p-mixed quermassintegral $W_{p,i}(K,L)$ $(i = 0,1,\cdots,n-1)$ is defined by

$$\frac{n-i}{p}W_{p,i}(K,L) = \lim_{\varepsilon \longrightarrow 0^+}\frac{W_i(K +_p \varepsilon \cdot L) - W_i(K)}{\varepsilon}.$$

Obviously, for $p = 1$, $W_{1,i}(K,L) = W_i(K,L)$ (see [17]). If $i = 0$, by (2.3) then the L_p-mixed quermassintegrals $W_{p,0}(K,L)$ is just the L_p-mixed volume $V_p(K,L)$, namely

$$W_{p,0}(K,L) = V_p(K,L).$$

In [17], Lutwak showed that for each $K \in \mathcal{K}_o^n$, $p \geqslant 1$, $i = 0, 1, \cdots, n-1$, there exist positive Borel measures $S_{p,i}(K, \cdot)$ on S^{n-1}, such that the L_p-mixed quermassintegrals $W_{p,i}(K,L)$ has the following integral representation

$$W_{p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L^p(v)dS_{p,i}(K,v) \tag{2.4}$$

for all $L \in \mathcal{K}_o^n$. Here $S_{p,i}(K,\cdot)$ is the L_p-mixed surface area measure of K. From (2.4), the integral representation of L_p-mixed volume $V_p(K,L)$ is given by

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L^p(v)dS_p(K,v). \tag{2.5}$$

From (2.2) and (2.4), we immediately have that for each $K \in \mathcal{K}_o^n$ and $p \geqslant 1$,

$$W_{p,i}(K,K) = W_i(K). \tag{2.6}$$

The Minkowski inequality for the L_p-mixed quermassintegrals $W_{p,i}$ can be stated that (see [17]):

THEOREM 2.A. *For $K, L \in \mathcal{K}_o^n$, and $p > 1$, $i = 0, 1, \cdots, n-1$, then*

$$W_{p,i}(K,L)^{n-i} \geqslant W_i(K)^{n-i-p} W_i(L)^p, \tag{2.7}$$

with equality if and only if K and L are dilates.

An immediate consequence of inequality (2.7) is that (see [17])

THEOREM 2.B. *For $K, L \in \mathcal{K}_o^n$, $n - i \neq p > 1$ and $i = 0, 1, \cdots, n-1$, if for any $Q \in \mathcal{K}_o^n$,*

$$W_{p,i}(K,Q) = W_{p,i}(L,Q) \quad or \quad W_{p,i}(Q,K) = W_{p,i}(Q,L),$$

then $K = L$.

3. An equivalent conclusion of L_p-mixed projection bodies

In this section, we will give an equivalent conclusion of L_p-mixed projection bodies, i.e., we give the proof of Theorem 1.1.

Proof of Theorem 1.1. From (1.3), we have for all $u \in S^{n-1}$,

$$\begin{aligned} h^p_{\Pi_{p,i}K}(u) &= \frac{1}{(n+p)c_{n,p}\omega_n}\int_{S^{n-1}} |u\cdot v|^p dS_{p,i}(K,v) \\ &= \frac{1}{(n+p)c_{n,p}\omega_n}\int_{S^{n-1}} |u\cdot(-v)|^p dS_{p,i}(K,-v) \\ &= \frac{1}{(n+p)c_{n,p}\omega_n}\int_{S^{n-1}} |u\cdot v|^p dS_{p,i}(-K,v) = h^p_{\Pi_{p,i}(-K)}(u). \end{aligned}$$

This yields

$$\Pi_{p,i}K = \Pi_{p,i}(-K). \tag{3.1}$$

Using (3.1), we know for all $u \in S^{n-1}$,

$$\begin{aligned} h^p_{\Pi_{p,i}K}(u) &= \frac{1}{2}h^p_{\Pi_{p,i}K}(u) + \frac{1}{2}h^p_{\Pi_{p,i}(-K)}(u) \\ &= \frac{1}{2(n+p)c_{n,p}\omega_n}\int_{S^{n-1}} |\, u\cdot v\,|^p [dS_{p,i}(K,v) + dS_{p,i}(-K,v)] \\ &= \frac{1}{2(n+p)c_{n,p}\omega_n}\int_{S^{n-1}} |\, u\cdot v\,|^p [dS_{p,i}(K,v) + dS_{p,i}(K,-v)]. \end{aligned} \tag{3.2}$$

Thus, if $\Pi_{p,i}K = \Pi_{p,i}L$, by (3.2) then for all $u \in S^{n-1}$,

$$\int_{S^{n-1}} |\, u\cdot v\,|^p [dS_{p,i}(K,v) + dS_{p,i}(K,-v) - dS_{p,i}(L,v) - dS_{p,i}(L,-v)] = 0.$$

Let

$$\mu(v) = S_{p,i}(K,v) + S_{p,i}(K,-v) - S_{p,i}(L,v) - S_{p,i}(L,-v),$$

we see that $\mu(v)$ is finite even Borel measure and

$$\int_{S^{n-1}} |\, u\cdot v\,|^p d\mu(v) = 0$$

for all $u \in S^{n-1}$. Hence $\mu(v) = 0$, i.e.

$$S_{p,i}(K,v) + S_{p,i}(K,-v) = S_{p,i}(L,v) + S_{p,i}(L,-v) \tag{3.3}$$

for all $v \in S^{n-1}$.

But $Q \in \mathscr{K}^n_s$ gives $h_Q(v) = h_{-Q}(v) = h_Q(-v)$ for all $v \in S^{n-1}$, thus by (2.4) we get

$$W_{p,i}(K,Q) = \frac{1}{n}\int_{S^{n-1}} h^p_Q(v)dS_{p,i}(K,v)$$

and

$$\begin{aligned} W_{p,i}(K,Q) &= \frac{1}{n}\int_{S^{n-1}} h^p_Q(-v)dS_{p,i}(K,-v) \\ &= \frac{1}{n}\int_{S^{n-1}} h^p_Q(v)dS_{p,i}(K,-v). \end{aligned}$$

Therefore, combining with (3.3), we have that for $Q \in \mathscr{K}^n_s$,

$$\begin{aligned} W_{p,i}(K,Q) &= \frac{1}{2n}\int_{S^{n-1}} h^p_Q(v)[dS_{p,i}(K,v) + dS_{p,i}(K,-v)] \\ &= \frac{1}{2n}\int_{S^{n-1}} h^p_Q(v)[dS_{p,i}(L,v) + dS_{p,i}(L,-v)] = W_{p,i}(L,Q). \end{aligned}$$

Conversely, for $Q \in \mathscr{K}^n_s$, let $Q = [-u,u]$ for all $u \in S^{n-1}$, then for all $v \in S^{n-1}$, $h_Q(v) = |\, u\cdot v\,|$. Thus

$$\begin{aligned} W_{p,i}(K,Q) &= \frac{1}{n}\int_{S^{n-1}} h^p_Q(v)dS_{p,i}(K,v) \\ &= \frac{1}{n}\int_{S^{n-1}} |\, u\cdot v\,|^p\, dS_{p,i}(K,v) \\ &= \frac{1}{n}(n+p)c_{n,p}\omega_n h^p_{\Pi_{p,i}K}(u). \end{aligned}$$

From this, if for any $Q \in \mathscr{K}_s^n$,

$$W_{p,i}(K,Q) = W_{p,i}(L,Q),$$

then for all $u \in S^{n-1}$,

$$h^p_{\Pi_{p,i}K}(u) = h^p_{\Pi_{p,i}L}(u).$$

This gives $\Pi_{p,i}K = \Pi_{p,i}L$. $\square$

As an application of Theorems 1.1, we get the following interesting fact.

THEOREM 3.1. *Let $i = 0,1,\cdots,n-1$ and $n-i \neq p > 1$. If $K,L \in \mathscr{K}_s^n$ and $\Pi_{p,i}K = \Pi_{p,i}L$, then $K = L$.*

Proof. Using Theorem 1.1, if $\Pi_{p,i}K = \Pi_{p,i}L$, then for any $Q \in \mathscr{K}_s^n$,

$$W_{p,i}(K,Q) = W_{p,i}(L,Q).$$

Since $K,L \in \mathscr{K}_s^n$, thus using Theorem 2.B, we obtain $K = L$. $\square$

4. The Shephard type problems

The Shephard problems for projection bodies were shown in [5]. Ryabogin and Zvavitch in [25] gave the Shephard type problems of L_p-projection bodies. Recently, Wang and Wan in [33] researched the Shephard type problems for general L_p-projection bodies. Here we will give the Shephard type problems for the L_p-mixed projection bodies which are stated by Theorems 1.2–1.3.

LEMMA 4.1. *If $K,L \in \mathscr{K}_o^n$, $p \geqslant 1$ and $i,j = 0,1,\cdots,n-1$, then*

$$W_{p,i}(K,\Pi_{p,j}L) = W_{p,j}(L,\Pi_{p,i}K). \tag{4.1}$$

Proof. Using formula (2.4) and definition (1.3), we have that

$$\begin{aligned} W_{p,i}(K,\Pi_{p,j}L) &= \frac{1}{n}\int_{S^{n-1}} h^p_{\Pi_{p,j}L}(u)dS_{p,i}(K,u) \\ &= \frac{1}{n}\int_{S^{n-1}} \frac{1}{(n+p)c_{n,p}\omega_n}\int_{S^{n-1}} |\, u\cdot v\,|^p dS_{p,j}(L,v)dS_{p,i}(K,u) \\ &= \frac{1}{n}\int_{S^{n-1}} h^p_{\Pi_{p,i}K}(v)dS_{p,j}(L,v) \\ &= W_{p,j}(L,\Pi_{p,i}K). \quad \square \end{aligned}$$

Proof of Theorem 1.2. Since $\Pi_{p,i}K \subseteq \Pi_{p,i}L$, thus by (2.4) we know for any $M \in \mathscr{K}_o^n$,

$$W_{p,j}(M,\Pi_{p,i}K) \leqslant W_{p,j}(M,\Pi_{p,i}L),$$

this together with (4.1), then

$$W_{p,i}(K,\Pi_{p,j}M) \leqslant W_{p,i}(L,\Pi_{p,j}M).$$

Hence, for any L_p-mixed projection body Q, taking $Q=\Pi_{p,j}M$, we get

$$W_{p,i}(K,Q) \leqslant W_{p,i}(L,Q),$$

this is (1.7). According to Theorem 2.B, we see that equality holds in (1.7) if and only if $K=L$ for $n-i\neq p$, this equality condition implies $\Pi_{p,i}K=\Pi_{p,i}L$. □

Let $Q=L$ in Theorem 1.2, and together with the Minkowski's inequality (2.7) of the L_p-mixed quermassintegrals, we easily get Theorem 1.A.

Using (4.1), we can prove a reversed form of Theorem 1.2 as follows:

THEOREM 4.1. *Let $K,L\in\mathscr{K}_o^n$, $i,j=0,1,\cdots,n-1$ and $n-i\neq p>1$. If for any L_p-mixed projection body Q,*

$$W_{p,i}(K,Q) \leqslant W_{p,i}(L,Q), \tag{4.2}$$

then

$$W_j(\Pi_{p,i}K) \leqslant W_j(\Pi_{p,i}L). \tag{4.3}$$

Equality hold in (4,2) and (4.3) if and only if $K=L$.

Proof. Since for any L_p-mixed projection body Q,

$$W_{p,i}(K,Q) \leqslant W_{p,i}(L,Q),$$

thus let $Q=\Pi_{p,j}M$ ($j=0,1,\cdots,n-1$) for any $M\in\mathscr{K}_o^n$, we have

$$W_{p,i}(K,\Pi_{p,j}M) \leqslant W_{p,i}(L,\Pi_{p,j}M),$$

this together with (4.1), then

$$W_{p,j}(M,\Pi_{p,i}K) \leqslant W_{p,j}(M,\Pi_{p,i}L).$$

Taking $M=\Pi_{p,i}L$ in above inequality and using inequality (2.7), we get

$$W_j(\Pi_{p,i}L) \geqslant W_{p,j}(\Pi_{p,i}L,\Pi_{p,i}K) \geqslant W_j(\Pi_{p,i}L)^{\frac{n-p-j}{n-j}} W_j(\Pi_{p,i}K)^{\frac{p}{n-j}}. \tag{4.4}$$

According to the equality condition of inequality (2.7), we see that equality holds in second inequality of (4.4) if and only if $\Pi_{p,i}K$ and $\Pi_{p,i}L$ are dilates. From (4.4), we give (4.3).

By Theorem 2.B we know that equality holds in (4.2) if and only if $K=L$ for $n-i\neq p$, this means that equality holds in first inequality of (4.4) if and only if $K=L$. But $K=L$ implies $\Pi_{p,i}K$ and $\Pi_{p,i}L$ are dilates, hence equality hold in (4.2) and (4.3) if and only if $K=L$. □

Proof of Theorem 1.3. Since $\Pi_{p,i}K = \Pi_{p,i}L$, thus, by Theorem 1.1 we know for any $Q \in \mathscr{K}_s^n$,

$$W_{p,i}(K,Q) = W_{p,i}(L,Q). \tag{4.5}$$

But $L \in \mathscr{K}_s^n$, then let $Q = L$ in (4.5), and use (2.6) and inequality (2.7), we have

$$W_i(L) = W_{p,i}(K,L) \geqslant W_i(K)^{\frac{n-i-p}{n-i}} W_i(L)^{\frac{p}{n-i}}, \tag{4.6}$$

i.e.,

$$W_i(K)^{\frac{n-i-p}{n-i}} \leqslant W_i(L)^{\frac{n-i-p}{n-i}}.$$

Thus for $0 \leqslant i < n-p$,

$$W_i(K) \leqslant W_i(L);$$

for $n-p < i < n$,

$$W_i(K) \geqslant W_i(L).$$

This give (1.8) and (1.9).

According to the equality condition of inequality (2.7), we see that equality holds in (4.6) if and only if K and L are dilates. Therefore, let $K = \lambda L$, by $\Pi_{p,i}K = \Pi_{p,i}L$ and (1.5) we see $\lambda = 1$, i.e., $K = L$. Hence, equality hold in (1.8) and (1.9) if and only if $K = L$. $\square$

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