Journal of
Mathematical
Inequalities
Volume 8, Number 4 (2014), 889–897 doi:10.7153/jmi-08-67

SOME NORM INEQUALITIES FOR COMMUTATORS WITH SYMBOL FUNCTION IN MORREY SPACES

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(*Communicated by J. Pečarić*)

Abstract. A version of Dini condition is introduced. Some boundedness of commutators with symbol functions belong to Morrey spaces are discussed. The results can be seen as continuity of our recent work [28].

1. Introduction and main results

Let $-n/p \leqslant \beta < 1$ and $1 \leqslant p < \infty$. Then the Morrey-Campanato space (called Campanato for simplicity) $C^{p,\beta}(\mathbb{R}^n)$ was defined by the norm

$$\|f\|_{C^{p,\beta}(\mathbb{R}^n)} = \sup_B \|f\|_{C^{p,\beta}(B)} := \sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \left(\frac{1}{|B|} \int_B |f - f_B|^p dx \right)^{1/p},$$

where $f_B = \frac{1}{|B|}\int_B f(x)dx$, B denotes any ball contained in $\mathbb{R}^n$ and $|B|$ is the Lebesgue measure of B. Campanato spaces are useful tools in the regularity theory of PDEs as a result of their better structures, which allow to give an integral characterization of the spaces of Hölder continuous functions. This leads to a generalization of the classical Sobolev embedding theorems, some of this work, see [18] and [19] for example. It is also well known that $C^{1,1/p-1}$ is the dual space of Hardy space $H^p(\mathbb{R}^n)$ $(0 < p < 1)$ (see [31]). For a recent account of the theory on $C^{p,\beta}(\mathbb{R}^n)$, we refer the reader to [12], [24], [33] and the references therein. Combining the statements in [7], [24] and [31], we have

$$C^{p,\beta}(\mathbb{R}^n) \begin{cases} = BMO(\mathbb{R}^n), & \beta = 0; \\ = Lip_\beta(\mathbb{R}^n), & 0 < \beta < 1; \\ \supset M^{p,\beta}(\mathbb{R}^n), & -n/p \leqslant \beta < 0, \end{cases}$$

where $BMO(\mathbb{R}^n)$ is the bounded mean oscillation space with the norm

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_B \frac{1}{|B|}\int_B |b - b_B| dx,$$

Mathematics subject classification (2010): Primary 42B20; Secondary 42B25.
Keywords and phrases: Morrey space, commutator, rough kernel.
This work was partially supported by NSF of China (Grant Nos. 10931001, 11271175, 11301249), NSF of Shandong Province (Grant No. ZR2012AQ026), the Fundamental Research Funds for the Central Universities (Grant No. 2012CXQT09), the AMEP and DYSP of Linyi University and the Key Laboratory of Mathematics and Complex System (Beijing Normal University), Ministry of Education, China.

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Paper JMI-08-67

$Lip_\beta(\mathbb{R}^n)$ is the Lipschitz space with the equivalent norm

$$\|f\|_{Lip_\beta(\mathbb{R}^n)} \approx \sup_B \frac{1}{|B|^{1+\frac{\beta}{n}}} \int_B |f(x)-f_B|dx \approx \sup_B \left(\frac{1}{|B|^{1+\frac{q\beta}{n}}} \int_B |f(x)-f_B|^q dx\right)^{1/q},$$

for $1 \leqslant q \leqslant \infty$ and $M^{p,\beta}(\mathbb{R}^n)$ is the Morrey space which original form was first introduced by Morrey [23] to investigate the local behavior of solutions to the second order elliptic PDE:

$$\|f\|_{M^{p,\beta}(\mathbb{R}^n)} = \sup_B \|f\|_{M^{p,\beta}(B)} = \sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \left(\frac{1}{|B|}\int_B |f(x)|^p dx\right)^{1/p}.$$

Let b be a locally integrable function on $\mathbb{R}^n$ and let T be an integral operator. Then the commutator operator formed by T and b was denoted by

$$T_b(f) := bTf - T(bf).$$

The function b was also called the symbol function of T_b. The investigation of the operator T_b begin with Calderón-Zygmund pioneering study of the operator T (see [3] and [5]). They found that the theory of commutators play an important role in studying the regularity of solutions to elliptic PDEs of the second order. The well-posedness of solutions to many PDEs can be attributed to the corresponding commutator's boundedness for singular integral operators. However, this topic exceeds the scope of this paper, for more information about this, see for example [4], [10], [13] and [30]. Especially in [30], the authors simplify the proof of the famous Wu's theorem on Navier-Stokes equations greatly by some estimates of commutators which were obtained by Yan in his Ph.D. thesis [32] (see also Lu and Yan's work in [21]). Since $L^\infty(\mathbb{R}^n) \subsetneq BMO(\mathbb{R}^n)$, the boundedness of T_b is worse than T (for example, the singularity, see also [26]). Therefore, many authors want to know whether T_b shares the similar boundedness with T. Many authors are interested in the study of commutators when the symbol functions b belong to BMO spaces and Lipschitz spaces. For some of this classical works, we refer the reader to [1], [16], [20] and [25].

Compared to the rich and significant results about commutators with symbol functions belong to BMO spaces and Lipschitz spaces, there are some works for the case of Morrey spaces. Recently, in [28], we gave some creative characterizations of Campanato spaces via the boundedness of commutators associated with the Calderón-Zygmund singular integral operator by some new methods instead of the sharp maximal function theorem. However, the ideas used in the characterizations of [28] depend heavily on the smoothness of the corresponding kernel functions. It is well known that the regularity of solutions to some elliptic PDEs with smooth boundary can attribute to the boundedness of corresponding commutators with smooth kernel in some sense. One question arose naturally: What happens when the boundary condition be weakened? The answer to the question need to study the boundedness of commutators with rough kernels, which motivates us to extend our results in [28] to the rough kernel case. Let us first recall some basic definitions about singular integral operators with rough kernels.

Let T_Ω be the Calderón-Zygmund singular integral operator

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy,$$

where Ω satisfies the homogeneous condition of degree 0 and

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0. \tag{1.1}$$

Here, $d\sigma$ is the normalized Lebesgue measure and $x' = x/|x|$. We say a function $\Omega(x')$ on S^{n-1} satisfies a version of L^q-Dini condition if

$$\Omega \in L^q(S^{n-1}), \quad 1 \leqslant q < \infty, \tag{1.2}$$

$$\int_0^1 \frac{w_q(\delta)}{\delta^2} d\delta < \infty, \tag{1.3}$$

where $w_q(\delta)$ is called the integral continuous modulus of Ω with degree q:

$$w_q(\delta) = \sup_{\|\rho\|<\delta} \left(\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^q d\sigma(x') \right)^{1/q},$$

here ρ is a rotation in $\mathbb{R}^n$ and $\|\rho\| = \sup |\rho x' - x'| : x' \in S^{n-1}$. When Ω satisfies some size conditions, the kernel of the operator T_Ω has no regularity, and so the operator T_Ω is called rough singular integral operator. In recent years, a variety of operators related to the singular integrals for Calderón-Zygmund, but lacking the smoothness required in the classical theory, have been studied. For some corresponding works, we refer the reader to [8], [14], [22], [27] and the references therein. The operator $T_{\Omega,b}$, whose kernel has the additional roughness due to the presence of b, was first studied by R. Fefferman ([15]) and subsequently by many other authors, such as M. Christ ([6]) and J. Duoandikoetxea et al. ([11]).

In this paper, highly inspired by the above statements, we set up some boundedness of $T_{\Omega,b}$ with b belongs to Morrey space under the L^q-Dini conditions (1.2)–(1.3).

We are now in a position to state our main results as:

THEOREM 1.1. *Let $-n/p \leqslant \beta < 0, 1 < \max\{q', n/(1-\beta)\} < p < \infty$, $1 < p_i < \infty, -n/p_i \leqslant \beta_i < 0$, $1/p = \sum_{i=1}^2 1/p_i$ and $\beta = \sum_{i=1}^2 \beta_i$, $i = 1,2$. If $b \in C^{p_1,\beta_1}(\mathbb{R}^n)$ and Ω satisfies the L^q-Dini condition, then $T_{\Omega,b}$ is a bounded operator from $M^{p_2,\beta_2}(\mathbb{R}^n)$ to $C^{p,\beta}(\mathbb{R}^n)$.*

THEOREM 1.2. *Let $q' < p < \infty$, $0 < \alpha < 1, -n/p \leqslant \beta < 0$, $1/s = 1/p - \alpha/n$, $b \in Lip_\alpha$ and Ω satisfies the L^q-Dini condition. Then $T_{\Omega,b}$ is a bounded operator from $M^{p,\beta}(\mathbb{R}^n)$ to $M^{s,\alpha+\beta}(\mathbb{R}^n)$.*

Theorem 1.1 and Theorem 1.2 can be seen as extensions of [28, Theorem 1.1 and Theorem 1.2] to the rough kernel case.

REMARK 1.1. Unlike $\beta \geqslant 0$, there are essential difficulties to deal with the case $\beta < 0$. Therefore, we have been working under the assumption that Ω satisfies the L^q-Dini conditions (1.2)–(1.3) instead of the classical L^q-conditions (with (1.3) replaced by $\int_0^1 \frac{w_q(\delta)}{\delta} d\delta < \infty$ or $\int_0^1 \log(\frac{1}{\delta}) \frac{w_q(\delta)}{\delta} d\delta < \infty$). In our judgement, this condition cannot be weakened since in the case of $\beta < 0$, there need a 1 factor contribution to guarantee the series's convergence(see the estimate for the term J_3 in Section 2.1) instead of a small ε $(\varepsilon > 0)$ factor for the case of $\beta \geqslant 0$.

Throughout this paper, for $x_0 \in \mathbb{R}^n$, $r > 0$ and $\lambda > 0$, $B = B(x_0, r)$ denotes the ball centered at x_0 with radius r and $\lambda B = B(x_0, \lambda r)$. C is a constant which may change from line to line. We will prove Theorem 1.1 and Theorem 1.2 in Section 2.

2. Proofs of the main results

2.1. Proof of Theorem 1.1

We begin this subsection with some lemmas about the estimates of operators on Morrey spaces, which will be used in the proofs of our main results.

LEMMA 2.1. [29] Let p, β and Ω be as in Theorem 1.1. Then T_Ω is bounded from $M^{p,\beta}(\mathbb{R}^n)$ to $M^{p,\beta}(\mathbb{R}^n)$.

LEMMA 2.2. [28] Let $p, p_1, p_2, \beta, \beta_1, \beta_2$ and b be as in Theorem 1.1. Then

$$\|(b - b_B) f \chi_B\|_{L^p(\mathbb{R}^n)} \leqslant |B|^{1/p+\beta/n} \|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} \|f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)}.$$

LEMMA 2.3. [28] Suppose that $B_* \subset B$ and $b \in C^{p_1,\beta_1}(\mathbb{R}^n)$ with $1 < p_1 < \infty, -n/p_1 \leqslant \beta_1 < 0$. Then the following estimate holds

$$|b_{B_*} - b_B| \leqslant C \|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} |B_*|^{\beta_1/n}.$$

LEMMA 2.4. [9] *Foe $R > 0$, there exists a constant $C > 0$ such that*

$$\left(\int_{|x-y|<R} |\Omega(x-y)|^q dy \right)^{1/q} \leqslant C R^{n/q} \|\Omega\|_{L^q(S^{n-1})}.$$

The following imbedding theorem for L^p spaces over domains with finite volume is very useful in the estimate of inequalities, which can be found in many books on the Sobolev spaces.

LEMMA 2.5. *Suppose that $|\Omega| = \int_\Omega 1 dx < \infty$ and $1 \leqslant p \leqslant q \leqslant \infty$. If $f \in L^q(\Omega)$, then $f \in L^p(\Omega)$ and*

$$\|f\|_{L^p(\Omega)} \leqslant C |\Omega|^{1/p - 1/q} \|f\|_{L^q(\Omega)}.$$

Inspired by the idea in [17], we can get the following estimates for the kernel $\Omega(x)$ on circles.

LEMMA 2.6. *Suppose that* Ω *satisfies* (1.1) *and the* L^q*-Dini condition* (1.2)–(1.3). *Then for any* $R>0$ *and* $x\in\mathbb{R}^n$, *when* $|x|<R/2$, *there exists constant* $C>0$ *such that*

$$\left(\int_{R<|y|\leqslant 2R}\left|\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(y)}{|y|^n}\right|^q dy\right)^{1/q}\leqslant CR^{-n/q'}\frac{|x|}{R}\left\{1+\int_{\frac{|x|}{R}}^{\frac{2|x|}{R}}\frac{w_q(\delta)}{\delta^2}d\delta\right\}. \quad (2.1)$$

Proof. We conclude from the fact $|x-y|\sim|y|$ that

$$\left|\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(y)}{|y|^n}\right|\leqslant C\left\{|\Omega(y)|\frac{|x|}{|y|^{n+1}}+\frac{|\Omega(y-x)-\Omega(y)|}{|y|^n}\right\},$$

hence that

$$\begin{aligned}&\left(\int_{R<|y|\leqslant 2R}\left|\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(y)}{|y|^n}\right|^q dy\right)^{1/q}\\ &\leqslant C\left(\int_{R<|y|\leqslant 2R}\left|\frac{|\Omega(y)||x|}{|y|^{n+1}}\right|^q dy\right)^{1/q}+C\left(\int_{R<|y|\leqslant 2R}\left|\frac{|\Omega(y-x)-\Omega(y)|}{|y|^n}\right|^q dy\right)^{1/q}\\ &:=I_1+I_2.\end{aligned}$$

It follows from $\Omega\in L^q(S^{n-1})$ and Lemma 2.4 that

$$I_1\leqslant C\|\Omega\|_{L^q(S^{n-1})}|x|R^{-(n+1)}R^{n/q}=CR^{-n/q'}\frac{|x|}{R}.$$

For the term I_2, by the method of rotation, we have that

$$\begin{aligned}I_2&\leqslant C\left(\int_R^{2R}t^{-nq+n-1}\int_{S^{n-1}}|\Omega(ty'-x)-\Omega(y')|^q d\sigma(y')dt\right)^{1/q}\\ &\leqslant CR^{-n/q'}\left(\int_R^{2R}\int_{S^{n-1}}\left|\Omega\left(\frac{y'-\alpha}{|y'-\alpha|}\right)-\Omega(y')\right|^q d\sigma(y')\frac{dt}{t}\right)^{1/q},\end{aligned}$$

where $\alpha=x/t$. Applying a result of Calderón-Weiss-Zygmund (see [2]) to $|\alpha|=|x|/t<1/2$ and $1\leqslant q<\infty$, we conclude that

$$\begin{aligned}\int_{S^{n-1}}\left|\Omega\left(\frac{y'-\alpha}{|y'-\alpha|}\right)-\Omega(y')\right|^q d\sigma(y')&\leqslant C\sup_{\|\rho\|<\alpha}\int_{S^{n-1}}|\Omega(\rho x')-\Omega(x')|^q d\sigma(x')\\ &\leqslant Cw_q^q\left(\frac{|x|}{R}\right).\end{aligned}$$

Here we use the property that $w(\delta)$ is nondecreasing in δ. Thus

$$\begin{aligned}I_2&\leqslant CR^{-n/q'}w_q\left(\frac{|x|}{R}\right)\left(\int_R^{2R}t^{-1}dt\right)^{1/q}\leqslant C'R^{-n/q'}w_q\left(\frac{|x|}{R}\right)(\ln 2)^{1/q}\\ &\leqslant C'R^{-n/q'}w_q\left(\frac{|x|}{R}\right)\int_{\frac{|x|}{R}}^{\frac{2|x|}{R}}\frac{d\delta}{\delta}\leqslant C'R^{-n/q'}\frac{|x|}{R}\int_{\frac{|x|}{R}}^{\frac{2|x|}{R}}w_q(\delta)\frac{2d\delta}{\delta^2}\\ &\leqslant C''R^{-n/q'}\frac{|x|}{R}\int_{\frac{|x|}{R}}^{\frac{2|x|}{R}}w_q(\delta)/\delta^2 d\delta.\end{aligned}$$

On account of the above remarks, we have obtained (2.1). $\square$

Now, we come to the proof of Theorem 1.1. For a ball $B=B(x_0,r)\subset\mathbb{R}^n$. Take $f\in M^{p_2,\beta_2}(\mathbb{R}^n)$ and $f_1=f\chi_{2B}$, $f_2=f-f_1$. After noticing $T_{\Omega,b}f=T_{\Omega,(b-b_B)}f$, we have

$$\begin{aligned}
&\left(\frac{1}{|B|^{1+p\beta/n}}\int_B|T_{\Omega,b}f(y)-(T_{\Omega,b}f)_B|^p dy\right)^{1/p}\\
&=\left(\frac{1}{|B|^{1+p\beta/n}}\int_B|T_{\Omega,(b-b_B)}f(y)-(T_{\Omega,(b-b_B)}f)_B|^p dy\right)^{1/p}\\
&\leqslant\left(\frac{1}{|B|^{1+p\beta/n}}\int_B|T_{\Omega,(b-b_B)}f(y)-T_\Omega(b-b_B)f_2(x_0)|^p dy\right)^{1/p}\\
&\leqslant\left(\frac{1}{|B|^{1+p\beta/n}}\int_B|(b-b_B)T_\Omega f(y)|^p dy\right)^{1/p}\\
&+\left(\frac{1}{|B|^{1+p\beta/n}}\int_B|T_\Omega(b-b_B)f_1(y)|^p dy\right)^{1/p}\\
&\quad+\left(\frac{1}{|B|^{1+p\beta/n}}\int_B|(T_\Omega(b-b_B)f_2)(y)-(T_\Omega(b-b_B)f_2)(x_0)|^p\, dy\right)^{1/p}\\
&:=J_1+J_2+J_3.
\end{aligned}$$

Combining Hölder's inequality with Lemma 2.1, we can assert that

$$\begin{aligned}
J_1&\leqslant\frac{1}{|B|^{1/p+\beta/n}}\left(\int_B|b(y)-b_B|^{p_1}dy\right)^{1/p_1}\left(\int_B|T_\Omega f(y)|^{p_2}dy\right)^{1/p_2}\\
&\leqslant C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|T_\Omega f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)}\\
&\leqslant C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)}.
\end{aligned}$$

Lemma 2.2 shows that we can estimate J_2 as

$$J_2\leqslant\frac{1}{|B|^{1/p+\beta/n}}\|(b-b_B)f_1\|_{L^p}\leqslant C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)}.$$

We now turn to the estimate for the term J_3. It is easy to check that

$$\begin{aligned}
&|(T_\Omega(b-b_B)f_2)(y)-T_\Omega((b-b_B)f_2)(x_0)|\\
&=\left|\int_{R^n}\left(\frac{\Omega(y-z)}{|y-z|^n}-\frac{\Omega(x_0-z)}{|x_0-z|^n}\right)(b(z)-b_B)f_2(z)dz\right|\\
&\leqslant C\sum_{k=2}^{\infty}\int_{(2^kB)\setminus(2^{k-1}B)}\left|\frac{\Omega(y-z)}{|y-z|^n}-\frac{\Omega(x_0-z)}{|x_0-z|^n}\right|(|b(z)-b_{2^kB}|+|b_B-b_{2^kB}|)|f(z)|dz\\
&:=K_1+K_2.
\end{aligned}$$

By Lemma 2.5, we have

$$
\begin{aligned}
K_1 &\leqslant \sum_{k=2}^{\infty}\left(\int_{(2^kB)\setminus(2^{k-1}B)}\left|\frac{\Omega(y-z)}{|y-z|^n}-\frac{\Omega(x_0-z)}{|x_0-z|^n}\right|^q dz\right)^{1/q}\left(\int_{2^kB}|(b-b_{2^kB})f(z)|^{q'}dz\right)^{1/q'}\\
&\leqslant \sum_{k=2}^{\infty}\frac{1}{2^k}\left(1+\int_{1/2^k}^{1/2^{k-1}}\frac{w_q(\delta)}{\delta^2}d\delta\right)\left(\frac{1}{|2^kB|}\int_{2^kB}|(b(z)-b_{2^kB})f(z)|^{q'}dz\right)^{1/q'}\\
&\leqslant C\sum_{k=2}^{\infty}\frac{1}{2^k}\left\|M(|(b-b_{2^kB})f|^{q'})^{1/q'}\right\|_{L^p}
\end{aligned}
$$

and

$$
\begin{aligned}
K_2 &\leqslant \sum_{k=2}^{\infty}\left(\int_{2^kB\setminus 2^{k-1}B}\left|\frac{\Omega(y-z)}{|y-z|^n}-\frac{\Omega(x_0-z)}{|x_0-z|^n}\right|^q dz\right)^{1/q}\left(\int_{2^kB}|(b_B-b_{2^kB})f(z)|^{q'}dz\right)^{1/q'}\\
&\leqslant C\sum_{k=2}^{\infty}\frac{1}{2^k}\left(\frac{1}{|2^kB|}\int_{2^kB}|(b_B-b_{2^kB})f(z)|^{q'}dz\right)^{1/q'}.
\end{aligned}
$$

Having disposed of this preliminary step, we can now obtain that

$$
\begin{aligned}
J_3 &\leqslant \frac{1}{|B|^{1/p+\beta/n}}\sum_{k=2}^{\infty}\frac{1}{2^k}\|M(|(b-b_{2^kB})f|^{q'})^{1/q'}\|_{L^p}\\
&\quad+\frac{1}{|B|^{1/p+\beta/n}}\sum_{k=2}^{\infty}\frac{1}{2^k}\left(\int_B\left|\left(\frac{1}{|2^kB|}\int_{2^kB}|b_B-b_{2^kB}|f(z)||^{q'}dz\right)^{1/q'}\right|^p dy\right)^{1/p}\\
&\leqslant \frac{1}{|B|^{1/p+\beta/n}}\sum_{k=2}^{\infty}\frac{1}{2^k}\|(b-b_{2^kB})f\|_{L^p}+\frac{1}{|B|^{1/p+\beta/n}}\sum_{k=2}^{\infty}\frac{1}{2^k}\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|(M(|f|^{q'}))^{1/q'}\|_{L^p}\\
&\leqslant C\sum_{k=2}^{\infty}\frac{1}{2^{k(1-1/p-\beta/n)}}\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)}\\
&\quad+C\sum_{k=2}^{\infty}\frac{1}{2^{k(1-1/p-\beta_2/n)}}\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)}\\
&\leqslant C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)},
\end{aligned}
$$

where we have used Lemma 2.3 and Lemma 2.5. We have thus proved Theorem 1.1. □

2.2. Proof of Theorem 1.2

After noticing

$$
\begin{aligned}
|T_{\Omega,b}f(x)| &\leqslant \int_{\mathbb{R}^n}|b(x)-b(y)|\left|\frac{\Omega(x-y)}{|x-y|^n}\right||f(y)|dy\\
&\leqslant \int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^{n-\alpha}}dy\\
&\leqslant I_\alpha(|f(x)|),
\end{aligned}
$$

the proof of Theorem 1.2 is a by-product of the following lemma

LEMMA 2.7. [29] Let p, s, β, α and Ω be as in Theorem 1.2. Then I_α is bounded from $M^{p,\beta}(\mathbb{R}^n)$ to $M^{s,\alpha+\beta}(\mathbb{R}^n)$.

Acknowledgements. The authors thank the anonymous referees cordially for their valuable suggestions on this paper.

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(Received July 31, 2013)

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