BEST POSSIBLE INEQUALITIES BETWEEN GENERALIZED LOGARITHMIC MEAN AND WEIGHTED GEOMETRIC MEAN OF GEOMETRIC, SQUARE–ROOT, AND ROOT–SQUARE MEANS

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Abstract. We establish two optimal double inequalities among generalized logarithmic mean $L_p(a,b)$, geometric mean $G(a,b)$, square-root $N(a,b)$, and root-square mean $S(a,b)$.

1. Introduction

For $p \in R$, the generalized logarithmic mean $L_p(a,b)$ of two positive numbers a and b with $a \neq b$ is defined as follow:

$$L_p(a,b) = \begin{cases} \left[\dfrac{a^{p+1} - b^{p+1}}{(p+1)(a-b)} \right]^{1/p}, & p \neq 0, \ p \neq -1, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a} \right)^{1/(b-a)}, & p = 0, \\ \dfrac{a-b}{\log a - \log b}, & p = -1. \end{cases} \quad (1.1)$$

It is well known that $L_p(a,b)$ is continuous and strictly increasing with respect to $p \in R$ for fixed $a,b > 0$ with $a \neq b$.

Recently, the generalized logarithmic mean has been the subject of intensive research. Many remarkable inequalities and monotonicity results for the generalized logarithmic mean can be found in the literature [1, 4, 5, 7, 11, 16–19]. It might be surprising that the generalized logarithmic mean has applications in physics, economics, and even in meteorology [8, 13–15]. Let

$$H(a,b) = \frac{2ab}{a+b}, \ \ G(a,b) = \sqrt{ab}, \ \ N(a,b) = (\tfrac{\sqrt{a}+\sqrt{b}}{2})^2, \ \ L(a,b) = \tfrac{a-b}{\log a - \log b},$$
$$I(a,b) = \tfrac{1}{e}(\tfrac{b^b}{a^a})^{1/(b-a)}, \ \ A(a,b) = \tfrac{a+b}{2}, \ \ and \ \ S(a,b) = \sqrt{\tfrac{a^2+b^2}{2}} \quad (1.2)$$

be the harmonic, geometric, square-root, logarithmic, identric, arithmetic, and root-square means of two positive real numbers a and b with $a \neq b$, respectively. Then

$$\min\{a,b\} < H(a,b) < G(a,b) = L_{-2}(a,b) < L(a,b) = L_{-1}(a,b) < N(a,b)$$
$$= L_{-\frac{1}{2}}(a,b) < I(a,b) = L_0(a,b) < A(a,b) = L_1(a,b) < S(a,b) < \max\{a,b\}. \tag{1.3}$$

For $p \in R$, the pth power mean $M_p(a,b)$ of two positive numbers a and b with $a \neq b$ is defined by

$$M_p(a,b) = \begin{cases} (\dfrac{a^p + b^p}{2})^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.4}$$

In [2], Alzer and Janous established the following sharp double inequality (see also [3], page 350):

$$M_{\frac{\log 2}{\log 3}} \leqslant \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b) \leqslant M_{\frac{2}{3}}(a,b) \tag{1.5}$$

for all real numbers $a, b > 0$.

In [12], Mao proved

$$M_{\frac{1}{3}} \leqslant \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b) \leqslant M_{\frac{1}{2}}(a,b) \tag{1.6}$$

for all real numbers $a, b > 0$, and the constant $\frac{1}{3}$ in the left side inequality cannot be improved.

In [6, 9, 20], the authors presented the bounds for $L(a,b)$ and $I(a,b)$ in terms of $A(a,b)$ and $G(a,b)$ as follows:

$$G^{\frac{2}{3}}(a,b) A^{\frac{1}{3}}(a,b) < L(a,b) < \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b),$$
$$\frac{1}{3}G(a,b) + \frac{2}{3}A(a,b) < I(a,b) \tag{1.7}$$

for all $a, b > 0$ with $a \neq b$.

In [10], Long and Chu presented the bounds for $\alpha A(a,b) + (1-\alpha)G(a,b)$ in term of $L_p(a,b)$.

THEOREM A. *Let $\alpha \in (0,1)$ and $a, b > 0$ with $a \neq b$, then*
(1) $L_{3\alpha-2}(a,b) = \alpha A(a,b) + (1-\alpha)G(a,b)$ for $\alpha = \frac{1}{2}$;
(2) $L_{3\alpha-2}(a,b) < \alpha A(a,b) + (1-\alpha)G(a,b)$ for $\alpha \in (0, \frac{1}{2})$, and $L_{3\alpha-2}(a,b) > \alpha A(a,b) + (1-\alpha)G(a,b)$ for $\alpha \in (\frac{1}{2}, 1)$, moreover, in each case, the bound $L_{3\alpha-2}(a,b)$ for the sum $\alpha A(a,b) + (1-\alpha)G(a,b)$ is optimal.

The main purpose of this paper is, for $\alpha \in (0,1)$, to present the optimal bounds for $G^\alpha(a,b)N^{1-\alpha}(a,b)$ and $G^\alpha(a,b)S^{1-\alpha}(a,b)$ in terms of $L_p(a,b)$.

2. Lemmas

In order to establish our main results we need seven lemmas, which we present in this section.

LEMMA 1. *Let $t > 1$ and*

$$
\begin{aligned}
g_1(t) = {} & (1-\alpha)(3\alpha-2)t^{3\alpha+1} - 2(1-\alpha)(3\alpha+1)t^{3\alpha} - \alpha(3\alpha+1)t^{3\alpha-1} \\
& + \alpha(3\alpha+1)t^2 + (1-\alpha)(3\alpha+1)t - (1-\alpha)(3\alpha-2).
\end{aligned}
\tag{2.1}
$$

Then

(1) $g_1(t) > 0$ for $\alpha \in (0,\frac{1}{3}) \cup (\frac{2}{3}, 1)$, and

(2) $g_1(t) < 0$ for $\alpha \in (\frac{1}{3}, \frac{2}{3})$.

Proof. Simple computations yield

$$
\begin{aligned}
g_1'(t) = {} & (3\alpha+1)[(1-\alpha)(3\alpha-2)t^{3\alpha} - 6\alpha(1-\alpha)t^{3\alpha-1} \\
& - \alpha(3\alpha-1)t^{3\alpha-2} + 2\alpha t + (1-\alpha)],
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
g_1''(t) = {} & \alpha(3\alpha+1)[3(1-\alpha)(3\alpha-2)t^{3\alpha-1} - 6(1-\alpha)(3\alpha-1)t^{3\alpha-2} \\
& - (3\alpha-1)(3\alpha-2)t^{3\alpha-3} + 2],
\end{aligned}
\tag{2.3}
$$

$$
g_1(1) = g_1'(1) = g_1''(1) = 0,
\tag{2.4}
$$

$$
g_1'''(t) = 3\alpha(1-\alpha)(3\alpha+1)(3\alpha-1)(3\alpha-2)(t-1)^2 t^{3\alpha-4}.
\tag{2.5}
$$

(1) If $\alpha \in (0,\frac{1}{3}) \cup (\frac{2}{3}, 1)$, then (2.5) implies $g_1'''(t) > 0$ for $t > 1$. Thus $g_1''(t)$ is strictly increasing in $(1, +\infty)$. Therefore, Lemma 1(1) follows from (2.4) together with the monotonicity of $g_1''(t)$.

(2) If $\alpha \in (\frac{1}{3}, \frac{2}{3})$, then (2.5) implies $g_1'''(t) < 0$ for $t > 1$. Thus $g_1''(t)$ is strictly decreasing in $(1, +\infty)$. Therefore, Lemma 1(2) follows from (2.4) together with the monotonicity of $g_1''(t)$. $\square$

LEMMA 2. *If $t > 1$, then*

$$
\log(t^2 - 1) - \log(2\log t) - \frac{1}{3}\log t - \frac{4}{3}\log\frac{t+1}{2} > 0.
\tag{2.6}
$$

Proof. Let $\varphi_1(t) = \log(t^2 - 1) - \log(2\log t) - \frac{1}{3}\log t - \frac{4}{3}\log\frac{t+1}{2}$. Then simple computations lead to

$$
\lim_{t \to 1^+} \varphi_1(t) = 0,
\tag{2.7}
$$

$$
\varphi_1'(t) = \frac{t^2 + 4t + 1}{3t(t^2-1)\log t}\psi_1(t),
\tag{2.8}
$$

where

$$
\psi_1(t) = \log t + \frac{3(1-t^2)}{t^2 + 4t + 1},
\tag{2.9}
$$

and

$$\psi_1(1) = 0, \tag{2.10}$$

$$\psi_1'(t) = \frac{(t-1)^4}{t(t^2+4t+1)^2} > 0 \tag{2.11}$$

for $t > 1$. Therefore, Lemma 2 follows from (2.7), (2.8) and (2.10) together with (2.11). $\square$

LEMMA 3. *Let $t > 1$ and*

$$G_1(t) = \alpha t^{\frac{2\alpha}{\alpha-2}+2} + 2(1-\alpha)t^{\frac{2\alpha}{\alpha-2}+1} - 2(1-\alpha)t - \alpha. \tag{2.12}$$

Then

    (1) $G_1(t) > 0$ for $\alpha \in (0, \frac{2}{3})$, and
    (2) $G_1(t) < 0$ for $\alpha \in (\frac{2}{3}, 1)$.

Proof. Simple computations lead to

$$G_1'(t) = \alpha\Big(\frac{2\alpha}{\alpha-2}+2\Big)t^{\frac{2\alpha}{\alpha-2}+1} + 2(1-\alpha)\Big(\frac{2\alpha}{\alpha-2}+1\Big)t^{\frac{2\alpha}{\alpha-2}} - 2(1-\alpha), \tag{2.13}$$

$$G_1(1) = G_1'(1) = 0, \tag{2.14}$$

$$G_1''(t) = \frac{4\alpha(1-\alpha)(2-3\alpha)}{(\alpha-2)^2}(t-1)t^{\frac{2\alpha}{\alpha-2}-1}. \tag{2.15}$$

(1) If $\alpha \in (0, \frac{2}{3})$, then $G_1''(t) > 0$ for $t > 1$. Thus $G_1'(t)$ is strictly increasing in $(1, +\infty)$. Therefore, Lemma 3(1) follows from (2.14) together with the monotonicity of $G_1'(t)$.

(2) If $\alpha \in (\frac{2}{3}, 1)$, then $G_1''(t) < 0$ for $t > 1$. Thus $G_1'(t)$ is strictly decreasing in $(1, +\infty)$. Therefore, Lemma 3(2) follows from (2.14) together with the monotonicity of $G_1'(t)$. $\square$

LEMMA 4. *Let $t > 1$ and*

$$\begin{aligned}
G_2(t) = {} & \alpha t^{\frac{\alpha}{\alpha-2}+3} + 2(1-\alpha)t^{\frac{\alpha}{\alpha-2}+2} + (2-\alpha)t^{\frac{\alpha}{\alpha-2}+1} \\
& - (2-\alpha)t^2 - 2(1-\alpha)t - \alpha.
\end{aligned} \tag{2.16}$$

Then $G_2(t) > 0$ for all $\alpha \in (0, 1)$.

Proof. Let $\alpha \in (0, 1)$ and $t > 1$. Simple computations yield

$$\begin{aligned}
G_2'(t) = {} & \alpha\Big(\frac{\alpha}{\alpha-2}+3\Big)t^{\frac{\alpha}{\alpha-2}+2} + 2(1-\alpha)\Big(\frac{\alpha}{\alpha-2}+2\Big)t^{\frac{\alpha}{\alpha-2}+1} \\
& + (2-\alpha)\Big(\frac{\alpha}{\alpha-2}+1\Big)t^{\frac{\alpha}{\alpha-2}} - 2(2-\alpha)t - 2(1-\alpha),
\end{aligned} \tag{2.17}$$

$$G_2''(t) = \alpha\Big(\frac{\alpha}{\alpha-2}+3\Big)\Big(\frac{\alpha}{\alpha-2}+2\Big)t^{\frac{\alpha}{\alpha-2}+1} + 2(1-\alpha)\Big(\frac{\alpha}{\alpha-2}+2\Big)$$
$$\cdot\Big(\frac{\alpha}{\alpha-2}+1\Big)t^{\frac{\alpha}{\alpha-2}} + (2-\alpha)\Big(\frac{\alpha}{\alpha-2}+1\Big)\frac{\alpha}{\alpha-2}t^{\frac{\alpha}{\alpha-2}-1} - 2(2-\alpha), \tag{2.18}$$

$$G_2(1) = G_2'(1) = G_2''(1) = 0, \tag{2.19}$$

$$G_2'''(t) = \Big(\frac{\alpha}{\alpha-2}+1\Big)t^{\frac{\alpha}{\alpha-2}-2}H(t), \tag{2.20}$$

where

$$H(t) = \alpha\Big(\frac{\alpha}{\alpha-2}+3\Big)\Big(\frac{\alpha}{\alpha-2}+2\Big)t^2 + 2(1-\alpha)\Big(\frac{\alpha}{\alpha-2}+2\Big)\frac{\alpha}{\alpha-2}t$$
$$-\alpha\Big(\frac{\alpha}{\alpha-2}-1\Big), \tag{2.21}$$

and

$$H(1) = \frac{2\alpha(5-3\alpha)}{2-\alpha} > 0, \tag{2.22}$$

$$H'(t) = 2\Big(\frac{\alpha}{\alpha-2}+2\Big)\Big[\alpha\Big(\frac{\alpha}{\alpha-2}+3\Big)t + (1-\alpha)\frac{\alpha}{\alpha-2}\Big], \tag{2.23}$$

$$H'(1) = 2\alpha\frac{5-3\alpha}{2-\alpha}\Big(\frac{\alpha}{\alpha-2}+2\Big) > 0, \tag{2.24}$$

$$H''(t) = 2\alpha\Big(\frac{\alpha}{\alpha-2}+3\Big)\Big(\frac{\alpha}{\alpha-2}+2\Big) > 0. \tag{2.25}$$

From (2.25) we clearly see that $H'(t)$ is strictly increasing in $(1,+\infty)$. Therefore Lemma 4 follows from (2.19), (2.20), (2.22) and (2.24) together with the monotonicity of $H'(t)$. $\square$

LEMMA 5. *Let $t > 1$ and*

$$g_2(t) = \alpha t^{8-6\alpha} - (1-\alpha)(2-3\alpha)^{-1}(3\alpha+1)t^{7-6\alpha} + (2-\alpha)t^{6-6\alpha}$$
$$+(1-\alpha)(2-3\alpha)^{-1}(3\alpha-5)t^{5-6\alpha} + (1-\alpha)(2-3\alpha)^{-1} \tag{2.26}$$
$$\cdot(5-3\alpha)t^3 - (2-\alpha)t^2 + (1-\alpha)(2-3\alpha)^{-1}(3\alpha+1)t - \alpha.$$

Then
   (1) $g_2(t) > 0$ *for $\alpha \in [\frac{5-\sqrt{19}}{6}, \frac{2}{3}) \cup (\frac{2}{3}, \frac{5}{6})$, and*
   (2) $g_2(t) < 0$ *for $\alpha \in (\frac{5}{6}, 1)$.*

Proof. Let $\alpha \in [\frac{5-\sqrt{19}}{6}, \frac{2}{3}) \cup (\frac{2}{3}, \frac{5}{6}) \cup (\frac{5}{6}, 1)$. Simple computations lead to

$$g_2'(t) = 2\alpha(4-3\alpha)t^{7-6\alpha} - (1-\alpha)(2-3\alpha)^{-1}(3\alpha+1)(7-6\alpha)t^{6-6\alpha}$$
$$+6(1-\alpha)(2-\alpha)t^{5-6\alpha} + (1-\alpha)(2-3\alpha)^{-1}(3\alpha-5)(5-6\alpha)$$
$$\cdot t^{4-6\alpha} + 3(1-\alpha)(2-3\alpha)^{-1}(5-3\alpha)t^2 - 2(2-\alpha)t \tag{2.27}$$
$$+(1-\alpha)(2-3\alpha)^{-1}(3\alpha+1),$$

$$g_2''(t) = 2[\alpha(4-3\alpha)(7-6\alpha)t^{6-6\alpha} - 3(1-\alpha)^2(2-3\alpha)^{-1}(3\alpha+1)$$
$$\cdot(7-6\alpha)t^{5-6\alpha} + 3(1-\alpha)(2-\alpha)(5-6\alpha)t^{4-6\alpha} + (1-\alpha)(3\alpha-5) \quad (2.28)$$
$$\cdot(5-6\alpha)t^{3-6\alpha} + 3(1-\alpha)(2-3\alpha)^{-1}(5-3\alpha)t - (2-\alpha)],$$

$$g_2'''(t) = 6(1-\alpha)[2\alpha(4-3\alpha)(7-6\alpha)t^{5-6\alpha} - (1-\alpha)(2-3\alpha)^{-1}(3\alpha+1)$$
$$\cdot(7-6\alpha)(5-6\alpha)t^{4-6\alpha} + 2(2-\alpha)(5-6\alpha)(2-3\alpha)t^{3-6\alpha} \quad (2.29)$$
$$+(3\alpha-5)(5-6\alpha)(1-2\alpha)t^{2-6\alpha} + (2-3\alpha)^{-1}(5-3\alpha)],$$

$$g_2(1) = g_2'(1) = g_2''(1) = g_2'''(1) = 0, \quad (2.30)$$

$$g_2^{(4)}(t) = 12(1-\alpha)(5-6\alpha)t^{1-6\alpha}h(t), \quad (2.31)$$

where

$$h(t) = \alpha(4-3\alpha)(7-6\alpha)t^3 - (1-\alpha)(3\alpha+1)(7-6\alpha)t^2$$
$$+3(2-\alpha)(2-3\alpha)(1-2\alpha)t + (3\alpha-5)(1-2\alpha)(1-3\alpha), \quad (2.32)$$

and

$$h(1) = 0, \quad (2.33)$$

$$h'(t) = 3\alpha(4-3\alpha)(7-6\alpha)t^2 - 2(1-\alpha)(3\alpha+1)(7-6\alpha)t$$
$$+3(2-\alpha)(2-3\alpha)(1-2\alpha), \quad (2.34)$$

$$h'(1) = 2(5+\sqrt{19}-6\alpha)\left(\alpha - \frac{5-\sqrt{19}}{6}\right) \geqslant 0, \quad (2.35)$$

$$h''(t) = 2(7-6\alpha)[3\alpha(4-3\alpha)t - (1-\alpha)(3\alpha+1)], \quad (2.36)$$

$$h''(1) = 2(7-6\alpha)(5+\sqrt{19}-6\alpha)\left(\alpha - \frac{5-\sqrt{19}}{6}\right) \geqslant 0, \quad (2.37)$$

$$h'''(t) = 6\alpha(7-6\alpha)(4-3\alpha) > 0. \quad (2.38)$$

It follows from (2.33), (2.35) and (2.37) together with (2.38) that

$$h(t) > 0, \quad (2.39)$$

for $t > 1$.

(1) If $\alpha \in [\frac{5-\sqrt{19}}{6}, \frac{2}{3}) \cup (\frac{2}{3}, \frac{5}{6})$, then (2.31) and (2.39) imply that

$$g_2^{(4)}(t) > 0, \quad (2.40)$$

for $t > 1$. Therefore Lemma 5(1) follows from (2.30) together with (2.40).

(2) If $\alpha \in (\frac{5}{6}, 1)$, then (2.31) and (2.39) imply that

$$g_2^{(4)}(t) < 0, \quad (2.41)$$

for $t > 1$. Therefore Lemma 5(2) follows from (2.30) together with (2.41). $\quad\square$

LEMMA 6. *If $t > 1$, then*

$$\frac{t}{t-1}\log t - \frac{1}{3}\log t - \frac{1}{6}\log(t^2+1) + \frac{1}{6}\log 2 - 1 > 0. \tag{2.42}$$

Proof. Let $\varphi_2(t) = \frac{t}{t-1}\log t - \frac{1}{3}\log t - \frac{1}{6}\log(t^2+1) + \frac{1}{6}\log 2 - 1$. Then simple computations lead to

$$\lim_{t \to 1^+} \varphi_2(t) = 0, \tag{2.43}$$

$$\varphi_2'(t) = \frac{\psi_2(t)}{(t-1)^2}, \tag{2.44}$$

where

$$\psi_2(t) = \frac{-2t^4 + 4t^3 - 3t^2 + 2t - 1}{3t(t^2+1)} - \log t + t - 1, \tag{2.45}$$

and

$$\psi_2(1) = 0, \tag{2.46}$$

$$\psi_2'(t) = \frac{(t-1)^4(t^2+t+1)}{3t^2(t^2+1)^2} > 0, \tag{2.47}$$

for $t > 1$. Therefore, Lemma 6 follows from (2.43), (2.44) and (2.46) together with (2.47). □

LEMMA 7. *If $t > 1$, then*

$$\log(t-1) - \log(\log t) - \frac{5}{12}\log t - \frac{1}{12}\log(t^2+1) + \frac{1}{12}\log 2 > 0. \tag{2.48}$$

Proof. Let $\varphi_3(t) = \log(t-1) - \log(\log t) - \frac{5}{12}\log t - \frac{1}{12}\log(t^2+1) + \frac{1}{12}\log 2$. Then simple computations yield

$$\lim_{t \to 1^+} \varphi_3(t) = 0, \tag{2.49}$$

$$\varphi_3'(t) = \frac{5t^3 + 7t^2 + 7t + 5}{12t(t-1)(t^2+1)\log t} \psi_3(t), \tag{2.50}$$

where

$$\psi_3(t) = \log t - \frac{12(t^3 - t^2 + t - 1)}{5t^3 + 7t^2 + 7t + 5}, \tag{2.51}$$

and

$$\psi_3(1) = 0, \tag{2.52}$$

$$\psi_3'(t) = \frac{(t-1)^4(25t^2 + 26t + 25)}{t(5t^3 + 7t^2 + 7t + 5)^2} > 0, \tag{2.53}$$

for $t > 1$. Therefore, Lemma 7 follows from (2.49), (2.50) and (2.52) together with (2.53). □

3. Main results

THEOREM 1. *Let $a,b > 0$ with $a \neq b$. Then*
(1) $L_{-\frac{1+3\alpha}{2}}(a,b) = G^\alpha(a,b)N^{1-\alpha}(a,b) = L_{\frac{2}{\alpha-2}}(a,b)$ *for* $\alpha = \frac{2}{3}$;
(2) $L_{-\frac{1+3\alpha}{2}}(a,b) > G^\alpha(a,b)N^{1-\alpha}(a,b) > L_{\frac{2}{\alpha-2}}(a,b)$ *for* $\alpha \in (0,\frac{2}{3})$ *and*
$L_{-\frac{1+3\alpha}{2}}(a,b) < G^\alpha(a,b)N^{1-\alpha}(a,b) < L_{\frac{2}{\alpha-2}}(a,b)$ *for* $\alpha \in (\frac{2}{3},1)$, *and the parameters*
$-\frac{1+3\alpha}{2}$ *and* $\frac{2}{\alpha-2}$ *cannot be improved in either case.*

Proof. (1) If $\alpha = \frac{2}{3}$, then simple computations lead to

$$L_{-\frac{1+3\alpha}{2}}(a,b) = L_{-\frac{2}{\alpha-2}}(a,b) = L_{-\frac{3}{2}}(a,b)$$
$$= (\sqrt{ab})^{\frac{2}{3}}\left[\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2\right]^{\frac{1}{3}} = G^\alpha(a,b)N^{1-\alpha}(a,b). \tag{3.1}$$

(2) Without loss of generality, we assume that $a > b$. Let $t = \sqrt{\frac{a}{b}} > 1$.

Firstly, we compare $L_{-\frac{1+3\alpha}{2}}(a,b)$ with $G^\alpha(a,b)N^{1-\alpha}(a,b)$. We divide the proof into two cases.

Case 1. If $\alpha \in (0,\frac{1}{3}) \cup (\frac{1}{3},\frac{2}{3}) \cup (\frac{2}{3},1)$, then (1.1) yields

$$\log L_{-\frac{1+3\alpha}{2}}(a,b) - \log[G^\alpha(a,b)N^{1-\alpha}(a,b)]$$
$$= \log L_{-\frac{1+3\alpha}{2}}(t^2,1) - \log[G^\alpha(t^2,1)N^{1-\alpha}(t^2,1)] \tag{3.2}$$
$$= f_1(t),$$

where

$$f_1(t) = \frac{2}{1+3\alpha}\log\frac{2(t^{1-3\alpha}-1)}{(1-3\alpha)(t^2-1)} - \alpha\log t - 2(1-\alpha)\log\frac{t+1}{2}. \tag{3.3}$$

Simple computations lead to

$$\lim_{t\to 1^+} f_1(t) = 0, \tag{3.4}$$

$$f_1'(t) = \frac{g_1(t)}{(1+3\alpha)t(t^2-1)(t^{1-3\alpha}-1)}, \tag{3.5}$$

where $g_1(t)$ have been denoted by (2.1).

From (3.5) and Lemma 1 we clearly see $f_1'(t) > 0$ for $\alpha \in (0,\frac{1}{3}) \cup (\frac{1}{3},\frac{2}{3})$ and $f_1'(t) < 0$ for $\alpha \in (\frac{2}{3},1)$. Hence, for $\alpha \in (0,\frac{1}{3}) \cup (\frac{1}{3},\frac{2}{3})$ $f_1(t)$ is strictly increasing in $(1,+\infty)$, and for $\alpha \in (\frac{2}{3},1)$ $f_1(t)$ is strictly decreasing in $(1,+\infty)$.

Therefore, inequalities $L_{-\frac{1+3\alpha}{2}}(a,b) > G^\alpha(a,b)N^{1-\alpha}(a,b)$ for $\alpha \in (0,\frac{1}{3}) \cup (\frac{1}{3},\frac{2}{3})$ and $L_{-\frac{1+3\alpha}{2}}(a,b) < G^\alpha(a,b)N^{1-\alpha}(a,b)$ for $\alpha \in (\frac{2}{3},1)$ follow from (3.2) and (3.4) together with the monotonicity of $f_1(t)$.

Case 2. If $\alpha = \frac{1}{3}$, then from (1.1) we have

$$
\begin{aligned}
&\log L_{-\frac{1+3\alpha}{2}}(a,b) - \log[G^\alpha(a,b)N^{1-\alpha}(a,b)] \\
&= \log L_{-1}(t^2,1) - \log[G^{\frac{1}{3}}(t^2,1)N^{\frac{2}{3}}(t^2,1)] \\
&= \log(t^2-1) - \log(2\log t) - \frac{1}{3}\log t - \frac{4}{3}\log\frac{t+1}{2}.
\end{aligned}
\tag{3.6}
$$

Therefore inequality $L_{-\frac{1+3\alpha}{2}}(a,b) > G^\alpha(a,b)N^{1-\alpha}(a,b)$ follows from (3.6) and Lemma 2.

Secondly, we compare $L_{\frac{2}{\alpha-2}}(a,b)$ with $G^\alpha(a,b)N^{1-\alpha}(a,b)$.

From (1.1), one has

$$
\begin{aligned}
&\log L_{\frac{2}{\alpha-2}}(a,b) - \log[G^\alpha(a,b)N^{1-\alpha}(a,b)] \\
&= \log L_{\frac{2}{\alpha-2}}(t^2,1) - \log[G^\alpha(t^2,1)N^{1-\alpha}(t^2,1)] \\
&= F_1(t),
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
F_1(t) = \frac{\alpha-2}{2}&\left[\log(t^{\frac{2\alpha}{\alpha-2}}-1) - \log(t^2-1) - \log\frac{\alpha}{\alpha-2}\right] \\
&- \alpha\log t - 2(1-\alpha)\log\frac{t+1}{2}.
\end{aligned}
\tag{3.8}
$$

Simple computations lead to

$$
\lim_{t\to 1^+} F_1(t) = 0,
\tag{3.9}
$$

$$
F_1'(t) = \frac{G_1(t)}{t(t^2-1)(t^{\frac{2\alpha}{\alpha-2}}-1)},
\tag{3.10}
$$

where $G_1(t)$ have been denoted by (2.12).

From (3.10) and Lemma 3 we derive that $F_1'(t) < 0$ for $\alpha \in (0,\frac{2}{3})$ and $F_1'(t) > 0$ for $\alpha \in (\frac{2}{3},1)$. Hence, for $\alpha \in (0,\frac{2}{3})$ $F_1(t)$ is strictly decreasing in $(1,+\infty)$, and for $\alpha \in (\frac{2}{3},1)$ $F_1(t)$ is strictly increasing in $(1,+\infty)$.

Therefore, inequalities $G^\alpha(a,b)N^{1-\alpha}(a,b) > L_{\frac{2}{\alpha-2}}(a,b)$ for $\alpha \in (0,\frac{2}{3})$ and $G^\alpha(a,b)N^{1-\alpha}(a,b) < L_{\frac{2}{\alpha-2}}(a,b)$ for $\alpha \in (\frac{2}{3},1)$ follow from (3.7) and (3.9) together with the monotonicity of $F_1(t)$.

At last, we prove that the parameters $-\frac{1+3\alpha}{2}$ and $\frac{2}{\alpha-2}$ cannot be improved in either case.

The following two cases will complete the proof for the optimality of parameter $\frac{2}{\alpha-2}$.

Case 1. If $\alpha \in (0, \frac{2}{3})$, then for any $\varepsilon \in (0, \frac{\alpha}{2-\alpha})$, one has

$$\lim_{t \to +\infty} \frac{L_{\frac{2}{\alpha-2}+\varepsilon}(1, t^2)}{G^{\alpha}(1, t^2)N^{1-\alpha}(1, t^2)}$$

$$= \lim_{t \to +\infty} \left\{ \left[\frac{\frac{\alpha}{2-\alpha} - \varepsilon}{1 - t^{-2(\frac{\alpha}{2-\alpha} - \varepsilon)}} \left(1 - \frac{1}{t^2}\right) \right]^{\frac{2-\alpha}{2-\varepsilon(2-\alpha)}} \left(2 - \frac{2}{t+1}\right)^{2(1-\alpha)} t^{\frac{\varepsilon(2-\alpha)^2}{2-\varepsilon(2-\alpha)}} \right\} \quad (3.11)$$

$$= +\infty.$$

Equation (3.11) implies that for any $\varepsilon \in (0, \frac{\alpha}{2-\alpha})$, there exists a sufficiently large $T_1 = T_1(\varepsilon, \alpha) > 1$, such that $L_{\frac{2}{\alpha-2}+\varepsilon}(1, t^2) > G^{\alpha}(1, t^2)N^{1-\alpha}(1, t^2)$ for $t \in (T_1, +\infty)$.

Case 2. If $\alpha \in (\frac{2}{3}, 1)$, then for any $\varepsilon > 0$, one has

$$\lim_{t \to +\infty} \frac{G^{\alpha}(1, t^2)N^{1-\alpha}(1, t^2)}{L_{\frac{2}{\alpha-2}-\varepsilon}(1, t^2)}$$

$$= \lim_{t \to +\infty} \left\{ \left[\frac{\frac{\alpha}{2-\alpha} + \varepsilon}{1 - t^{-2(\frac{\alpha}{2-\alpha} + \varepsilon)}} \left(1 - \frac{1}{t^2}\right) \right]^{\frac{\alpha-2}{2+\varepsilon(2-\alpha)}} \left(\frac{t+1}{2t}\right)^{2(1-\alpha)} t^{\frac{\varepsilon(2-\alpha)^2}{2+\varepsilon(2-\alpha)}} \right\} \quad (3.12)$$

$$= +\infty.$$

Equation (3.12) implies that for any $\varepsilon > 0$, there exists a sufficiently large $T_2 = T_2(\varepsilon, \alpha) > 1$, such that $G^{\alpha}(1, t^2)N^{1-\alpha}(1, t^2) > L_{\frac{2}{\alpha-2}-\varepsilon}(1, t^2)$ for $t \in (T_2, +\infty)$.

The following three cases will complete the proof for the optimality of parameter $-\frac{1+3\alpha}{2}$.

Case A. If $\alpha = \frac{1}{3}$, then for any $\varepsilon > 0$ and $x > 0$, one has

$$[L_{-\frac{1+3\alpha}{2}-\varepsilon}((1+x)^2, 1)]^{-(1+\varepsilon)} - [G^{\alpha}((1+x)^2, 1)N^{1-\alpha}((1+x)^2, 1)]^{-(1+\varepsilon)}$$

$$= \frac{\tau(x)}{\varepsilon x(x+2)}, \quad (3.13)$$

where

$$\tau(x) = 1 - (1+x)^{-2\varepsilon} - 2^{\frac{4}{3}(1+\varepsilon)}\varepsilon x(1+x)^{-\frac{1}{3}(1+\varepsilon)}(2+x)^{-\frac{1}{3}(1+4\varepsilon)}. \quad (3.14)$$

Upon letting $x \longrightarrow 0^+$, the Taylor expansion leads to

$$g_1(x) = \frac{1}{3}\varepsilon^2(1+\varepsilon)x^3 + o(x^3). \quad (3.15)$$

Equations (3.13) and (3.15) imply that for any $\varepsilon > 0$, there exists a sufficiently small $\delta_1 = \delta_1(\varepsilon) > 0$, such that $L_{-\frac{1+3\alpha}{2}-\varepsilon}((1+x)^2, 1) < G^{\alpha}((1+x)^2, 1)N^{1-\alpha}((1+x)^2, 1)$ for $x \in (0, \delta_1)$.

Case B. For $\alpha \in (0,\frac{1}{3}) \cup (\frac{1}{3},\frac{2}{3})$ and $x > 0 (x \longrightarrow 0)$, One has (For notation simplicity, we write $\lambda = 1 + 3\alpha + 2\varepsilon$.)

$$
\begin{aligned}
&[L_{-\frac{1+3\alpha}{2}-\varepsilon}((1+x)^2,1)]^{-\frac{1}{2}\lambda} - [G^\alpha((1+x)^2,1)N^{1-\alpha}((1+x)^2,1)]^{-\frac{1}{2}\lambda} \\
&= \frac{2(1+x)^{2-\lambda} - (2-\lambda)2^{\lambda(1-\alpha)}x(1+x)^{-\frac{1}{2}\alpha\lambda}(2+x)^{\lambda(\alpha-1)+1} - 2}{(2-\lambda)x(x+2)} \\
&= \frac{\varepsilon[1-(3\alpha+2\varepsilon)^2]x^3 + o(x^3)}{6(2-\lambda)x(x+2)},
\end{aligned}
\tag{3.16}
$$

If $\alpha \in (0,\frac{1}{3})$, then for any $\varepsilon \in (0,\frac{1-3\alpha}{2})$, equality (3.16) implies that there exists a sufficiently small $\delta_2 = \delta_2(\varepsilon) > 0$ such that $L_{-\frac{1+3\alpha}{2}-\varepsilon}((1+x)^2,1) < G^\alpha((1+x)^2,1)N^{1-\alpha}((1+x)^2,1)$ for $x \in (0,\delta_2)$.

If $\alpha \in (\frac{1}{3},\frac{2}{3})$, then for any $\varepsilon > 0$, equality (3.16) implies that there exists a sufficiently small $\delta_3 = \delta_3(\varepsilon) > 0$ such that $L_{-\frac{1+3\alpha}{2}-\varepsilon}((1+x)^2,1) < G^\alpha((1+x)^2,1)N^{1-\alpha}((1+x)^2,1)$ for $x \in (0,\delta_3)$.

Case C. If $\alpha \in (\frac{2}{3},1)$, then for any $\varepsilon \in (0,\frac{3\alpha-1}{2})$ and $x > 0 (x \longrightarrow 0)$, one has (For notation simplicity, we write $\mu = 2\varepsilon - 3\alpha - 1$.)

$$
\begin{aligned}
&[L_{-\frac{1+3\alpha}{2}+\varepsilon}((1+x)^2,1)]^{\frac{1}{2}\mu} - [G^\alpha((1+x)^2,1)N^{1-\alpha}((1+x)^2,1)]^{\frac{1}{2}\mu} \\
&= \frac{2(1+x)^{\mu+2} - 2^{\mu(\alpha-1)}(\mu+2)x(1+x)^{\frac{\alpha\mu}{2}}(2+x)^{\mu(1-\alpha)+1} - 2}{(\mu+2)x(2+x)} \\
&= \frac{-\varepsilon[(3\alpha-2\varepsilon)^2-1]x^3 + o(x^3)}{6(3\alpha-2\varepsilon-1)x(2+x)}.
\end{aligned}
\tag{3.17}
$$

Equality (3.17) implies that for any $\varepsilon \in (0,\frac{3\alpha-1}{2})$, there exists a sufficiently small $\delta_4 = \delta_4(\varepsilon) > 0$, such that $L_{-\frac{1+3\alpha}{2}-\varepsilon}((1+x)^2,1) > G^\alpha((1+x)^2,1)N^{1-\alpha}((1+x)^2,1)$ for $x \in (0,\delta_4)$. $\square$

THEOREM 2. *Let $a,b > 0$ with $a \neq b$. Then*

(1) $L_{\frac{2}{\alpha-2}}(a,b) < G^\alpha(a,b)S^{1-\alpha}(a,b)$ *for $\alpha \in (0,1)$, and the parameter $\frac{2}{\alpha-2}$ cannot be improved;*

(2) $G^\alpha(a,b)S^{1-\alpha}(a,b) < L_{2(2-3\alpha)}(a,b)$ *for $\alpha \in [\frac{5-\sqrt{19}}{6},1)$, and the parameter $2(2-3\alpha)$ cannot be improved.*

Proof. Without loss of generality, we assume that $a > b$. Let $t = \frac{a}{b} > 1$.

(1) Firstly, we prove that

$$
L_{\frac{2}{\alpha-2}}(a,b) < G^\alpha(a,b)S^{1-\alpha}(a,b),
\tag{3.18}
$$

for $\alpha \in (0,1)$.

From (1.1) we have

$$
\begin{aligned}
&\log L_{\frac{2}{\alpha-2}}(a,b) - \log[G^\alpha(a,b)S^{1-\alpha}(a,b)] \\
&= \log L_{\frac{2}{\alpha-2}}(t,1) - \log[G^\alpha(t,1)S^{1-\alpha}(t,1)] \\
&= F_2(t),
\end{aligned}
\tag{3.19}
$$

where

$$
\begin{aligned}
F_2(t) = {} & \frac{\alpha-2}{2}\left[\log\left(t^{\frac{\alpha}{\alpha-2}}-1\right) - \log(t-1) - \log\frac{\alpha}{\alpha-2}\right] \\
& - \frac{\alpha}{2}\log t - \frac{1-\alpha}{2}\log\frac{t^2+1}{2}.
\end{aligned}
\tag{3.20}
$$

Simple computations lead to

$$
\lim_{t\to 1^+} F_2(t) = 0,
\tag{3.21}
$$

$$
F_2'(t) = \frac{G_2(t)}{2t(t-1)(t^2+1)(t^{\frac{\alpha}{\alpha-2}}-1)},
\tag{3.22}
$$

where $G_2(t)$ have been denoted by (2.16).

From (3.22) and Lemma 4 we clearly see that $F_2'(t) < 0$ for $\alpha \in (0,1)$. Therefore, inequality (3.18) follows from (3.19) and (3.21) together with $F_2'(t) < 0$.

In the following, we prove that the parameter $\frac{2}{\alpha-2}$ cannot be improved.

For $\alpha \in (0,1)$ and any $\varepsilon \in (0,\frac{\alpha}{2-\alpha})$, one has

$$
\begin{aligned}
&\lim_{t\to+\infty} \frac{L_{\frac{2}{\alpha-2}+\varepsilon}(t,1)}{G^\alpha(t,1)S^{1-\alpha}(t,1)} \\
&= \lim_{t\to+\infty} \left\{ \left[\frac{1-t^{-(\frac{\alpha}{2-\alpha}-\varepsilon)}}{(\frac{\alpha}{2-\alpha}-\varepsilon)(1-\frac{1}{t})}\right]^{\frac{2-\alpha}{\varepsilon(2-\alpha)-2}} \left(\frac{1}{2}+\frac{1}{2t^2}\right)^{\frac{\alpha-1}{2}} t^{\frac{\varepsilon(2-\alpha)^2}{2[2-\varepsilon(2-\alpha)]}} \right\} \\
&= +\infty.
\end{aligned}
\tag{3.23}
$$

Equation (3.23) implies that for any $\varepsilon \in (0,\frac{\alpha}{2-\alpha})$, there exists a sufficiently large $T = T(\varepsilon,\alpha) > 1$, such that $L_{\frac{2}{\alpha-2}+\varepsilon}(t,1) > G^\alpha(t,1)S^{1-\alpha}(t,1)$ for $t \in (T,+\infty)$.

(2) Firstly, we prove that

$$
G^\alpha(a,b)S^{1-\alpha}(a,b) < L_{2(2-3\alpha)}(a,b),
\tag{3.24}
$$

for $\alpha \in [\frac{5-\sqrt{19}}{6},1)$. We divide the proof into three cases.

Case 1. If $\alpha \in [\frac{5-\sqrt{19}}{6},\frac{2}{3}) \cup (\frac{2}{3},\frac{5}{6}) \cup (\frac{5}{6},1)$, then (1.1) yields

$$
\begin{aligned}
&\log L_{2(2-3\alpha)}(a,b) - \log[G^\alpha(a,b)S^{1-\alpha}(a,b)] \\
&= \log L_{2(2-3\alpha)}(t,1) - \log[G^\alpha(t,1)S^{1-\alpha}(t,1)] \\
&= f_2(t),
\end{aligned}
\tag{3.25}
$$

where

$$f_2(t) = \frac{1}{2(2-3\alpha)} \log \frac{t^{5-6\alpha}-1}{(5-6\alpha)(t-1)} - \frac{1}{2}\left[\alpha \log t + (1-\alpha) \log \frac{t^2+1}{2}\right]. \quad (3.26)$$

Simple computations lead to

$$\lim_{t \to 1^+} f_2(t) = 0, \quad (3.27)$$

$$f_2'(t) = \frac{g_2(t)}{2t(t-1)(t^2+1)(t^{5-6\alpha}-1)}, \quad (3.28)$$

where $g_2(t)$ have been denoted by (2.26).

From (3.28) and Lemma 5 we affirm that $f_2'(t) > 0$ for $\alpha \in [\frac{5-\sqrt{19}}{6}, \frac{2}{3}) \cup (\frac{2}{3}, \frac{5}{6}) \cup (\frac{5}{6}, 1)$. Therefore, inequality (3.24) follows from (3.25) and (3.27) together with $f_2'(t) > 0$.

Case 2. If $\alpha = \frac{2}{3}$, then from (1.1) one has

$$\log L_{2(2-3\alpha)}(a,b) - \log[G^\alpha(a,b)S^{1-\alpha}(a,b)]$$
$$= \log L_0(t,1) - \log[G^{\frac{2}{3}}(t,1)S^{\frac{1}{3}}(t,1)] \quad (3.29)$$
$$= \frac{t}{t-1}\log t - \frac{1}{3}\log t - \frac{1}{6}\log(t^2+1) + \frac{1}{6}\log 2 - 1.$$

From (3.29) and Lemma 6 we know that the inequality (3.24) is correct.

Case 3. If $\alpha = \frac{5}{6}$, then simple computations lead to

$$\log L_{2(2-3\alpha)}(a,b) - \log[G^\alpha(a,b)S^{1-\alpha}(a,b)]$$
$$= \log L_{-1}(t,1) - \log[G^{\frac{5}{6}}(t,1)S^{\frac{1}{6}}(t,1)] \quad (3.30)$$
$$= \log t - 1 - \log(\log t) - \frac{5}{12}\log t - \frac{1}{12}\log(t^2+1) + \frac{1}{12}\log 2.$$

From (3.30) and Lemma 7 we deem that the inequality (3.24) is established.

Finally, we prove that the parameter $2(2-3\alpha)$ cannot be improved.

For $\alpha \in [\frac{5-\sqrt{19}}{6}, 1)$ and $x > 0(x \longrightarrow 0)$, one has (For notation simplicity, we write $\lambda = \frac{2(2-\alpha)-\varepsilon}{2}$.)

$$[L_{2(2-3\alpha)-\varepsilon}(1+x,1)]^{2\lambda} - [G^\alpha(1+x,1)S^{1-\alpha}(1+x,1)]^{2\lambda}$$
$$= \frac{1}{x}\left\{\frac{(1+x)^{2\lambda+1}-1}{2\lambda+1} - 2^{(\alpha-1)\lambda}x(1+x)^{\alpha\lambda}[(1+x)^2+1]^{(1-\alpha)\lambda}\right\} \quad (3.31)$$
$$= \frac{1}{x}\left\{\frac{\varepsilon}{4}[\varepsilon - 2(2-3\alpha)]x^3 + o(x^3)\right\}.$$

We divide the proof into three cases.

Case A. If $\alpha \in [\frac{5-\sqrt{19}}{6}, \frac{2}{3})$, then for any $\varepsilon \in (0, 2(2-3\alpha))$, equality (3.31) implies that there exists a sufficiently small $\delta_1 = \delta_1(\varepsilon) > 0$ such that $L_{2(2-3\alpha)-\varepsilon}(1+x,1) < G^\alpha(1+x,1)S^{1-\alpha}(1+x,1)$ for $x \in (0, \delta_1)$.

Case B. If $\alpha = \frac{2}{3}$, then for any $\varepsilon > 0$, equality (3.31) implies that there exists a sufficiently small $\delta_2 = \delta_2(\varepsilon) > 0$ such that $L_{2(2-3\alpha)-\varepsilon}(1+x,1) < G^\alpha(1+x,1)S^{1-\alpha}(1+x,1)$ for $x \in (0,\delta_2)$.

Case C. If $\alpha \in (\frac{2}{3},1)$, then for any $\varepsilon \in (0,2(3\alpha-2))$, equality (3.31) implies that there exists a sufficiently small $\delta_3 = \delta_3(\varepsilon) > 0$ such that $L_{2(2-3\alpha)-\varepsilon}(1+x,1) < G^\alpha(1+x,1)S^{1-\alpha}(1+x,1)$ for $x \in (0,\delta_3)$. $\square$

REMARK 1. For $\alpha \in (0,\frac{5-\sqrt{19}}{6})$, although we cannot find the least parameter p_{min} such that $G^\alpha(a,b)S^{1-\alpha}(a,b) < L_{p_{min}}(a,b)$, but we can give the estimate $2(2-3\alpha) < p_{min} < \frac{1-6\alpha}{\alpha}$.

In fact, if $\alpha \in (0,\frac{5-\sqrt{19}}{6})$ and $p = \frac{1-6\alpha}{\alpha}$, then (1.1) leads to

$$\begin{aligned}
&\log L_p(a,b) - \log[G^\alpha(a,b)S^{1-\alpha}(a,b)] \\
&= \log L_{\frac{1-6\alpha}{\alpha}}(t,1) - \log[G^\alpha(t,1)S^{1-\alpha}(t,1)] \qquad (3.32)\\
&= f(t),
\end{aligned}$$

where

$$f(t) = \frac{\alpha}{1-6\alpha}\{\log(t^{\frac{1-5\alpha}{\alpha}} - 1) - \log[\frac{1-5\alpha}{\alpha}(t-1)]\} - \frac{\alpha}{2}\log t - \frac{1-\alpha}{2}\log\frac{t^2+1}{2}. \quad (3.33)$$

Simple computations lead to

$$\lim_{t \to 1^+} f(t) = 0, \qquad (3.34)$$

$$f'(t) = \frac{g(t)}{2(1-6\alpha)t(t-1)(t^2+1)(t^{\frac{1-5\alpha}{\alpha}} - 1)}, \qquad (3.35)$$

where

$$\begin{aligned}
g(t) =\ & \alpha(1-6\alpha)t^{\frac{1}{\alpha}-2} - 3\alpha(1-2\alpha)t^{\frac{1}{\alpha}-3} + (1-6\alpha)(2-\alpha)t^{\frac{1}{\alpha}-4} \\
& -(6\alpha^2-11\alpha+2)t^{\frac{1}{\alpha}-5} + (6\alpha^2-11\alpha+2)t^3 - (1-6\alpha)(2-\alpha)t^2 \quad (3.36)\\
& +3\alpha(1-2\alpha)t - \alpha(1-6\alpha),
\end{aligned}$$

and

$$\begin{aligned}
g'(t) =\ & \alpha(1-6\alpha)(\tfrac{1}{\alpha}-2)t^{\frac{1}{\alpha}-3} - 3\alpha(1-2\alpha)(\tfrac{1}{\alpha}-3)t^{\frac{1}{\alpha}-4} \\
& +(1-6\alpha)(2-\alpha)(\tfrac{1}{\alpha}-4)t^{\frac{1}{\alpha}-5} - (6\alpha^2-11\alpha+2)(\tfrac{1}{\alpha}-5)t^{\frac{1}{\alpha}-6} \quad (3.37)\\
& +3(6\alpha^2-11\alpha+2)t^2 - 2(1-6\alpha)(2-\alpha)t + 3\alpha(1-2\alpha),
\end{aligned}$$

$$\begin{aligned}
g''(t) =\ & \alpha(1-6\alpha)(\tfrac{1}{\alpha}-2)(\tfrac{1}{\alpha}-3)t^{\frac{1}{\alpha}-4} - 3\alpha(1-2\alpha)(\tfrac{1}{\alpha}-3)(\tfrac{1}{\alpha}-4)t^{\frac{1}{\alpha}-5} \\
& +(1-6\alpha)(2-\alpha)(\tfrac{1}{\alpha}-4)(\tfrac{1}{\alpha}-5)t^{\frac{1}{\alpha}-6} - (6\alpha^2-11\alpha+2) \quad (3.38)\\
& \cdot(\tfrac{1}{\alpha}-5)(\tfrac{1}{\alpha}-6)t^{\frac{1}{\alpha}-7} + 6(6\alpha^2-11\alpha+2)t - 2(1-6\alpha)(2-\alpha),
\end{aligned}$$

$$g'''(t) = \alpha^{-3}[\alpha(1-6\alpha)(1-2\alpha)(1-3\alpha)(1-4\alpha)t^{\frac{1}{\alpha}-5} - 3\alpha(1-2\alpha)$$
$$\cdot(1-3\alpha)(1-4\alpha)(1-5\alpha)t^{\frac{1}{\alpha}-6} + (1-6\alpha)(2-\alpha)(1-4\alpha)$$
$$\cdot(1-5\alpha)(1-6\alpha)t^{\frac{1}{\alpha}-7} - (6\alpha^2 - 11\alpha + 2)(1-5\alpha)(1-6\alpha)$$
$$\cdot(1-7\alpha)t^{\frac{1}{\alpha}-8}] + 6(6\alpha^2 - 11\alpha + 2), \tag{3.39}$$

$$g(1) = g'(1) = g''(1) = g'''(1) = 0, \tag{3.40}$$

$$g^{(4)}(t) = \alpha^{-4}(1-5\alpha)(1-6\alpha)t^{\frac{1}{\alpha}-9}k(t), \tag{3.41}$$

where

$$k(t) = \alpha(1-2\alpha)(1-3\alpha)(1-4\alpha)t^3 - 3\alpha(1-2\alpha)(1-3\alpha)(1-4\alpha)t^2$$
$$+(1-6\alpha)(2-\alpha)(1-4\alpha)(1-7\alpha)t - (6\alpha^2 - 11\alpha + 2) \tag{3.42}$$
$$\cdot(1-7\alpha)(1-8\alpha),$$

and

$$k(1) = 4\alpha(1-5\alpha)(5+\sqrt{19}-6\alpha)\left(\frac{5-\sqrt{19}}{6} - \alpha\right) > 0, \tag{3.43}$$

$$k'(t) = (1-4\alpha)[3\alpha(1-2\alpha)(1-3\alpha)t^2 - 6\alpha(1-2\alpha)(1-3\alpha)t$$
$$+(1-6\alpha)(2-\alpha)(1-7\alpha)], \tag{3.44}$$

$$k'(1) = 2(1-4\alpha)(1-5\alpha)(5+\sqrt{19}-6\alpha)\left(\frac{5-\sqrt{19}}{6} - \alpha\right) > 0, \tag{3.45}$$

$$k''(t) = 6\alpha(1-2\alpha)(1-3\alpha)(1-4\alpha)(t-1) > 0, \tag{3.46}$$

for $t > 1$. It follows from (3.32), (3.34), (3.35), (3.40), (3.41), (3.43) and (3.45) together with (3.46) that

$$L_{\frac{1-6\alpha}{\alpha}}(a,b) > G^\alpha(a,b)S^{1-\alpha}(a,b). \tag{3.47}$$

For $p = 2(2-3\alpha)$, thinking back (2.35) one has

$$h'(1) < 0, \tag{3.48}$$

for $\alpha \in (0, \frac{5-\sqrt{19}}{6})$. From (3.48) and the continuity of $h'(t)$ we know that there exists $\delta = \delta(\alpha) > 0$ such that $h'(t) < 0$ for $t \in (1, 1+\delta)$. This implies that, by (2.30), (2.31), (2.33), (3.25), (3.27) and (3.28),

$$L_{2(2-3\alpha)}(a,b) < G^\alpha(a,b)S^{1-\alpha}(a,b). \tag{3.49}$$

Therefore, the estimate $2(2-3\alpha) < p_{min} < \frac{1-6\alpha}{\alpha}$ follows from (3.47) and (3.49).

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