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RIGOROUS MULTIPLICATIVE PERTURBATION BOUNDS FOR THE GENERALIZED CHOLESKY FACTORIZATION AND THE CHOLESKY–LIKE FACTORIZATION

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Abstract. The generalized Cholesky factorization and the Cholesky-like factorization are two generalizations of the classic Cholesky factorization. In this paper, the rigorous multiplicative perturbation bounds for the two factorizations are derived using the matrix equation and the refined matrix equation approaches. The corresponding first-order multiplicative perturbation bounds, as special cases, are also presented.

1. Introduction

Let $\mathbb{R}^{m\times n}$ be the set of $m\times n$ real matrices and $\mathbb{R}_r^{m\times n}$ be the subset of $\mathbb{R}^{m\times n}$ consisting of matrices with rank r. Let I_r be the identity matrix of order r. For a matrix $A\in\mathbb{R}^{m\times n}$, we denote by A^T and $A[\langle i\rangle]$ the transpose and the i-th leading principal submatrix of A, respectively.

First, consider the following block matrix $K\in\mathbb{R}^{(m+n)\times(m+n)}$

$$K=\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}, \tag{1.1}$$

where $A\in\mathbb{R}_m^{m\times m}$ is symmetric positive definite, $B\in\mathbb{R}_n^{n\times m}$, and $C\in\mathbb{R}^{n\times n}$ is symmetric positive semi-definite. The matrix K frequently arises in the system called an augmented system or an equilibrium system [7]. For this matrix, there always exists the following factorization

$$K=LJ_{m+n}L^T, \tag{1.2}$$

where

$$L=\begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix},\ J_{m+n}=\begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix},$$

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$L_{11} \in \mathbb{R}_m^{m\times m}$ and $L_{22} \in \mathbb{R}_n^{n\times n}$ are lower triangular, and $L_{21} \in \mathbb{R}_n^{n\times m}$. This factorization is called the generalized Cholesky factorization and L is referred to as the generalized Cholesky factor [19].

Now, we consider the skew-symmetric matrix $B \in \mathbb{R}^{2n\times 2n}$. If all even leading principal submatrices of B are nonsingular, i.e., $B[\langle 2i\rangle]$ $(i = 1,\cdots,n)$ are nonsingular, then B has the following factorization

$$B = R^T \widehat{J}_{2n} R, \tag{1.3}$$

where $R = (r_{ij}) \in \mathbb{R}^{2n\times 2n}$ is upper triangular with $r_{2j-1,2j} = 0, r_{2j-1,2j-1} > 0$ and $r_{2j,2j} = \pm r_{2j-1,2j-1}$ for $j = 1,2,\cdots,n$, and

$$\widehat{J}_{2n} = \mathrm{diag}(J_0,\cdots,J_0),\ J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Thus, R has 2×2 blocks of the form $\begin{bmatrix} r & 0 \\ 0 & \pm r \end{bmatrix}$ running down the main diagonal. The factorization (1.3) is called the Cholesky-like factorization and R is referred to as the Cholesky-like factor [1].

For these two factorizations, some authors studied their algorithms, error analysis, and perturbation analysis [1, 5, 6, 8, 14, 16, 17, 19]. In this paper, using the classic and refined matrix equation approaches from [3], we consider the rigorous perturbation bounds for these two factorizations with respect to multiplicative perturbation. That is, the original matrices K and B are respectively perturbed to

$$\widetilde{K} = QKQ^T \tag{1.4}$$

and

$$\widetilde{B} = S^T BS, \tag{1.5}$$

where $Q \in \mathbb{R}^{(m+n)\times(m+n)}$ and $S \in \mathbb{R}^{2n\times 2n}$ are called the multiplicative perturbation matrices. The multiplicative perturbations naturally arise from matrix scaling, a technical often used to improve the conditioning of a matrix. So they have important applications. Of course, the multiplicative perturbation can be turned into additive perturbation. However, in this case, the perturbation will lose their nature and the obtained additive perturbation bounds will not reveal the special structure of multiplicative perturbation. There were many works on the multiplicative perturbation analysis in the past. For example, some authors considered the multiplicative perturbation analysis of the polar decomposition [9, 10], the eigendecomposition of a Hermitian matrix and the singular value decomposition [11–13], and the QR factorization [2]. Recently, Fang [5] presented some multiplicative perturbation bounds for the generalized Cholesky factorization using the classic matrix equation approach. These results will be improved in this paper.

To simplify the presentation, we now introduce some notation which will be used in this paper. Given a matrix $A \in \mathbb{R}_r^{m\times n}$, $\|A\|_2$ and $\|A\|_F$ denote its spectral norm and

Frobenius norm, respectively. From [18], we have

$$\|XYZ\|_2 \leqslant \|X\|_2 \|Y\|_2 \|Z\|_2, \; \|XYZ\|_F \leqslant \|X\|_2 \|Y\|_F \|Z\|_2, \tag{1.6}$$

whenever the matrix product XYZ is defined. In addition, if A is square and nonsingular, we denote its condition number as $\kappa(A) = \left\|A^{-1}\right\|_2 \|A\|_2$.

For any matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, define

$$\mathrm{up}(A) = \begin{bmatrix} \frac{1}{2}a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \frac{1}{2}a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2}a_{nn} \end{bmatrix}. \tag{1.7}$$

The symbol is taken from [3, 4]. Obviously,

$$\|\mathrm{up}(A)\|_F \leqslant \|A\|_F. \tag{1.8}$$

If $A^T = A$, then

$$\|\mathrm{up}\,(A)\|_F \leqslant (1/\sqrt{2})\,\|A\|_F. \tag{1.9}$$

The inequality can be found in [3] and [4]. Let $\mathbb{D}_n \in \mathbb{R}^{n\times n}$ denote the set of all $n\times n$ positive definite diagonal matrices. Then, for any $D_n = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{D}_n$,

$$\mathrm{up}\,(AD_n) = \mathrm{up}\,(A)\,D_n. \tag{1.10}$$

Furthermore, from [4, Lemma 5.1],

$$\|\mathrm{up}(A) + D_n^{-1}\mathrm{up}(A^T)D_n\|_F \leqslant \sqrt{1+\zeta_{D_n}^2}\|A\|_F, \quad \zeta_{D_n} = \max_{1\leqslant i<j\leqslant n}\{\delta_j/\delta_i\}. \tag{1.11}$$

For any $2n\times 2n$ skew-symmetric matrix $B = (b_{ij})$, define

$$\mathrm{upb}(B) = \begin{bmatrix} \frac{1}{2}B_{11} & B_{12} & \cdots & B_{1n} \\ & \frac{1}{2}B_{22} & \cdots & B_{2n} \\ & & \ddots & \vdots \\ & & & \frac{1}{2}B_{nn} \end{bmatrix}, \tag{1.12}$$

where

$$B_{ii} = \begin{bmatrix} 0 & b_{2i-1,2i} \\ -b_{2i-1,2i} & 0 \end{bmatrix}, \quad B_{ij} = \begin{bmatrix} b_{2i-1,2j-1} & b_{2i-1,2j} \\ b_{2i,2j-1} & b_{2i,2j} \end{bmatrix}, \quad i<j, \quad B_{ij} = -B_{ji}^T, \quad i>j.$$

The notation follows from [8]. Clearly,

$$\|\mathrm{upb}(B)\|_F \leqslant \|B\|_F. \tag{1.13}$$

Meanwhile, from [8], it follows that

$$\|\mathrm{upb}\,(B)\|_F \leqslant (1/\sqrt{2})\,\|B\|_F\,. \tag{1.14}$$

Let $\widehat{\mathbb{D}}_{2n} \in \mathbb{R}^{2n\times 2n}$ denote the set of all $2n\times 2n$ positive definite diagonal matrices with 2×2 main diagonal blocks of the form $\begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$, $d>0$. Then for any matrix

$$\widehat{D}_{2n} = \begin{bmatrix} D_{11} & & & \\ & D_{22} & & \\ & & \ddots & \\ & & & D_{nn} \end{bmatrix} \in \widehat{\mathbb{D}}_{2n}, \quad D_{ii} = \begin{bmatrix} d_i & 0 \\ 0 & d_i \end{bmatrix}, \quad d_i > 0, \quad i = 1,2,\ldots,n, \tag{1.15}$$

it is easy to verify that

$$\mathrm{upb}\,(B)\,\widehat{D}_{2n} = \mathrm{upb}(B\widehat{D}_{2n}). \tag{1.16}$$

Moreover, the following property for “upb” also holds.

LEMMA 1.1. *For any matrix* $C \in \mathbb{R}^{2n\times 2n}$ *and* $\widehat{D}_{2n} \in \widehat{\mathbb{D}}_{2n}$ *defined by* (1.15)*,*

$$\phi \equiv \|\mathrm{upb}(C) + \widehat{D}_{2n}^{-1}\mathrm{upb}(C^T)\widehat{D}_{2n}\|_F \leqslant \sqrt{1+\zeta^2_{\widehat{D}_{2n}}}\,\|C\|_F, \tag{1.17}$$

where $\zeta_{\widehat{D}_{2n}} = \max\limits_{1\leqslant i<j\leqslant n}\{d_j/d_i\}$.

Proof. Obviously,

$$\begin{aligned}\phi^2 &= \sum_{i=1}^{n}\|C_{ii}\|_F^2 + \sum_{j=2}^{n-1}\sum_{i=1}^{j-1}\|C_{ij} + D_{ii}^{-1}C_{ji}D_{jj}\|_F^2 \\ &\leqslant \sum_{i=1}^{n}\|C_{ii}\|_F^2 + \sum_{j=2}^{n-1}\sum_{i=1}^{j-1}\left(\|C_{ij}\|_F + (d_i^{-1}d_j)\|C_{ji}\|_F\right)^2.\end{aligned}$$

This result with $\left(\|C_{ij}\|_F + (d_i^{-1}d_j)\|C_{ji}\|_F\right)^2 \leqslant (1+(d_i^{-1}d_j)^2)(\|C_{ij}\|_F^2 + \|C_{ji}\|_F^2)$, which is derived by the Cauchy-Schwarz theorem, leads to

$$\begin{aligned}\phi^2 &\leqslant \sum_{i=1}^{n}\|C_{ii}\|_F^2 + \sum_{j=2}^{n-1}\sum_{i=1}^{j-1}(1+(d_i^{-1}d_j)^2)(\|C_{ij}\|_F^2 + \|C_{ji}\|_F^2) \\ &\leqslant \|C\|_F + \zeta^2_{\widehat{D}_{2n}}\sum_{j=2}^{n-1}\sum_{i=1}^{j-1}(\|C_{ij}\|_F^2 + \|C_{ji}\|_F^2) \\ &\leqslant (1+\zeta^2_{\widehat{D}_{2n}})\|C\|_F^2.\end{aligned}$$

Taking the square root gives (1.17). □

In addition, the following two lemmas are also necessary later in this paper.

LEMMA 1.2. [3] *Let* $a,b>0$. *Let* $c(\cdot)$ *be a continuous function of a parameter* $t\in[0,1]$ *such that* $b^2-4ac(t)>0$ *holds for all* t. *Suppose that a continuous function* $x(t)$ *satisfies the quadratic inequality* $ax(t)^2-bx(t)+c(t)\geqslant 0$. *If* $c(0)=x(0)=0$, *then* $x(1)\leqslant(1/2a)(b-\sqrt{b^2-4ac(1)})$.

LEMMA 1.3. [8] *If* $R=(r_{ij})\in\mathbb{R}^{2n\times 2n}$ *is upper triangular and has* 2×2 *blocks of the form* $\begin{bmatrix} r_i & 0\\ 0 & r_i\end{bmatrix}$ *running down the main diagonal, then* $B=\widehat{J_{2n}}R-(\widehat{J_{2n}}R)^T$ *is skew-symmetric matrix and has the* 2×2 *main diagonal blocks* $\begin{bmatrix} 0 & 2r_i\\ -2r_i & 0\end{bmatrix}$. *Moreover,*

$$\widehat{J_{2n}}R=\mathrm{upb}(B). \tag{1.18}$$

2. Perturbation bounds for the generalized Cholesky factorization

In this section, we consider the rigorous multiplicative perturbation bounds for the generalized Cholesky factorization. The main results are given in the following theorem.

THEOREM 2.1. *Suppose that* $K\in\mathbb{R}^{(m+n)\times(m+n)}$ *is defined by* (1.1) *and has the factorization* (1.2). *Let* $Q=I_{m+n}+E\in\mathbb{R}^{(m+n)\times(m+n)}$. *If*

$$\kappa(L)\|E\|_F<(\sqrt{6}-2)/2, \tag{2.1}$$

then $\widetilde{K}=QKQ^T$ *has the unique generalized Cholesky factorization*

$$\widetilde{K}=QKQ^T=(L+\Delta L)J_{m+n}(L+\Delta L)^T, \tag{2.2}$$

and

$$\frac{\|\Delta L\|_F}{\|L\|_2}\leqslant(\sqrt{3}+\sqrt{6})\inf_{D_{m+n}\in\mathbb{D}_{m+n}}\sqrt{1+\zeta^2_{D_{m+n}}}\,\kappa(D_{m+n}L^{-1})\|Q-I_{m+n}\|_F. \tag{2.3}$$

Proof. For any $t\in[0,1]$, let $Q(t)=I_{m+n}+tE$. Then considering (1.2),

$$\begin{aligned}&(I_{m+n}+tE)K(I_{m+n}+tE)^T\\ &=L(J_{m+n}+tJ_{m+n}L^TE^TL^{-T}+tL^{-1}ELJ_{m+n}+t^2L^{-1}ELJ_{m+n}L^TE^TL^{-T})L^T.\end{aligned}$$

Using (1.6) and (2.1), we have

$$\begin{aligned}&\|tJ_{m+n}L^TE^TL^{-T}+tL^{-1}ELJ_{m+n}+t^2L^{-1}ELJ_{m+n}L^TE^TL^{-T}\|_F\\ &\leqslant 2\|L^{-1}EL\|_F+\|L^{-1}EL\|_F^2\leqslant 2\kappa(L)\|E\|_F+\kappa^2(L)\|E\|_F^2<1/2<1.\end{aligned}$$

According to [17, Lemma 2.1] i.e., [15, Lemma 6], we have

$$J_{m+n}+tJ_{m+n}L^TE^TL^{-T}+tL^{-1}ELJ_{m+n}+t^2L^{-1}ELJ_{m+n}L^TE^TL^{-T}=(I+\Gamma)J_{m+n}(I+\Gamma^T),$$

where Γ is lower triangular with zero diagonals. Thus, we obtain that $\widetilde{K}(t) = Q(t)KQ^T(t)$ has the unique generalized Cholesky factorization

$$\begin{aligned}\widetilde{K}(t) = Q(t)KQ^T(t) &= (I_{m+n} + tE)K(I_{m+n} + tE)^T \\ &= (L + \Delta L(t))J_{m+n}(L + \Delta L(t))^T,\end{aligned} \tag{2.4}$$

which, with $\Delta L(1) = \Delta L$, implies (2.2).

Next, we consider (2.3). From (2.4) and (1.2), it follows that

$$\begin{aligned}tLJ_{m+n}L^TE^T + tELJ_{m+n}L^T + t^2ELJ_{m+n}L^TE^T = LJ_{m+n}\Delta L^T(t) + \Delta L(t)J_{m+n}L^T \\ + \Delta L(t)J_{m+n}\Delta L^T(t).\end{aligned}$$

Premultipling the above equation by L^{-1} and postmultipling it by L^{-T} leads to

$$\begin{aligned}J_{m+n}\Delta L^T(t)L^{-T} + L^{-1}\Delta L(t)J_{m+n} &= J_{m+n}\Delta L^T(t)L^{-T} + (J_{m+n}\Delta L^T(t)L^{-T})^T \\ &= J_{m+n}L^TE^TL^{-T} + tL^{-1}ELJ_{m+n} + t^2L^{-1}ELJ_{m+n}L^TE^TL^{-T} \\ &\quad - L^{-1}\Delta L(t)J_{m+n}\Delta L^T(t)L^{-T}.\end{aligned} \tag{2.5}$$

Since $J_{m+n}\Delta L^T(t)L^{-T}$ is upper triangular, by the symbol (1.7), from (2.5), we have

$$\begin{aligned}J_{m+n}\Delta L^T(t)L^{-T} = \mathrm{up}\left(tJ_{m+n}L^TE^TL^{-T} + tL^{-1}ELJ_{m+n}\right) \\ + \mathrm{up}\left(t^2L^{-1}ELJ_{m+n}L^TE^TL^{-T} - L^{-1}\Delta L(t)J_{m+n}\Delta L^T(t)L^{-T}\right).\end{aligned} \tag{2.6}$$

Taking the Frobenius norm on the both sides of (2.6) and using (1.6) and (1.9) gives

$$\begin{aligned}\|L^{-1}\Delta L(t)\|_F &\leqslant (1/\sqrt{2})\|tL^{-1}ELJ_{m+n} + tJ_{m+n}L^TE^TL^{-T}\|_F + (1/\sqrt{2})\|tL^{-1}ELJ_{m+n}\|_F^2 \\ &\quad + (1/\sqrt{2})\|L^{-1}\Delta L(t)\|_F^2 \\ &\leqslant \sqrt{2}\|tL^{-1}EL\|_F + (1/\sqrt{2})\|tL^{-1}EL\|_F^2 + (1/\sqrt{2})\|L^{-1}\Delta L(t)\|_F^2.\end{aligned} \tag{2.7}$$

Let $x(t) = \|L^{-1}\Delta L(t)\|_F$ and $c(t) = 2\|tL^{-1}EL\|_F + \|tL^{-1}EL\|_F^2$. Then (2.7) can be rewritten as

$$x^2(t) - \sqrt{2}x(t) + c(t) \geqslant 0.$$

Since

$$\Delta = 2 - 4c(t) \geqslant 2 - 8\|L^{-1}EL\|_F - 4\|L^{-1}EL\|_F^2 \geqslant 2 - 8\kappa(L)\|E\|_F - 4\kappa^2(L)\|E\|_F^2 > 0,$$

$x(t)$ and $c(t)$ are continuous with $t \in [0,1]$, and $c(0) = x(0) = 0$, from Lemma 1.2, it is seen that

$$\|L^{-1}\Delta L\|_F \leqslant (1/\sqrt{2})\left(1 - \sqrt{1 - 4\|L^{-1}EL\|_F - 2\|L^{-1}EL\|_F^2}\right). \tag{2.8}$$

Let $L = \widehat{L}D_{m+n}$ for any $D_{m+n} \in \mathbb{D}_{m+n}$. Then, from (2.6) with $t = 1$ and (1.10), we have

$$J_{m+n}\Delta L^T \widehat{L}^{-T} = \mathrm{up}\left(J_{m+n}L^T E^T \widehat{L}^{-T} + D_{m+n}^{-1}(\widehat{L}^{-1}ELJ_{m+n})D_{m+n}\right) + \mathrm{up}\left(L^{-1}ELJ_{m+n}L^T E^T \widehat{L}^{-T} - L^{-1}\Delta L J_{m+n}\Delta L^T \widehat{L}^{-T}\right). \tag{2.9}$$

Taking the Frobnius norm on the both sides of (2.9) and using (1.11), (1.8), and (1.6) yields

$$\|\widehat{L}^{-1}\Delta L\|_F \leqslant \sqrt{1+\zeta_{D_{m+n}}^2}\|\widehat{L}^{-1}EL\|_F + \|L^{-1}EL\|_F\|\widehat{L}^{-1}EL\|_F + \|L^{-1}\Delta L\|_F\|\widehat{L}^{-1}\Delta L\|_F. \tag{2.10}$$

Considering (2.8), (2.1), (1.6), and $\sqrt{1+\zeta_{D_{m+n}}^2} > 1$, from (2.10), we obtain

$$\|\widehat{L}^{-1}\Delta L\|_F \leqslant \frac{\sqrt{2}\left(\sqrt{1+\zeta_{D_{m+n}}^2} + \kappa(L)\|E\|_F\right)\|\widehat{L}^{-1}EL\|_F}{\sqrt{2}-1+\sqrt{1-4\|L^{-1}EL\|_F - 2\|L^{-1}EL\|_F^2}} \tag{2.11}$$

$$\leqslant (2+\sqrt{2})\left(\sqrt{1+\zeta_{D_{m+n}}^2} + \kappa(L)\|E\|_F\right)\|\widehat{L}^{-1}EL\|_F \tag{2.12}$$

$$\leqslant (\sqrt{3}+\sqrt{6})\sqrt{1+\zeta_{D_{m+n}}^2}\|\widehat{L}^{-1}EL\|_F. \tag{2.13}$$

Combining (2.13) with

$$\|\Delta L\|_F \leqslant \|\widehat{L}\|_2\|\widehat{L}^{-1}\Delta L\|_F \quad \text{and} \quad E = Q - I_{m+n}, \tag{2.14}$$

and noting (1.6) leads to (2.3). □

REMARK 2.1. If

$$\kappa(L)\|E\|_F < \sqrt{2}-1, \tag{2.15}$$

we can verify that

$$2\kappa(L)\|E\|_F + \kappa^2(L)\|E\|_F^2 < 1,$$

which guarantees that $\widetilde{K}(t) = Q(t)KQ^T(t)$ has the unique generalized Cholesky factorization (2.4). As a result, the condition of the existence and uniqueness of the generalized Cholesky factorization (2.2) can be weakened to (2.15).

REMARK 2.2. With (2.11), (2.12) and (2.14), the following rigorous multiplicative perturbation bounds can also be derived:

$$\frac{\|\Delta L\|_F}{\|L\|_2} \leqslant \frac{\sqrt{2}\inf_{\widehat{D}_{m+n}\in\widehat{\mathbb{D}}_{m+n}}\left(\sqrt{1+\zeta_{D_{m+n}}^2}+\kappa(L)\|Q-I_{m+n}\|_F\right)\kappa(D_{m+n}L^{-1})\|Q-I_{m+n}\|_F}{\sqrt{2}-1+\sqrt{1-4\kappa(L)\|Q-I_{m+n}\|_F-2\kappa^2(L)\|Q-I_{m+n}\|_F^2}} \tag{2.16}$$

$$\leqslant (2+\sqrt{2})\inf_{D_{m+n}\in\mathbb{D}_{m+n}}(\sqrt{1+\zeta_{D_{m+n}}^2}+\kappa(L)\|Q-I_{m+n}\|_F)\kappa(D_{m+n}L^{-1})\|Q-I_{m+n}\|_F. \tag{2.17}$$

In comparison, the bound (2.16) is better than (2.17), which in turn is better than (2.3). However, the above two bounds are more complicated.

REMARK 2.3. We can derive the first-order multiplicative perturbation bound from (2.16) as follows:

$$\begin{aligned}\frac{\|\Delta L\|_F}{\|L\|_2} &\lesssim \inf_{\widehat{D}_{m+n}\in\widehat{\mathbb{D}}_{m+n}}\sqrt{1+\zeta_{D_{m+n}}^2}\kappa(D_{m+n}L^{-1})\|Q-I_{m+n}\|_F \\ &= \inf_{\widehat{D}_{m+n}\in\widehat{\mathbb{D}}_{m+n}}\sqrt{1+\zeta_{D_{m+n}}^2}\kappa(D_{m+n}L^{-1})\|E\|_F. \end{aligned} \tag{2.18}$$

Obviously, (2.3) is a constant multiple of (2.18).

REMARK 2.4. The following rigorous multiplicative perturbation bound is presented in [5, Theorem 2]:

$$\frac{\|\Delta L\|_F}{\|L\|_2} \leqslant \|Q-I_{m+n}\|_F. \tag{2.19}$$

Clearly, it is better than (2.3). However, the bound is only valid for the multiplicative perturbation matrices which are lower triangular with positive diagonal elements. In this case, the exact value of ΔL can be got easily by considering the fact that $\Delta L = QL - L$. While (2.3) is valid for all of multiplicative perturbation matrices whenever they satisfy the condition (2.1). Moreover, numerical experiment indicated that the bounds (2.3) and (2.19) may have the same order of magnitude. For example, let $L = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}$ and $D_{m+n} = \mathrm{diag}(\gamma,\gamma)$ with $0<\gamma<1$. Then we have that $\sqrt{1+\zeta_{D_{m+n}}^2}\kappa(D_{m+n}^{-1}L) = \sqrt{2}$.

REMARK 2.5. The following first-order multiplicative perturbation bounds are presented in [5, Theorem 3]:

$$\frac{\|\Delta L(t)\|_F}{\|L\|_2} \lesssim \sqrt{2}\|L^{-1}\|_2^2\|t\Delta Q\|_2\|K\|_F, \tag{2.20}$$

$$\frac{\|\Delta L(t)\|_F}{\|L\|_2} \lesssim \sqrt{2}\|L^{-1}\|_2^2\|t\Delta Q\|_F\|K\|_2, \tag{2.21}$$

where $t \in (-\varepsilon, \varepsilon)$ and ε is assumed to be small enough. Obviously, if we set $t\Delta Q = E$, the two bounds are the same as (2.18) in form. However, the bound (2.18) can be much better than (2.20) and (2.21). For example, let $L = \begin{bmatrix} 1 & 0 \\ \gamma & \gamma \end{bmatrix}$ and $D_{m+n} = \mathrm{diag}(1, \gamma)$ with $0 < \gamma \ll 1$. Then we have that $\sqrt{2}\|L^{-1}\|_2^2\|K\|_2 = O(1/\gamma^2)$, $\sqrt{2}\|L^{-1}\|_2^2\|K\|_F = O(1/\gamma)$ and $\sqrt{1+\zeta^2_{D_{m+n}}}\kappa(D^{-1}_{m+n}L) = O(\gamma)$. These results show that the bound (2.18) can be arbitrarily smaller than (2.20) and (2.21). Furthermore, in [5, Theorem 3], the author only presents the existence of ε instead of a accurate value of ε. While, the condition (2.1) above clearly presents the constraint on the multiplicative perturbation matrix.

3. Perturbation bounds for the Cholesky-like factorization

Now we consider the rigorous multiplicative perturbation bounds for the Cholesky-like factorization. Some detailed deductions are omitted since they are similar to those for the generalized Cholesky factorization.

THEOREM 3.1. *Suppose that the skew-symmetric matrix $B \in \mathbb{R}^{2n\times 2n}$ has the Cholesky-like factorization* (1.3). *Let $S = I_{2n} + F \in \mathbb{R}^{2n\times 2n}$. If*

$$\kappa(R)\|F\|_F < (\sqrt{6}-2)/2, \tag{3.1}$$

then

$$\widetilde{B} = S^T BS = (I_{2n}+F)^T B(I_{2n}+F) = (R+\Delta R)^T \widehat{J}_{2n}(R+\Delta R), \tag{3.2}$$

and

$$\frac{\|\Delta R\|_F}{\|R\|_2} \leqslant (\sqrt{3}+\sqrt{6}) \inf_{\widehat{D}_{2n}\in\widehat{\mathbb{D}}_{2n}} \sqrt{1+\zeta^2_{\widehat{D}_{2n}}}\kappa(\widehat{D}_{2n}R^{-1})\|S-I_{2n}\|_F. \tag{3.3}$$

Proof. For any $t \in [0,1]$, let $S(t) = I_{2n} + tF \in \mathbb{R}^{2n\times 2n}$. Then

$$S^T(t)BS(t) = (I_{2n}+tF)^T B(I_{2n}+tF) = R^T(\widehat{J}_{2n}+M(t))R, \tag{3.4}$$

where $M(t) = t\widehat{J}_{2n}RFR^{-1} + tR^{-T}F^TR^T\widehat{J}_{2n} + t^2R^{-T}F^TR^T\widehat{J}_{2n}RFR^{-1}$. Using (1.6) and (3.1), we have

$$\|M(t)\|_2 < 1/2 < 1,$$

which implies (see [18])

$$\|M(t)[\langle 2k\rangle]\|_2 < 1, \quad k = 1, 2, \dots, n.$$

Obviously, $\|\widehat{J}_{2n}[\langle 2k\rangle]\|_2 = 1$. So $\|M(t)[\langle 2k\rangle]\widehat{J}_{2n}[\langle 2k\rangle]\|_2 < 1$. As a result, $I_{2k} - M(t)[\langle 2k\rangle]\widehat{J}_{2n}[\langle 2k$ is nonsingular. Therefore,

$$(\widehat{J}_{2n}+M(t))[\langle 2k\rangle] = \widehat{J}_{2n}[\langle 2k\rangle](I_{2k} - M(t)[\langle 2k\rangle]\widehat{J}_{2n}[\langle 2k\rangle])$$

is also nonsingular since $\widehat{J}_{2n}[\langle 2k\rangle]$ is nonsingular. Furthermore, $\widehat{J}_{2n}+M(t)$ is skew-symmectric. Thus, from [1], we have

$$\widehat{J}_{2n}+M(t)=\widehat{R}^T(t)\widehat{J}_{2n}\widehat{R}(t), \tag{3.5}$$

where $\widehat{R}(t)$ is upper triangular with 2×2 main diagonal blocks:

$$\begin{bmatrix} \widehat{r}_{ii}(t) & 0 \\ 0 & \pm\widehat{r}_{ii}(t) \end{bmatrix}, \quad \widehat{r}_{ii}(t)>0, \quad i=1,2,\ldots,n.$$

Substituting (3.5) into (3.4) gives

$$S^T(t)BS(t)=(I_{2n}+tF)^TB(I_{2n}+tF)=(\widehat{R}(t)R)^T\widehat{J}_{2n}(\widehat{R}(t)R). \tag{3.6}$$

Considering the structures of R and $\widehat{R}(t)$, it is easy to check that (3.6) is the Cholesky-like factorization of $S^T(t)BS(t)$. We can rewritten (3.6) as

$$S^T(t)BS(t)=(I_{2n}+tF)^TB(I_{2n}+tF)=(R+\Delta R(t))^T\widehat{J}_{2n}(R+\Delta R(t)), \tag{3.7}$$

where $R+\Delta R(t)=\widehat{R}(t)R$. Setting $\Delta R(1)=\Delta R$, from (3.7), we obtain (3.2).

Next, we consider (3.3). Similar to the proof of Theorem 2.1, from (3.7), we get

$$\begin{aligned} &t\widehat{J}_{2n}RFR^{-1}+tR^{-T}F^TR^T\widehat{J}_{2n}+t^2R^{-T}F^TR^T\widehat{J}_{2n}RFR^{-1} \\ &=\widehat{J}_{2n}\Delta R(t)R^{-1}+R^{-T}\Delta R^T(t)\widehat{J}_{2n}+R^{-T}\Delta R^T(t)\widehat{J}_{2n}\Delta R(t)R^{-1}. \end{aligned}$$

Considering the forms of $\widehat{J}_{2n}$ and R and by Lemma 1.3 and the symbol defined by (1.12), from the above equation we have

$$\begin{aligned} \widehat{J}_{2n}\Delta R(t)R^{-1}=&\operatorname{upb}\left(t\widehat{J}_{2n}RFR^{-1}+tR^{-T}F^TR^T\widehat{J}_{2n}+t^2R^{-T}F^TR^T\widehat{J}_{2n}RFR^{-1}\right) \\ &-\operatorname{upb}\left(R^{-T}\Delta R^T(t)\widehat{J}_{2n}\Delta R(t)R^{-1}\right). \end{aligned} \tag{3.8}$$

Taking the Frobenuis norm on the both sides of (3.8) and using (1.14) and (1.6) yields

$$\|\Delta R(t)R^{-1}\|_F\leqslant\sqrt{2}\|tRFR^{-1}\|_F+(1/\sqrt{2})\|tRFR^{-1}\|_F^2+(1/\sqrt{2})\|\Delta R(t)R^{-1}\|_F^2. \tag{3.9}$$

Similar to the proof of Theorem 2.1, from (3.9), we have

$$\|\Delta RR^{-1}\|_F\leqslant(1/\sqrt{2})\left(1-\sqrt{1-4\|RFR^{-1}\|_F-2\|RFR^{-1}\|_F^2}\right). \tag{3.10}$$

Let $R=\widehat{R}\widehat{D}_{2n}$ with $\widehat{D}_{2n}\in\widehat{\mathbb{D}}_{2n}$. Then from (3.8) with $t=1$ and using (1.16), it follows that

$$\begin{aligned} \widehat{J}_{2n}\Delta R\widehat{R}^{-1}=&\operatorname{upb}\left(\widehat{J}_{2n}RF\widehat{R}^{-1}+D_{2n}^{-1}(\widehat{R}^{-T}F^TR^T\widehat{J}_{2n})D_{2n}\right) \\ &+\operatorname{upb}\left(R^{-T}F^TR^T\widehat{J}_{2n}RF\widehat{R}^{-1}-R^{-T}\Delta R^T\widehat{J}_{2n}\Delta R\widehat{R}^{-1}\right). \end{aligned} \tag{3.11}$$

Taking the Frobenius norm on the both sides of (3.11), by Lemma 1.1 and using (1.6) and (1.13), we get

$$\|\Delta R\widehat{R}^{-1}\|_F \leqslant \sqrt{1+\zeta^2_{\widehat{D}_{2n}}}\|RF\widehat{R}^{-1}\|_F + \|RFR^{-1}\|_F\|RF\widehat{R}^{-1}\|_F + \|\Delta RR^{-1}\|_F\|\Delta R\widehat{R}^{-1}\|_F,$$

which combined with (3.10) and $\sqrt{1+\zeta^2_{\widehat{D}_{2n}}} > 1$ yields

$$\begin{aligned}\|\Delta R\widehat{R}^{-1}\|_F &\leqslant \frac{\sqrt{2}\left(\sqrt{1+\zeta^2_{\widehat{D}_{2n}}}+\kappa(R)\|F\|_F\right)\|RF\widehat{R}^{-1}\|_F}{\sqrt{2}-1+\sqrt{1-4\|RFR^{-1}\|_F-2\|RFR^{-1}\|_F^2}},\\ &\leqslant (2+\sqrt{2})\left(\sqrt{1+\zeta^2_{\widehat{D}_{2n}}}+\kappa(R)\|F\|_F\right)\|RF\widehat{R}^{-1}\|_F,\\ &\leqslant (\sqrt{3}+\sqrt{6})\sqrt{1+\zeta^2_{\widehat{D}_{2n}}}\|RF\widehat{R}^{-1}\|_F. \end{aligned} \tag{3.12}$$

Combining (3.12) with $\|\Delta R\|_F \leqslant \|\Delta R\widehat{R}^{-1}\|_F\|\widehat{R}\|_2$ and $F = S - I_{2n}$, and using (1.6) leads to (3.3). □

REMARK 3.1. The proof of this theorem shows that the condition of the existence of the Cholesky-like factorization (3.2) can be weakened to:

$$\kappa(R)\|F\|_F < \sqrt{2}-1. \tag{3.13}$$

REMARK 3.2. Similar to Remarks 2.2 and 2.3, we can get the following rigorous and first-order multiplicative perturbation bounds for the Cholesky-like factorization:

$$\frac{\|\Delta R\|_F}{\|R\|_2} \leqslant \frac{\sqrt{2}\inf\limits_{\widehat{D}_{2n}\in\widehat{\mathbb{D}}_{2n}}\left(\sqrt{1+\zeta^2_{\widehat{D}_{2n}}}+\kappa(R)\|F\|_F\right)\kappa(D_{2n}R^{-1})\|F\|_F}{\sqrt{2}-1+\sqrt{1-4\kappa(R)\|F\|_F-2\kappa^2(R)\|F\|_F^2}}, \tag{3.14}$$

$$\frac{\|\Delta R\|_F}{\|R\|_2} \leqslant (2+\sqrt{2})\inf_{\widehat{D}_{2n}\in\widehat{\mathbb{D}}_{2n}}\left(\sqrt{1+\zeta^2_{\widehat{D}_{2n}}}+\kappa(R)\|S-I_{2n}\|_F\right)\kappa(\widehat{D}_{2n}R^{-1})\|S-I_{2n}\|_F, \tag{3.15}$$

$$\frac{\|\Delta R\|_F}{\|R\|_2} \lesssim \inf_{\widehat{D}_{2n}\in\widehat{\mathbb{D}}_{2n}}\sqrt{1+\zeta^2_{\widehat{D}_{2n}}}\kappa(\widehat{D}_{2n}R^{-1})\|S-I_{2n}\|_F. \tag{3.16}$$

Clearly, the difference between the bounds (3.3) and (3.16) is a constant $\sqrt{3}+\sqrt{6}$. And in comparison, the bound (3.14) is better than (3.15), which in turn is better than (3.3). However, the bounds (3.15) and (3.16) are more complicated.

REMARK 3.3. In [8, Theorem 2.2] and [8, Theorem 2.3], the authors presented the following first-order and rigorous perturbation bounds for the Cholesky-like factorization with respect to additive perturbation:

$$\frac{\|\Delta R\|_F}{\|R\|_2} \lesssim \frac{1}{\sqrt{2}}\|R^{-1}\|_2^2\|B\|_F\varepsilon, \tag{3.17}$$

$$\frac{\|\Delta R\|_F}{\|R\|_2} \leqslant \frac{1}{2}(1-\sqrt{1-4\|R^{-1}\|_2^2\|\Delta B\|_F}), \tag{3.18}$$

where $\varepsilon \geqslant \frac{\|\Delta B\|_F}{\|B\|_F}$, ΔB is the perturbation matrix, and the condition for the bound (3.18) to hold is

$$4\|R^{-1}\|_2^2\|\Delta B\|_F < 1. \tag{3.19}$$

After turning the multiplicative perturbation into the additive perturbation, we find that the bounds (3.16) and (3.3) can be much smaller than the ones (3.17) and (3.18). The following is a simple example. Let $R = F = \widehat{D}_{2n} = \mathrm{diag}(1,1,\gamma,\gamma)$ with $0 < \gamma \ll 1$. Then

$$\sqrt{1+\zeta_{\widehat{D}_{2n}}^2}\,\kappa(\widehat{D}_{2n}R^{-1})\|S-I_{2n}\|_F = \Theta(\gamma),$$
$$\frac{1}{\sqrt{2}}\|R^{-1}\|_2^2\|B\|_F\varepsilon \geqslant \frac{1}{\sqrt{2}}\|R^{-1}\|_2^2\|\Delta B\|_F = \Theta(\frac{1}{\gamma}),$$
$$\frac{1}{2}(1-\sqrt{1-4\|R^{-1}\|_2^2\|\Delta B\|_F}) = \Theta(\frac{1}{\gamma}),$$

which demonstrate the fact that we expect.

4. Concluding remarks

In this paper, some new rigorous multiplicative perturbation bounds and the corresponding first-order perturbation bounds for the generalized Cholesky factorization and the Cholseky-like factorization are obtained. In comparison, these bounds either can be much shaper than the ones presented in [5, 8] or have a broader range of applications.

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