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OPTIMAL CONVEX COMBINATIONS BOUNDS OF CENTROIDAL AND HARMONIC MEANS FOR WEIGHTED GEOMETRIC MEAN OF LOGARITHMIC AND IDENTRIC MEANS

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Abstract. In this paper, optimal convex combination bounds of centroidal and harmonic means for weighted geometric mean of logarithmic and identric means are proved. We find the greatest value $\lambda(\alpha)$ and the least value $\Delta(\alpha)$ for each $\alpha \in (0,1)$ such that the double inequality:

$$\lambda C(a,b) + (1-\lambda)H(a,b) < L^{\alpha}(a,b)I^{1-\alpha}(a,b) < \Delta C(a,b) + (1-\Delta)H(a,b)$$

holds for all $a,b > 0$ with $a \neq b$. Here, $C(a,b)$, $H(a,b)$, $L(a,b)$ and $I(a,b)$ denote centroidal, harmonic, logarithmic and identric means of two positive numbers a and b, respectively.

1. Introduction

Recently, means have been the subject of intensive research. In particular, many remarkable inequalities for the centroidal, harmonic, logarithmic and identric means can be found in the literature [4], [11], [12].

We recall some definitions.

The centroidal, harmonic, logarithmic, identric, and weighted geometric means of two positive real numbers a, b, $a \neq b$, are defined, respectively, as follows:

$$C(a,b) = \frac{2(a^2+ab+b^2)}{3(a+b)},$$

$$H(a,b) = \frac{2ab}{(a+b)},$$

$$L(a,b) = \frac{a-b}{\log a - \log b},$$

$$I(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{(a-b)}},$$

$$G_{\alpha}(a,b) = a^{\alpha}b^{1-\alpha} \quad \text{for} \quad 0 \leqslant \alpha \leqslant 1.$$

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Means have many applications not only in in mathematics, but in physics, economics, meteorology,... (see for example [5], [7], [8]).

It is well-known that the following inequalities hold:

$$H(a,b) < L(a,b) < I(a,b) < C(a,b) \quad \text{for positive} \quad a \neq b. \tag{1}$$

In the paper [4], authors inspired by (1), proved the following theorems:

THEOREM 1.

$$\alpha_1 C(a,b) + (1-\alpha_1)H(a,b) < L(a,b) < \beta_1 C(a,b) + (1-\beta_1)H(a,b) \tag{2}$$

holds for all $a,b > 0$, *with* $a \neq b$ *if and only if* $\alpha_1 \leqslant 0$, $\beta_1 \geqslant 1/2$.

THEOREM 2.

$$\alpha_2 C(a,b) + (1-\alpha_2)H(a,b) < I(a,b) < \beta_2 C(a,b) + (1-\beta_2)H(a,b) \tag{3}$$

holds for all $a,b > 0$, *with* $a \neq b$ *if and only if* $\alpha_2 \leqslant 3/(2e) = 0.551819$, $\beta_2 \geqslant 5/8$.

Similar double inequality was proved by Alzer and Qiu [1]:

$$\alpha A(a,b) + (1-\alpha)G(a,b) < I(a,b) < \beta A(a,b) + (1-\beta)G(a,b) \tag{4}$$

holds for all $a,b > 0$, with $a \neq b$ if and only if $\alpha \leqslant 2/3$, $\beta \geqslant 2/e = 0.73575$.

From results of [4], it is natural to ask what is the greatest function $\lambda(\alpha)$, and the least function $\Delta(\alpha)$, for $0 \leqslant \alpha \leqslant 1$ such that the double inequality:

$$\begin{gathered} \lambda(\alpha)C(a,b) + (1-\lambda(\alpha))H(a,b) < L^{\alpha}(a,b)I^{1-\alpha}(a,b) \\ < \Delta(\alpha)C(a,b) + (1-\Delta(\alpha))H(a,b) \end{gathered}$$

holds for all $a,b > 0$ with $a \neq b$, $0 \leqslant \alpha \leqslant 1$. The purpose of this paper is to find the optimal functions $\lambda(\alpha)$, $\Delta(\alpha)$. For some other details about means, see [1]–[12] and the related references cited there in.

2. Main results

LEMMA 1. *Let*

$$g(t,\alpha) = \frac{3}{2et^{\frac{t}{1-t}}}\left(3 + t - \alpha\frac{(1+t)(t\ln t - t + 1)}{t\ln t} - (1-\alpha)\frac{(1+t)(\ln t - t + 1)}{1-t}\right) \tag{5}$$

for $0 < t < 1$, $0 \leqslant \alpha \leqslant 1$. *Then* $g(t,0) > 0$, $g(t,1) > 0$ *for* $0 < t < 1$.

Proof. $g(t,0) > 0$ follows from $s^*(t,0) > 0$ (see Lemma 2).

From (5) with $\alpha = 1$ we conclude that

$$(3+t)t\ln t - (1+t)(t\ln t - t + 1) < 0. \tag{6}$$

Some calculation gives that (6) leads to the evident inequality $2t\ln t-(1-t)^2<0$. □

Denote

$$h(t)=\frac{(1-t)et^{\frac{t}{1-t}}}{-\ln t}\quad \text{for}\quad t\in(0,1). \tag{7}$$

LEMMA 2. *Let*

$$s^*(t,\alpha)=\alpha\ln(h(t))-\ln\left(\frac{3(1+t)}{g(t,\alpha)}\right)=\alpha\ln\left(\frac{e(1-t)t^{\frac{t}{1-t}}}{-\ln t}\right)-\ln\left(\frac{3(1+t)}{g(t,\alpha)}\right)$$

for $0<t<1,\ 0\leqslant\alpha\leqslant 1$. *Then* $s^*(t,0)>0,\ s^*(t,1)>0,\ s^{*''}_{\alpha,\alpha}(t,\alpha)<0$ *for* $0<t<1$, $0<\alpha<1$.

Proof. From

$$s^{*'}_{\alpha}(t,\alpha)=\ln(h(t))+\frac{g'_{\alpha}(t,\alpha)}{g(t,\alpha)}$$

we have

$$s^{*''}_{\alpha,\alpha}(t,\alpha)=-\frac{g'^2_{\alpha}(t,\alpha)}{g^2(t,\alpha)}<0.$$

Now we show $s^*(t,0)>0$. The inequality is equivalent to

$$u(t)=g(t,0)-3(1+t)>0\quad \text{for}\quad t\in(0,1). \tag{8}$$

Inequality (8) will be proved if we show

$$(3+t)(1-t)-(1+t)(\ln t-t+1)>2e(1+t)(1-t)t^{\frac{t}{1-t}}. \tag{9}$$

Rewriting inequality (9) we obtain

$$r(t)=2(1-t)-(1+t)\ln t-2(1-t^2)e^{1+\frac{t\ln t}{1-t}}>0\quad \text{for}\quad t\in(0,1).$$

Denote

$$v(t)=\ln(2(1-t)-(1+t)\ln t)-\ln(2(1-t^2))-\frac{1-t+t\ln t}{1-t}. \tag{10}$$

Because $v(1)=0$ to show (10) it suffices to prove $v'(t)<0$ for $t\in(0,1)$.

Simple calculation gives

$$v'(t)=-\frac{3t+1+t\ln t}{t(2(1-t)-(1+t)\ln t)}+\frac{2t}{1-t^2}-\frac{1-t+\ln t}{(1-t)^2}.$$

The inequality $v'(t)<0$ is equivalent to

$$w(t)=\ln^2 t-\frac{2(1-t)}{1+t}\ln t-\frac{(1-t)^2(t^2+6t+1)}{t(1+t)^2}<0.$$

Simple calculation leads to $w(t) < 0$ if and only if

$$\left(\ln t - \frac{1-t}{1+t}\right)^2 < \frac{(1-t)^2}{(1+t)^2}\left(\frac{t^2+7t+1}{t}\right).$$

From this we have that, it suffices to show that

$$-\frac{1+t}{1-t}\ln t < \sqrt{\frac{t^2+7t+1}{t}} - 1. \tag{11}$$

Inequality (11) is equivalent to

$$o(t) = \frac{1-t}{1+t}\left(\sqrt{\frac{t^2+7t+1}{t}} - 1\right) + \ln t > 0.$$

We show that $o'(t) < 0$ for $t \in (0,1)$. Simple calculation gives

$$o'(t) = \frac{2\sqrt{t^2+7t+1}}{(1+t)^2} - \frac{2(t^2+7t+1)}{(1+t)^2\sqrt{t}} + \frac{(1-t)(t^2-1)}{(1+t)2t\sqrt{t}} + \frac{\sqrt{t^2+7t+1}}{t}.$$

To prove inequality (8) we first show that

$$(t^2+7t+1)(1+4t+t^2)^2 < \frac{1}{4t}(1+4t+26t^2+4t^3+t^4)^2. \tag{12}$$

Inequality (12) can be rewriting as

$$m(t) = -t^8 - 4t^7 - 8t^6 + 84t^5 - 142t^4 + 84t^3 - 8t^2 - 4t - 1 < 0.$$

It is easy to see that

$$m(t) = -(1-t)^4(t^4+8t^3+34t^2+8t+1) < 0 \quad \text{for} \quad t \in (0,1).$$

Now we prove $s^*(t,1) > 0$ for $t \in (0,1)$. The inequality $s^*(t,1) > 0$ is equivalent to

$$h(t)g(t,1) - 3(1+t) > 0 \quad \text{for} \quad t \in (0,1). \tag{13}$$

Inequality (13) can be rewriting as

$$\ln^2 t + \frac{1-t}{1+t}\ln t - \frac{(1-t)^2}{2t} < 0.$$

Simple calculation leads to

$$\left(\ln t + \frac{1-t}{2(1+t)}\right)^2 < \frac{(1-t)^2}{4(1+t)^2}\left(\frac{2t^2+5t+2}{t}\right). \tag{14}$$

Inequality (14) will be shown if we prove that

$$-\ln t - \frac{1-t}{2(1+t)} < \frac{(1-t)}{2(1+t)}\sqrt{\frac{2t^2+5t+2}{t}}$$

because of $2(1+t)\ln t + 1 - t < 0$.

Indeed, if we denote $z(t) = \ln t + (1-t)/(2(1+t))$ then $z(1) = 0$ and $z'(t) = 2\ln t + 2/t + 1 > 0$. It follows from $z'(1) = 3$ and $z''(t) = 2(t-1)/(t^2) < 0$. Denote

$$a(t) = \frac{1-t}{2(1+t)}\left(\sqrt{\frac{2t^2+5t+2}{t}}+1\right) + \ln t > 0.$$

From $a(1) = 0$ it suffices to show that $a'(t) < 0$ for $t \in (0,1)$. Simple calculation gives

$$a'(t) = -\frac{\sqrt{2t^2+5t+2}}{\sqrt{t}(1+t)^2} + \frac{t^2+t+1}{t(1+t)^2} + \frac{(1-t)\sqrt{t}(t^2-1)}{2t^2(1+t)\sqrt{2t^2+5t+2}}.$$

The inequality $a'(t) < 0$ is equivalent to

$$2\sqrt{t}(1+t+t^2)\sqrt{2t^2+5t+2} < 1 + 4t + 8t^2 + 4t^3 + t^4,$$

which can be rewriting as

$$4t(1+t+t^2)^2(2t^2+5t+2) < (1+4t+8t^2+4t^3+t^4)^2. \tag{15}$$

Easy computation leads that inequality (15) is

$$1 - 4t^2 + 6t^4 - 4t^6 + t^8 = (t^2-1)^4 > 0.$$

The proof is complete. $\square$

Our main result reads as follows

THEOREM 3. *The double inequality*

$$\lambda C(a,b) + (1-\lambda)H(a,b) < L^{\alpha}(a,b)I^{1-\alpha}(a,b) < \Delta C(a,b) + (1-\Delta)H(a,b) \tag{16}$$

holds for all $a,b > 0$ *with* $a \neq b$, $\alpha \in (0,1>$ *if and only if* $\lambda(\alpha) \leqslant 0$ *and* $\Delta(\alpha) \geqslant (5-\alpha)/8$.

Proof. Suppose $a,b > 0$ with $a > b$, $\alpha \in (0,1)$, $t = b/a < 1$. Using

$$\frac{C(a,b)}{a} = \frac{2(1+t+t^2)}{3(1+t)}, \quad \frac{H(a,b)}{a} = \frac{2t}{1+t},$$

$$\frac{L(a,b)}{a} = \frac{1-t}{-\ln t}, \quad \frac{I(a,b)}{a} = \frac{1}{et^{\frac{t}{1-t}}}$$

we can write inequality (16) in the form

$$\lambda(\alpha)\left(\frac{2(1+t+t^2)}{3(1+t)} - \frac{2t}{1+t}\right) < \left(\frac{1-t}{-\ln t}\right)^{\alpha}\left(\frac{1}{et^{\frac{t}{1-t}}}\right)^{1-\alpha} - \frac{2t}{1+t}$$
$$< \Delta(\alpha)\left(\frac{2(1+t+t^2)}{3(1+t)} - \frac{2t}{1+t}\right).$$

Denote

$$F(t,\alpha)=\frac{3(1+t)}{2(1-t)^2}\left(\left(\frac{1-t}{-\ln t}\right)^{\alpha}\left(\frac{1}{et^{\frac{t}{1-t}}}\right)^{1-\alpha}-\frac{2t}{1+t}\right). \tag{17}$$

We show $F_t'(t,\alpha)>0$, $\lambda(\alpha)=\lim_{t\to 0^+}F(t,\alpha)$ and $\Delta(\alpha)=\lim_{t\to 1^-}F(t,\alpha)$.

Rewriting (17) we have

$$F(t,\alpha)=\frac{3(1+t)}{2e(1-t)^2t^{\frac{t}{1-t}}}\left(\frac{(1-t)et^{\frac{t}{1-t}}}{-\ln t}\right)^{\alpha}-\frac{3t}{(1-t)^2}.$$

Using elementary calculations we obtain

$$F_t'(t,\alpha)=\frac{1}{(1-t)^3}\left(h(t)^{\alpha}g(t,\alpha)-3(1+t)\right),$$

where $h(t)$ is defined in (7) and $g(t,\alpha)$ is defined in (5). It implies that it suffices to prove $h(t)^{\alpha}g(t,\alpha)-3(1+t)>0$ for $t\in(0,1)$, $\alpha\in(0,1)$. It follows from Lemma 1 and Lemma 2, because of $0<h(t)<1$ is a increasing function for $t\in(0,1)$. Indeed, $h(0)=0$, $h(1)=1$ and

$$h'(t)=\frac{et^{\frac{t}{1-t}}}{t\ln^2 t}(1-t-t\ln^2 t)=\frac{et^{\frac{t}{1-t}}}{t\ln^2 t}Q(t)>0,$$

$Q(1)=0$, $Q'(t)=-(1+\ln^2 t)<0$. Now we find functions $\lambda(\alpha)$ and $\Delta(\alpha)$. Using $\lim_{t\to 0^+}e^{-\frac{t(1-\alpha)\ln t}{1-t}}=1$ we have $\lambda(\alpha)=0$ for all $\alpha\in(0,1)$.

Now we show $\Delta(\alpha)=(5-\alpha)/8$ for $\alpha\in(0,1)$.

We have

$$\Delta(\alpha)=\lim_{t\to 1^-}\frac{3}{2}\frac{(1+t)}{(1-t)^2}\left(\left(\frac{(1-t)}{-\ln t}\right)^{\alpha}\frac{t^{-\frac{t(1-\alpha)}{1-t}}}{e^{1-\alpha}}-\frac{2t}{1+t}\right).$$

Denote

$$S(t,\alpha)=\left(\frac{(1-t)}{-\ln t}\right)^{\alpha},\quad H(t,\alpha)=\frac{t^{-\frac{t(1-\alpha)}{1-t}}}{e^{1-\alpha}}. \tag{18}$$

Using Taylor's series for (18) and for the function $2t/(1+t)$ in the point $t=1$ for given $\alpha\in(0,1)$ we obtain

$$S(t,\alpha)=1-\frac{\alpha}{2}(1-t)+\frac{3\alpha^2-5\alpha}{24}(1-t)^2+s(\alpha)(1-t)^3,$$

$$H(t,\alpha)=1-\frac{1-\alpha}{2}(1-t)+\frac{3\alpha^2-2\alpha-1}{24}(1-t)^2+j(\alpha)(1-t)^3,$$

$$\frac{2t}{1+t}=1+\sum_{k=1}^{+\infty}\left(\frac{1}{2^k}-\frac{1}{2^{k-1}}\right)(1-t)^k=1-\frac{1}{2}(1-t)-\frac{1}{4}(1-t)^2+i(\alpha)(1-t)^3,$$

where $s(\alpha)$, $j(\alpha)$, $i(\alpha)$ are suitable functions. It implies

$$\Delta(\alpha) = \lim_{t\to 1^-} F(t,\alpha) = 3\lim_{t\to 1^-} \frac{1}{(1-t)^2}\left(S(t,\alpha)H(t,\alpha) - \frac{2t}{1+t}\right)$$
$$= 3\lim_{t\to 1^-} \frac{1}{(1-t)^2}\left(\frac{5-\alpha}{24}(1-t)^2\right) = \frac{5-\alpha}{8}.$$

The proof is complete. □

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