OPTIMAL CONVEX COMBINATIONS BOUNDS OF CENTROIDAL AND HARMONIC MEANS FOR WEIGHTED GEOMETRIC MEAN OF LOGARITHMIC AND IDENTRIC MEANS

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Abstract. In this paper, optimal convex combination bounds of centroidal and harmonic means for weighted geometric mean of logarithmic and identric means are proved. We find the greatest value $\lambda(\alpha)$ and the least value $\Delta(\alpha)$ for each $\alpha \in (0, 1)$ such that the double inequality:

$$\lambda C(a, b) + (1 - \lambda) H(a, b) < L^\alpha(a, b) I^{1-\alpha}(a, b) < \Delta C(a, b) + (1 - \Delta) H(a, b)$$

holds for all $a, b > 0$ with $a \neq b$. Here, $C(a, b)$, $H(a, b)$, $L(a, b)$ and $I(a, b)$ denote centroidal, harmonic, logarithmic and identric means of two positive numbers $a$ and $b$, respectively.

1. Introduction

Recently, means have been the subject of intensive research. In particular, many remarkable inequalities for the centroidal, harmonic, logarithmic and identric means can be found in the literature [4], [11], [12].

We recall some definitions.

The centroidal, harmonic, logarithmic, identric, and weighted geometric means of two positive real numbers $a, b, a \neq b$, are defined, respectively, as follows:

$$C(a, b) = \frac{2(a^2 + ab + b^2)}{3(a + b)},$$

$$H(a, b) = \frac{2ab}{a + b},$$

$$L(a, b) = \frac{a - b}{\log a - \log b},$$

$$I(a, b) = \frac{1}{e} \left( \frac{a^a}{b^b} \right)^{\frac{1}{a-b}},$$

$$G_\alpha(a, b) = a^\alpha b^{1-\alpha} \quad \text{for} \quad 0 \leq \alpha \leq 1.$$
Means have many applications not only in mathematics, but in physics, economics, meteorology,... (see for example [5], [7], [8]).

It is well-known that the following inequalities hold:

\[ H(a, b) < L(a, b) < I(a, b) < C(a, b) \text{ for positive } a \neq b. \quad (1) \]

In the paper [4], authors inspired by (1), proved the following theorems:

**Theorem 1.**

\[ \alpha_1 C(a, b) + (1 - \alpha_1) H(a, b) < L(a, b) < \beta_1 C(a, b) + (1 - \beta_1) H(a, b) \quad (2) \]

holds for all \( a, b > 0 \), with \( a \neq b \) if and only if \( \alpha_1 \leq 0, \beta_1 \geq 1/2 \).

**Theorem 2.**

\[ \alpha_2 C(a, b) + (1 - \alpha_2) H(a, b) < I(a, b) < \beta_2 C(a, b) + (1 - \beta_2) H(a, b) \quad (3) \]

holds for all \( a, b > 0 \), with \( a \neq b \) if and only if \( \alpha_2 \leq 3/(2e) = 0.551819, \beta_2 \geq 5/8 \).

Similar double inequality was proved by Alzer and Qiu [1]:

\[ \alpha A(a, b) + (1 - \alpha) G(a, b) < I(a, b) < \beta A(a, b) + (1 - \beta) G(a, b) \quad (4) \]

holds for all \( a, b > 0 \), with \( a \neq b \) if and only if \( \alpha \leq 2/3, \beta \geq 2/e = 0.73575 \).

From results of [4], it is natural to ask what is the greatest function \( \lambda(\alpha) \), and the least function \( \Delta(\alpha) \), for \( 0 \leq \alpha \leq 1 \) such that the double inequality:

\[
\lambda(\alpha) C(a, b) + (1 - \lambda(\alpha)) H(a, b) < L^\alpha(a, b) I^{1-\alpha}(a, b) < \Delta(\alpha) C(a, b) + (1 - \Delta(\alpha)) H(a, b)
\]

holds for all \( a, b > 0 \) with \( a \neq b \), \( 0 \leq \alpha \leq 1 \). The purpose of this paper is to find the optimal functions \( \lambda(\alpha), \Delta(\alpha) \). For some other details about means, see [1]–[12] and the related references cited there in.

### 2. Main results

**Lemma 1.** Let

\[
g(t, \alpha) = \frac{3}{2et^{1/t}} \left( 3 + t - \alpha \frac{(1+t)(t\ln t - t + 1)}{t\ln t} - (1 - \alpha) \frac{(1+t)(\ln t - t + 1)}{1-t} \right) \quad (5)
\]

for \( 0 < t < 1, 0 \leq \alpha \leq 1 \). Then \( g(t, 0) > 0 \), \( g(t, 1) > 0 \) for \( 0 < t < 1 \).

**Proof.** \( g(t, 0) > 0 \) follows from \( s^+(t, 0) > 0 \) (see Lemma 2).

From (5) with \( \alpha = 1 \) we conclude that

\[
(3 + t) t \ln t - (1 + t)(t \ln t - t + 1) < 0. \quad (6)
\]
Some calculation gives that (6) leads to the evident inequality $2t \ln t - (1-t)^2 < 0$. □

Denote

$$h(t) = \frac{(1-t)e^{t \frac{t}{1-t}}}{-\ln t} \quad \text{for} \quad t \in (0, 1).$$

(7)

LEMMA 2. Let

$$s^*(t, \alpha) = \alpha \ln(h(t)) - \ln \left( \frac{3(1+t)}{g(t, \alpha)} \right) = \alpha \ln \left( \frac{e(1-t)t^{\frac{t}{1-t}}}{-\ln t} \right) - \ln \left( \frac{3(1+t)}{g(t, \alpha)} \right)$$

for $0 < t < 1$, $0 \leq \alpha \leq 1$. Then $s^*(t, 0) > 0$, $s^*(t, 1) > 0$, $s''^*_{\alpha, \alpha}(t, \alpha) < 0$ for $0 < t < 1$, $0 < \alpha < 1$.

Proof. From

$$s'_\alpha(t, \alpha) = \ln(h(t)) + \frac{g'_\alpha(t, \alpha)}{g(t, \alpha)}$$

we have

$$s''_{\alpha, \alpha}(t, \alpha) = -\frac{g''_\alpha(t, \alpha)}{g(t, \alpha)} < 0.$$

Now we show $s^*(t, 0) > 0$. The inequality is equivalent to

$$u(t) = g(t, 0) - 3(1+t) > 0 \quad \text{for} \quad t \in (0, 1).$$

(8)

Inequality (8) will be proved if we show

$$(3+t)(1-t) - (1+t)(\ln t - t + 1) > 2e(1+t)(1-t)t^{\frac{t}{1-t}}.$$  

(9)

Rewriting inequality (9) we obtain

$$r(t) = 2(1-t) - (1+t) \ln t - 2(1-t^2)e^{1+\frac{\ln t}{1-t}} > 0 \quad \text{for} \quad t \in (0, 1).$$

Denote

$$v(t) = \ln(2(1-t) - (1+t) \ln t) - \ln(2(1-t^2)) - \frac{1-t+\ln t}{1-t}.$$ 

(10)

Because $v(1) = 0$ to show (10) it suffices to prove $v'(t) < 0$ for $t \in (0, 1)$.

Simple calculation gives

$$v'(t) = -\frac{3t+1+t\ln t}{t(2(1-t) - (1+t) \ln t)} + \frac{2t}{1-t^2} - \frac{1-t+\ln t}{(1-t)^2}.$$  

The inequality $v'(t) < 0$ is equivalent to

$$w(t) = \ln^2 t - \frac{2(1-t)}{1+t} \ln t - \frac{(1-t)^2(t^2 + 6t + 1)}{t(1+t)^2} < 0.$$
Simple calculation leads to \( w(t) < 0 \) if and only if

\[
(\ln t - \frac{1-t}{1+t})^2 < \frac{(1-t)^2}{(1+t)^2} \left( \frac{t^2 + 7t + 1}{t} \right).
\]

From this we have that, it suffices to show that

\[
-\frac{1+t}{1-t} \ln t < \sqrt{\frac{t^2 + 7t + 1}{t}} - 1.
\] (11)

Inequality (11) is equivalent to

\[
o(t) = \frac{1-t}{1+t} \left( \sqrt{\frac{t^2 + 7t + 1}{t}} - 1 \right) + \ln t > 0.
\]

We show that \( o'(t) < 0 \) for \( t \in (0, 1) \). Simple calculation gives

\[
o'(t) = \frac{2\sqrt{t^2 + 7t + 1}}{(1+t)^2} - \frac{2(t^2 + 7t + 1)}{(1+t)^2 \sqrt{t}} + \frac{(1-t)(t^2 - 1)}{(1+t)2t \sqrt{t}} + \frac{\sqrt{t^2 + 7t + 1}}{t}.
\]

To prove inequality (8) we first show that

\[
(t^2 + 7t + 1)(1 + 4t + t^2) < \frac{1}{4t} (1 + 4t + 26t^2 + 4t^3 + t^4)^2.
\] (12)

Inequality (12) can be rewriting as

\[
m(t) = -t^8 - 4t^7 - 8t^6 + 84t^5 - 142t^4 + 84t^3 - 8t^2 - 4t - 1 < 0.
\]

It is easy to see that

\[
m(t) = - (1-t)^4(t^4 + 8t^3 + 34t^2 + 8t + 1) < 0 \quad \text{for} \quad t \in (0, 1).
\]

Now we prove \( s^*(t, 1) > 0 \) for \( t \in (0, 1) \). The inequality \( s^*(t, 1) > 0 \) is equivalent to

\[
h(t) g(t, 1) - 3(1+t) > 0 \quad \text{for} \quad t \in (0, 1).
\] (13)

Inequality (13) can be rewriting as

\[
\ln^2 t + \frac{1-t}{1+t} \ln t - \frac{(1-t)^2}{2t} < 0.
\]

Simple calculation leads to

\[
\left( \ln t + \frac{1-t}{2(1+t)} \right)^2 < \frac{(1-t)^2}{4(1+t)^2} \left( \frac{2t^2 + 5t + 2}{t} \right).
\] (14)

Inequality (14) will be shown if we prove that

\[
- \ln t - \frac{1-t}{2(1+t)} < \frac{(1-t)}{2(1+t)} \sqrt{\frac{2t^2 + 5t + 2}{t}}
\]
because of $2(1+t)\ln t + 1 - t < 0$.

Indeed, if we denote $z(t) = \ln t + (1-t)/(2(1+t))$ then $z(1) = 0$ and $z'(t) = 2\ln t + 2/t + 1 > 0$. It follows from $z'(1) = 3$ and $z''(t) = 2(t-1)/(t^2) < 0$. Denote

$$a(t) = \frac{1-t}{2(1+t)} \left( \sqrt{\frac{2t^2 + 5t + 2}{t}} + 1 \right) + \ln t > 0.$$  

From $a(1) = 0$ it suffices to show that $a'(t) < 0$ for $t \in (0,1)$. Simple calculation gives

$$a'(t) = -\frac{\sqrt{2t^2 + 5t + 2}}{\sqrt{t}(1+t)^2} + \frac{t^2 + t + 1}{t(1+t)^2} + \frac{(1-t)\sqrt{t}(t^2 - 1)}{2t^2(1+t)\sqrt{2t^2 + 5t + 2}}.$$  

The inequality $a'(t) < 0$ is equivalent to

$$2\sqrt{t}(1+t + t^2)\sqrt{2t^2 + 5t + 2} < 1 + 4t + 8t^2 + 4t^3 + t^4,$$

which can be rewriting as

$$4t(1+t + t^2)^2(2t^2 + 5t + 2) < (1 + 4t + 8t^2 + 4t^3 + t^4)^2. \quad (15)$$

Easy computation leads that inequality (15) is

$$1 - 4t^2 + 6t^4 - 4t^6 + t^8 = (t^2 - 1)^4 > 0.$$  

The proof is complete. □

Our main result reads as follows

**THEOREM 3.** The double inequality

$$\lambda C(a,b) + (1 - \lambda)H(a,b) < L^\alpha(a,b) I^{1-\alpha}(a,b) < \Delta C(a,b) + (1 - \Delta)H(a,b) \quad (16)$$

holds for all $a,b > 0$ with $a \neq b$, $\alpha \in (0,1)$ if and only if $\lambda(\alpha) \leq 0$ and $\Delta(\alpha) \geq (5 - \alpha)/8$.

**Proof.** Suppose $a,b > 0$ with $a > b$, $\alpha \in (0,1)$, $t = b/a < 1$. Using

$$\frac{C(a,b)}{a} = \frac{2(1+t + t^2)}{3(1+t)}, \quad \frac{H(a,b)}{a} = \frac{2t}{1+t},$$

$$\frac{L(a,b)}{a} = \frac{1-t}{-\ln t}, \quad \frac{I(a,b)}{a} = \frac{1}{et^{t/t}},$$

we can write inequality (16) in the form

$$\lambda(\alpha) \left( \frac{2(1+t + t^2)}{3(1+t)} - \frac{2t}{1+t} \right) < \left( \frac{1-t}{-\ln t} \right)^\alpha \left( \frac{1}{et^{t/t}} \right)^{1-\alpha} - \frac{2t}{1+t}$$

$$< \Delta(\alpha) \left( \frac{2(1+t + t^2)}{3(1+t)} - \frac{2t}{1+t} \right).$$
Denote

\[ F(t, \alpha) = \frac{3(1+t)}{2(1-t)^2} \left( \left( \frac{1-t}{-\ln t} \right)^\alpha \left( \frac{1}{e^t-1} \right)^{1-\alpha} - \frac{2t}{1+t} \right). \]  \hspace{1cm} (17)

We show \( F'_1(t, \alpha) > 0, \lambda(\alpha) = \lim_{t \to 0^+} F(t, \alpha) \) and \( \Delta(\alpha) = \lim_{t \to 1^-} F(t, \alpha). \)

Rewriting (17) we have

\[ F(t, \alpha) = \frac{3(1+t)}{2e(1-t)^2} \left( \frac{(1-t)e^{t \frac{\alpha}{1-t}}}{-\ln t} \right)^\alpha - \frac{3t}{(1-t)^2}. \]

Using elementary calculations we obtain

\[ F'_1(t, \alpha) = \frac{1}{(1-t)^3} (h(t)^\alpha g(t, \alpha) - 3(1+t)), \]

where \( h(t) \) is defined in (7) and \( g(t, \alpha) \) is defined in (5). It implies that it suffices to prove \( h(t)^\alpha g(t, \alpha) - 3(1+t) > 0 \) for \( t \in (0,1), \alpha \in (0,1) \). It follows from Lemma 1 and Lemma 2, because of \( 0 < h(t) < 1 \) is a increasing function for \( t \in (0,1) \). Indeed, \( h(0) = 0, h(1) = 1 \) and

\[ h'(t) = \frac{et^{\frac{\alpha}{1-t}}}{t \ln^2 t} (1-t - t \ln^2 t) = \frac{et^{\frac{\alpha}{1-t}}}{t \ln^2 t} Q(t) > 0, \]

\( Q(1) = 0, Q'(t) = -(1+\ln^2 t) < 0. \) Now we find functions \( \lambda(\alpha) \) and \( \Delta(\alpha) \). Using \( \lim_{t \to 0^+} e^{-\frac{t(1-\alpha)^{1-t}}{1-t}} = 1 \) we have \( \lambda(\alpha) = 0 \) for all \( \alpha \in (0,1) \).

Now we show \( \Delta(\alpha) = (5-\alpha)/8 \) for \( \alpha \in (0,1) \).

We have

\[ \Delta(\alpha) = \lim_{t \to 1^-} \frac{3}{2} \frac{1+t}{(1-t)^2} \left( \frac{(1-t)^\alpha t^{-\frac{t(1-\alpha)}{1-t}}}{e^{1-\alpha} - 1} - \frac{2t}{1+t} \right). \]

Denote

\[ S(t, \alpha) = \frac{(1-t)^\alpha}{-\ln t}, \quad H(t, \alpha) = \frac{t^{\frac{t(1-\alpha)}{1-t}}}{e^{1-\alpha}}. \]  \hspace{1cm} (18)

Using Taylor’s series for (18) and for the function \( 2t/(1+t) \) in the point \( t = 1 \) for given \( \alpha \in (0,1) \) we obtain

\[ S(t, \alpha) = 1 - \frac{\alpha}{2} (1-t) + \frac{3\alpha^2 - 5\alpha}{24} (1-t)^2 + s(\alpha)(1-t)^3, \]

\[ H(t, \alpha) = 1 - \frac{1-\alpha}{2} (1-t) + \frac{3\alpha^2 - 2\alpha - 1}{24} (1-t)^2 + h(\alpha)(1-t)^3, \]

\[ \frac{2t}{1+t} = 1 + \sum_{k=1}^{\infty} \left( \frac{1}{2^k} - \frac{1}{2^{k-1}} \right) (1-t)^k = 1 - \frac{1}{2} (1-t) - \frac{1}{4} (1-t)^2 + i(\alpha)(1-t)^3, \]
where \( s(\alpha), j(\alpha), i(\alpha) \) are suitable functions. It implies

\[
\Delta(\alpha) = \lim_{t \to 1^-} F(t, \alpha) = 3 \lim_{t \to 1^-} \frac{1}{(1 - t)^2} \left( S(t, \alpha) H(t, \alpha) - \frac{2t}{1 + t} \right)
\]

\[
= 3 \lim_{t \to 1^-} \frac{1}{(1 - t)^2} \left( \frac{5 - \alpha}{24} (1 - t)^2 \right) = \frac{5 - \alpha}{8}.
\]

The proof is complete. \( \Box \)

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\textbf{REFERENCES}


