Journal of
Mathematical
Inequalities
Volume 8, Number 4 (2014), 947–961
doi:10.7153/jmi-08-72

UPPER AND LOWER BOUNDS FOR THE p–ANGULAR DISTANCE IN NORMED SPACES WITH APPLICATIONS

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(*Communicated by J. Pečarić*)

Abstract. For nonzero vectors x and y in the normed linear space $(X,\|\cdot\|)$ we can define the *p-angular distance* by

$$\alpha_p[x,y] := \left\| \|x\|^{p-1}x - \|y\|^{p-1}y \right\|.$$

In this paper we show among others that

$$\begin{aligned}&\frac{1}{2}\left|\left|\|x\|^{p-1}-\|y\|^{p-1}\right|\|x+y\| - \left(\|x\|^{p-1}+\|y\|^{p-1}\right)\|x-y\|\right|\\ &\leqslant \alpha_p[x,y]\\ &\leqslant \frac{1}{2}\left[\left|\|x\|^{p-1}-\|y\|^{p-1}\right|\|x+y\| + \left(\|x\|^{p-1}+\|y\|^{p-1}\right)\|x-y\|\right]\end{aligned}$$

for any $p\in\mathbb{R}$ and for any nonzero $x,y\in X$.

Some reverses of the triangle and the continuity of the norm inequalities are given as well.

Applications for functions f defined by power series in estimating the more general "distance" $\|f(\|x\|)x - f(\|y\|)y\|$ for certain $x,y\in X$ are also provided.

1. Introduction

Following [3, p. 403] or [12], for nonzero vectors x and y in the normed linear space $(X,\|\cdot\|)$ we define the *angular distance* $\alpha[x,y]$ between x and y by

$$\alpha[x,y] := \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

In 1958, Massera and Schäffer [12, Lemma 5.1] showed that

$$\alpha[x,y] \leqslant \frac{2\|x-y\|}{\max\{\|x\|,\|y\|\}}, \tag{1.1}$$

which is better than the *Dunkl-Williams inequality* [6]

$$\alpha[x,y] \leqslant \frac{4\|x-y\|}{\|x\|+\|y\|}. \tag{1.2}$$

Mathematics subject classification (2010): Primary 46B20, 46B99; Secondary 51M16.
Keywords and phrases: Angular distance, normed linear spaces, Massera-Schäffer's inequality, Dunkl-Williams' inequality, Hile's inequality, power series.

We notice that the *Massera-Schäffer inequality* was rediscovered by Gurariĭ in [7] (see also [13, p. 516]).

In [10], Maligranda obtained the double inequality (see also [11]):

$$\frac{\|x-y\| - |\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}} \leqslant \alpha[x,y] \leqslant \frac{\|x-y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}}. \tag{1.3}$$

The second inequality in (1.3) is better than Massera-Schäffer's inequality (1.1).

In the recent paper [10], L. Maligranda has also considered the p*-angular distance*

$$\alpha_p[x,y] := \left\| \|x\|^{p-1} x - \|y\|^{p-1} y \right\|$$

between the vectors x and y in the normed linear space $(X, \|\cdot\|)$ over the real or complex number field $\mathbb{K}$ and showed that

$$\alpha_p[x,y] \leqslant \|x-y\| \times \begin{cases} (2-p) \cdot \frac{\max\{\|x\|^p, \|y\|^p\}}{\max\{\|x\|, \|y\|\}} & \text{if } p \in (-\infty, 0) \text{ and } x,y \neq 0; \\ (2-p) \cdot \frac{1}{[\max\{\|x\|, \|y\|\}]^{1-p}} & \text{if } p \in [0,1] \text{ and } x,y \neq 0; \\ p \cdot [\max\{\|x\|, \|y\|\}]^{p-1} & \text{if } p \in (1, \infty). \end{cases} \tag{1.4}$$

The constants $2-p$ and p in (1.4) are best possible in the sense that they cannot be replaced by smaller quantities.

As pointed out in [10], the inequality (1.4) for $p \in [1, \infty)$ is better than the Bourbaki inequality obtained in 1965, [1, p. 257] (see also [13, p. 516]):

$$\alpha_p[x,y] \leqslant 3p \|x-y\| [\|x\| + \|y\|]^{p-1}, \quad x,y \in X. \tag{1.5}$$

The following results concerning upper bounds for the p-angular distance have been obtained by the author in [5]:

$$\alpha_p[x,y] \leqslant \begin{cases} \|x-y\| [\max\{\|x\|, \|y\|\}]^{p-1} + \left| \|x\|^{p-1} - \|y\|^{p-1} \right| \min\{\|x\|, \|y\|\} \\ \qquad \text{if } p \in (1, \infty); \\ \frac{\|x-y\|}{[\min\{\|x\|, \|y\|\}]^{1-p}} + \left| \|x\|^{1-p} - \|y\|^{1-p} \right| \min\left\{ \frac{\|x\|^p}{\|y\|^{1-p}}, \frac{\|y\|^p}{\|x\|^{1-p}} \right\} \\ \qquad \text{if } p \in [0,1]; \\ \frac{\|x-y\|}{[\min\{\|x\|, \|y\|\}]^{1-p}} + \frac{\left| \|x\|^{1-p} - \|y\|^{1-p} \right|}{\max\{\|x\|^{-p}\|y\|^{1-p}, \|y\|^{-p}\|x\|^{1-p}\}} \quad \text{if } p \in (-\infty, 0); \end{cases} \tag{1.6}$$

and

$$\alpha_p[x,y] \tag{1.7}$$
$$\leqslant \begin{cases} \|x-y\|\left[\min\{\|x\|,\|y\|\}\right]^{p-1}+\left|\|x\|^{p-1}-\|y\|^{p-1}\right|\max\{\|x\|,\|y\|\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } p\in(1,\infty); \\ \frac{\|x-y\|}{[\max\{\|x\|,\|y\|\}]^{1-p}}+\left|\|x\|^{1-p}-\|y\|^{1-p}\right|\max\left\{\frac{\|x\|^p}{\|y\|^{1-p}},\frac{\|y\|^p}{\|x\|^{1-p}}\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } p\in[0,1]; \\ \frac{\|x-y\|}{[\max\{\|x\|,\|y\|\}]^{1-p}}+\frac{\left|\|x\|^{1-p}-\|y\|^{1-p}\right|}{\min\left\{\|x\|^{-p}\|y\|^{1-p},\|y\|^{-p}\|x\|^{1-p}\right\}} \qquad \text{if } p\in(-\infty,0); \end{cases}$$

for any two nonzero vectors x,y in the normed linear space $(X,\|.\|)$.

The upper bounds for $\alpha_p[x,y]$ provided by (1.4), (1.6) and (1.7) have been compared in [5] to conclude that some of the later ones are better in certain cases. The details are omitted here.

Finally, we recall the results of G. N. Hile from [4]:

$$\alpha_p[x,y] \leqslant \frac{\|x\|^p-\|y\|^p}{\|x\|-\|y\|}\cdot\|x-y\|, \tag{1.8}$$

for $p\in[1,\infty)$ and $x,y\in X$ with $\|x\|\neq\|y\|$, and

$$\alpha_{-p-1}[x,y] \leqslant \frac{\|x\|^p-\|y\|^p}{\|x\|-\|y\|}\cdot\frac{\|x-y\|}{\|x\|^p\|y\|^p}, \tag{1.9}$$

for $p\in[1,\infty)$ and $x,y\in X\setminus\{0\}$ with $\|x\|\neq\|y\|$.

For other norm inequalities, see [8], [9] and [13].

In this paper we establish some new upper and lower bounds for the p-angular distance. Some reverses of the triangle and the continuity of the norm inequalities are given as well. Applications for functions f defined by power series in estimating the more general "distance" $\|f(\|x\|)x-f(\|y\|)y\|$ for certain $x,y\in X$ are also provided.

2. Some preliminary results

For a pair of scalars $(\gamma,\beta)\in\mathbb{K}^2$ we can introduce the following (γ,β)-*angular distance* $d_{(\gamma,\beta)}:X^2\to[0,\infty)$ given by

$$d_{(\gamma,\beta)}[x,y]:=\|\gamma x-\beta y\|. \tag{2.1}$$

When $\gamma=\|x\|^{p-1}$ and $\beta=\|y\|^{p-1}$, then we have

$$d_{\left(\|x\|^{p-1},\|y\|^{p-1}\right)}[x,y]=\alpha_p[x,y]$$

for $p\in\mathbb{R}$.

In [5] we have shown that, for any $(\gamma,\beta)\in\mathbb{K}^2$ and $x,y\in X$ we have the following upper bounds for the (γ,β)-angular distance:

$$d_{(\gamma,\beta)}[x,y]\leqslant\|x-y\|\max\{|\gamma|,|\beta|\}+|\gamma-\beta|\min\{\|x\|,\|y\|\} \tag{2.2}$$

and

$$d_{(\gamma,\beta)}[x,y]\leqslant\|x-y\|\min\{|\gamma|,|\beta|\}+|\gamma-\beta|\max\{\|x\|,\|y\|\} \tag{2.3}$$

respectively.

By adding these two upper bounds we have the symmetrical bound

$$d_{(\gamma,\beta)}[x,y]\leqslant\|x-y\|\cdot\frac{|\gamma|+|\beta|}{2}+|\gamma-\beta|\cdot\frac{\|x\|+\|y\|}{2}. \tag{2.4}$$

We also proved in [5] that the following lower bounds may be provided as well:

$$\|x-y\|\min\{|\gamma|,|\beta|\}-|\gamma-\beta|\min\{\|x\|,\|y\|\}\leqslant d_{(\gamma,\beta)}[x,y] \tag{2.5}$$

and

$$\|x-y\|\max\{|\gamma|,|\beta|\}-|\gamma-\beta|\max\{\|x\|,\|y\|\}\leqslant d_{(\gamma,\beta)}[x,y], \tag{2.6}$$

and, by addition

$$\|x-y\|\cdot\frac{|\gamma|+|\beta|}{2}-|\gamma-\beta|\cdot\frac{\|x\|+\|y\|}{2}\leqslant d_{(\gamma,\beta)}[x,y].$$

We provide now some different upper and lower bounds for the (γ,β)-*angular distance:*

LEMMA 1. *For any $(\gamma,\beta)\in\mathbb{K}^2$ and $x,y\in X$ we have the inequalities*

$$\begin{aligned}&\frac{1}{2}\left||\gamma-\beta|\,\|x+y\|-|\gamma+\beta|\,\|x-y\|\right|\\&\leqslant d_{(\gamma,\beta)}[x,y]\\&\leqslant\frac{1}{2}\left[|\gamma-\beta|\,\|x+y\|+|\gamma+\beta|\,\|x-y\|\right].\end{aligned} \tag{2.7}$$

Proof. We observe that for any $(\gamma,\beta)\in\mathbb{K}^2$ and $x,y\in X$ we have the key equality

$$\gamma x-\beta y=(\gamma-\beta)\frac{x+y}{2}+\frac{\gamma+\beta}{2}(x-y).$$

Utilising the triangle inequality we have

$$\begin{aligned}\|\gamma x-\beta y\|&=\left\|(\gamma-\beta)\frac{x+y}{2}+\frac{\gamma+\beta}{2}(x-y)\right\|\\&\leqslant\left\|(\gamma-\beta)\frac{x+y}{2}\right\|+\left\|\frac{\gamma+\beta}{2}(x-y)\right\|\\&=|\gamma-\beta|\left\|\frac{x+y}{2}\right\|+\left|\frac{\gamma+\beta}{2}\right|\|x-y\|\end{aligned}$$

and the second inequality in (2.7) is proved.

Utilising the continuity inequality of the norm, i.e., $\|u-v\| \geqslant |\|u\|-\|v\||$ we also have

$$\begin{aligned}
\|\gamma x-\beta y\| &= \left\|(\gamma-\beta)\frac{x+y}{2}-\frac{\gamma+\beta}{2}(y-x)\right\| \\
&\geqslant \left|\left\|(\gamma-\beta)\frac{x+y}{2}\right\|-\left\|\frac{\gamma+\beta}{2}(x-y)\right\|\right| \\
&= \left||\gamma-\beta|\left\|\frac{x+y}{2}\right\|-\left|\frac{\gamma+\beta}{2}\right|\|x-y\|\right|,
\end{aligned}$$

and the proof is complete. □

COROLLARY 1. *For any* $(\gamma,\beta)\in\mathbb{K}^2$ *and* $x,y\in X$ *we have the inequalities*

$$\left|d_{(\gamma,\beta)}[x,y]-\frac{1}{2}|\gamma-\beta|\|x+y\|\right| \leqslant \frac{1}{2}|\gamma+\beta|\|x-y\|$$

and

$$\left|d_{(\gamma,\beta)}[x,y]-\frac{1}{2}|\gamma+\beta|\|x-y\|\right| \leqslant \frac{1}{2}|\gamma-\beta|\|x+y\|.$$

REMARK 1. We observe that the upper bound for $d_{(\gamma,\beta)}[x,y]$ provided by (2.7) is better than the one provided by (2.4).

3. New bounds for the angular distance

We can state the following result providing upper and lower bounds for the angular distance:

THEOREM 1. *For any nonzero vectors* x *and* y *in the normed linear space* $(X,\|\cdot\|)$ *we have*

$$\begin{aligned}
0 &\leqslant \frac{1}{2\|y\|\|x\|}\max\{\|x+y\|[\|x-y\|-|\|x\|-\|y\||], \\
&\qquad |\|x\|-\|y\||[\|y\|+\|x\|-\|x+y\|]\} \\
&\leqslant \frac{1}{2\|y\|\|x\|}[(\|y\|+\|x\|)\|x-y\|-|\|x\|-\|y\||\|x+y\|] \\
&\leqslant \alpha[x,y] \\
&\leqslant \frac{1}{2\|y\|\|x\|}[|\|x\|-\|y\||\|x+y\|+(\|x\|+\|y\|)\|x-y\|] \\
&\leqslant \frac{1}{2\|y\|\|x\|}\min\{(\|x\|+\|y\|)[|\|x\|-\|y\||+\|x-y\|], \\
&\qquad \|x-y\|[\|x+y\|+\|x\|+\|y\|]\}.
\end{aligned} \tag{3.1}$$

Proof. If we take $\gamma = \frac{1}{\|x\|}$ and $\beta = \frac{1}{\|y\|}$ (in 2.7) then we get

$$\begin{aligned}&\frac{1}{2}\left|\left|\frac{1}{\|x\|}-\frac{1}{\|y\|}\right|\|x+y\|-\left|\frac{1}{\|x\|}+\frac{1}{\|y\|}\right|\|x-y\|\right|\\ &\leqslant \alpha[x,y]\\ &\leqslant \frac{1}{2}\left[\left|\frac{1}{\|x\|}-\frac{1}{\|y\|}\right|\|x+y\|+\left|\frac{1}{\|x\|}+\frac{1}{\|y\|}\right|\|x-y\|\right],\end{aligned}$$

which is equivalent with

$$\begin{aligned}&\frac{1}{2\|y\|\|x\|}\left|\left|\|x\|-\|y\|\right|\|x+y\|-(\|y\|+\|x\|)\|x-y\|\right|\\ &\leqslant \alpha[x,y]\\ &\leqslant \frac{1}{2\|x\|\|y\|}\left[\left|\|x\|-\|y\|\right|\|x+y\|+(\|x\|+\|y\|)\|x-y\|\right].\end{aligned}$$

We notice that, by the triangle inequality and by the continuity of norm inequality we have

$$\begin{aligned}&\left|\left|\|x\|-\|y\|\right|\|x+y\|-(\|y\|+\|x\|)\|x-y\|\right|\\ &=(\|y\|+\|x\|)\|x-y\|-\left|\|x\|-\|y\|\right|\|x+y\|\geqslant 0.\end{aligned}$$

We also have

$$\begin{aligned}&(\|y\|+\|x\|)\|x-y\|-\left|\|x\|-\|y\|\right|\|x+y\|\\ &\geqslant \|x+y\|\left[\|x-y\|-\left|\|x\|-\|y\|\right|\right]\geqslant 0\end{aligned}$$

and

$$\begin{aligned}&(\|y\|+\|x\|)\|x-y\|-\left|\|x\|-\|y\|\right|\|x+y\|\\ &\geqslant \left|\|x\|-\|y\|\right|\left[\|y\|+\|x\|-\|x+y\|\right]\geqslant 0\end{aligned}$$

which implies that

$$\begin{aligned}&(\|y\|+\|x\|)\|x-y\|-\left|\|x\|-\|y\|\right|\|x+y\|\\ &\geqslant \max\left\{\|x+y\|\left[\|x-y\|-\left|\|x\|-\|y\|\right|\right],\right.\\ &\qquad\left.\left|\|x\|-\|y\|\right|\left[\|y\|+\|x\|-\|x+y\|\right]\right\}\\ &\geqslant 0.\end{aligned}$$

These prove the first three inequalities in (3.1).

We also have that

$$\begin{aligned}&\left|\|x\|-\|y\|\right|\|x+y\|+(\|x\|+\|y\|)\|x-y\|\\ &\leqslant (\|x\|+\|y\|)\left[\left|\|x\|-\|y\|\right|+\|x-y\|\right]\end{aligned}$$

and

$$\begin{aligned} & |\|x\|-\|y\||\,\|x+y\|+(\|x\|+\|y\|)\,\|x-y\| \\ & \leqslant \|x-y\|\,[\|x+y\|+\|x\|+\|y\|] \end{aligned}$$

which implies that

$$\begin{aligned} & |\|x\|-\|y\||\,\|x+y\|+(\|x\|+\|y\|)\,\|x-y\| \\ & \leqslant \min\{(\|x\|+\|y\|)\,[|\|x\|-\|y\||+\|x-y\|]\,, \\ & \qquad \|x-y\|\,[\|x+y\|+\|x\|+\|y\|]\} \end{aligned}$$

and the last part of (3.1) is proved. □

PROBLEM 1. It is an open question for the author whether or not the upper bound for the angular distance provided by the inequality

$$\alpha[x,y] \leqslant \frac{1}{2\|y\|\,\|x\|}\,[|\|x\|-\|y\||\,\|x+y\|+(\|x\|+\|y\|)\,\|x-y\|]$$

is better than the one in (1.3). The same question applies for the lower bound.

4. Bounds for the 2-angular distance

The case $p=2$ in the p-angular distance is of interest. It generates the 2-angular distance which has the expression

$$\alpha_2[x,y] := \|\|x\|\,x-\|y\|\,y\| \tag{4.1}$$

and can be defined for any x,y in a normed linear space.

We observe that from Maligranda's result (1.4) we get the upper bound

$$\alpha_2[x,y] \leqslant 2\max\{\|x\|\,,\|y\|\}\,\|x-y\| \tag{4.2}$$

while from Hile's inequality we have

$$\alpha_2[x,y] \leqslant (\|x\|+\|y\|)\,\|x-y\|\,, \tag{4.3}$$

for any vectors x and y in the normed linear space $(X,\|\cdot\|)$.

Since

$$\frac{\|x\|+\|y\|}{2} \leqslant \max\{\|x\|\,,\|y\|\}$$

then (4.3) is a better inequality than (4.2).

Moreover, if we employ (2.2) and (2.3) for $\gamma=\|x\|$ and $\beta=\|y\|$, then we get

$$\alpha_2[x,y] \leqslant \|x-y\|\max\{\|x\|\,,\|y\|\}+|\|x\|-\|y\||\min\{\|x\|\,,\|y\|\}$$

and

$$\alpha_2[x,y] \leqslant \|x-y\|\min\{\|x\|\,,\|y\|\}+|\|x\|-\|y\||\max\{\|x\|\,,\|y\|\}$$

implying that

$$\alpha_2[x,y] \leqslant \min\{\|x-y\|\max\{\|x\|,\|y\|\}+|\|x\|-\|y\||\min\{\|x\|,\|y\|\}, \\ \|x-y\|\min\{\|x\|,\|y\|\}+|\|x\|-\|y\||\max\{\|x\|,\|y\|\}\} \tag{4.4}$$

for any vectors x and y in the normed linear space $(X,\|\cdot\|)$.

Since

$$\begin{aligned} &\|x-y\|\max\{\|x\|,\|y\|\}+|\|x\|-\|y\||\min\{\|x\|,\|y\|\} \\ &\leqslant \|x-y\|[\max\{\|x\|,\|y\|\}+\min\{\|x\|,\|y\|\}] \\ &= \|x-y\|(\|x\|+\|y\|) \end{aligned}$$

we can conclude that (4.4) is better that Hile's inequality (4.3).

Now, utilizing (2.5) and (2.6) we also have the lower bounds

$$\min\{\|x\|,\|y\|\}[\|x-y\|-|\|x\|-\|y\||] \leqslant \alpha_2[x,y] \tag{4.5}$$

and

$$\max\{\|x\|,\|y\|\}[\|x-y\|-|\|x\|-\|y\||] \leqslant \alpha_2[x,y]. \tag{4.6}$$

Obviously (4.6) is better than (4.5) and produces the following reverse of continuity of norm inequality

$$0 \leqslant \|x-y\|-|\|x\|-\|y\|| \leqslant \frac{1}{\max\{\|x\|,\|y\|\}}\alpha_2[x,y] \tag{4.7}$$

that holds for any nonzero vectors x and y in the normed linear space $(X,\|\cdot\|)$.

THEOREM 2. *For any vectors x and y in the normed linear space $(X,\|\cdot\|)$ we have*

$$\begin{aligned} 0 &\leqslant \max\{\|x+y\|[\|x-y\|-|\|x\|-\|y\||], \\ &\qquad |\|x\|-\|y\||[\|y\|+\|x\|-\|x+y\|]\} \\ &\leqslant (\|y\|+\|x\|)\|x-y\|-|\|x\|-\|y\||\|x+y\| \\ &\leqslant 2\alpha_2[x,y] \\ &\leqslant |\|x\|-\|y\||\|x+y\|+(\|x\|+\|y\|)\|x-y\| \\ &\leqslant \min\{(\|x\|+\|y\|)[|\|x\|-\|y\||+\|x-y\|], \\ &\qquad \|x-y\|[\|x+y\|+\|x\|+\|y\|]\}. \end{aligned} \tag{4.8}$$

Proof. If we take $\gamma = \|x\|$ and $\beta = \|y\|$ in the inequality (2.7) then we get

$$\begin{aligned} &\frac{1}{2}||\|x\|-\|y\||\|x+y\|-(\|x\|+\|y\|)\|x-y\|| \\ &\leqslant \alpha_2[x,y] \\ &\leqslant \frac{1}{2}[|\|x\|-\|y\||\|x+y\|+(\|x\|+\|y\|)\|x-y\|], \end{aligned}$$

which is equivalent with (see the proof of the above Theorem 1)

$$\begin{aligned}
&\frac{1}{2}\left[\left(\|x\|+\|y\|\right)\|x-y\|-\left|\|x\|-\|y\|\right|\|x+y\|\right] \\
&\leqslant \alpha_2[x,y] \\
&\leqslant \frac{1}{2}\left[\left|\|x\|-\|y\|\right|\|x+y\|+\left(\|x\|+\|y\|\right)\|x-y\|\right].
\end{aligned}$$

The rest follows as in the proof of the above Theorem 1 and the details are omitted. □

REMARK 2. From the inequality (4.8) we get the following reverse for the triangle inequality

$$0 \leqslant \|y\|+\|x\|-\|x+y\| \leqslant \frac{2}{\left|\|x\|-\|y\|\right|}\alpha_2[x,y] \tag{4.9}$$

that holds for any vectors x and y in the normed linear space $(X,\|\cdot\|)$ with $\|x\| \neq \|y\|$.

From the inequality (4.8) we also have the following reverse of continuity of norm inequality

$$0 \leqslant \|x-y\|-\left|\|x\|-\|y\|\right| \leqslant \frac{2}{\|x+y\|}\alpha_2[x,y] \tag{4.10}$$

that holds for any vectors x and y in the normed linear space $(X,\|\cdot\|)$ with $x \neq -y$.

Note that the inequality (4.7) is better than (4.10) since

$$\frac{1}{\max\{\|x\|,\|y\|\}} \leqslant \frac{2}{\|x+y\|}$$

for any vectors x and y in the normed linear space $(X,\|\cdot\|)$ with $x \neq -y$.

5. Bounds for the p-angular distance

If we write Lemma 1 for $\gamma = \|x\|^{p-1}$ and $\beta = \|y\|^{p-1}$ we can state the following result for the p-angular distance:

THEOREM 3. *For any nonzero vectors x and y in the normed linear space $(X,\|\cdot\|)$ we have*

$$\begin{aligned}
&\frac{1}{2}\left|\left|\|x\|^{p-1}-\|y\|^{p-1}\right|\|x+y\|-\left(\|x\|^{p-1}+\|y\|^{p-1}\right)\|x-y\|\right| \\
&\leqslant \alpha_p[x,y] \\
&\leqslant \frac{1}{2}\left[\left|\|x\|^{p-1}-\|y\|^{p-1}\right|\|x+y\|+\left(\|x\|^{p-1}+\|y\|^{p-1}\right)\|x-y\|\right]
\end{aligned} \tag{5.1}$$

for any $p \in \mathbb{R}$.

Now, for $s \in [-\infty, \infty]$ and $a, b > 0$, $a \neq b$, by following [2, p. 385], we can introduce the *s-generalized logarithmic means* by

$$L^{[s]}(a,b) := \begin{cases} \left(\dfrac{b^{s+1}-a^{s+1}}{(s+1)(b-a)}\right)^{1/s} & \text{if } s \neq -1, 0, \pm\infty; \\ \dfrac{b-a}{\ln b - \ln a} & \text{if } s = -1; \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)} & s = 0; \\ \max\{a,b\} & s = \infty; \\ \min\{a,b\} & s = -\infty. \end{cases}$$

The mapping $\mathbb{R} \ni s \to L^{[s]}(a,b)$ is *strictly increasing* and (see [2, p. 386])

$$\min\{a,b\} < L^{[s]}(a,b) < \max\{a,b\} \tag{5.2}$$

for any $s \in \mathbb{R}$ and $a, b > 0$, with $a \neq b$.

Utilising the properties of the s-generalized logarithmic means we proved the following lemma in [5]:

LEMMA 2. *For any two nonzero vectors $x, y \in X$ we have*

$$\begin{aligned} &(p-1)\left[\min\{\|x\|,\|y\|\}\right]^{p-2}\left|\|x\|-\|y\|\right| \\ &\quad \leqslant \left|\|x\|^{p-1}-\|y\|^{p-1}\right| \leqslant (p-1)\left|\|x\|-\|y\|\right|\left[\max\{\|x\|,\|y\|\}\right]^{p-2} \end{aligned} \tag{5.3}$$

if $p \in (2, \infty)$,

$$\begin{aligned} &(p-1)\frac{1}{\left[\max\{\|x\|,\|y\|\}\right]^{2-p}}\left|\|x\|-\|y\|\right| \\ &\quad \leqslant \left|\|x\|^{p-1}-\|y\|^{p-1}\right| \leqslant (p-1)\left|\|x\|-\|y\|\right|\frac{1}{\left[\min\{\|x\|,\|y\|\}\right]^{2-p}} \end{aligned} \tag{5.4}$$

if $p \in [1, 2]$, and

$$\begin{aligned} &(1-p)\frac{\|x\|^{1-p}\|y\|^{1-p}}{\left[\max\{\|x\|,\|y\|\}\right]^{2-p}}\left|\|x\|-\|y\|\right| \\ &\quad \leqslant \left|\|x\|^{1-p}-\|y\|^{1-p}\right| \leqslant (1-p)\left|\|x\|-\|y\|\right|\frac{\|x\|^{1-p}\|y\|^{1-p}}{\left[\min\{\|x\|,\|y\|\}\right]^{2-p}} \end{aligned} \tag{5.5}$$

if $p \in (-\infty, 1)$, respectively.

By utilizing this lemma and Theorem 3 we can state the following sequence of upper bounds for the p-angular distance:

$$\begin{aligned}
&\alpha_p[x,y] \qquad (5.6)\\
&\leqslant \frac{1}{2}\left[\left|\|x\|^{p-1}-\|y\|^{p-1}\right|\|x+y\|+\left(\|x\|^{p-1}+\|y\|^{p-1}\right)\|x-y\|\right]\\
&\leqslant (p-1)\,|\|x\|-\|y\||\,[\max\{\|x\|,\|y\|\}]^{p-2}\left\|\frac{x+y}{2}\right\|\\
&\quad+\frac{\|x\|^{p-1}+\|y\|^{p-1}}{2}\|x-y\|\\
&\leqslant [\max\{\|x\|,\|y\|\}]^{p-1}\left[(p-1)\,|\|x\|-\|y\||+\|x-y\|\right]\\
&\leqslant p\,[\max\{\|x\|,\|y\|\}]^{p-1}\|x-y\|
\end{aligned}$$

for $p \geqslant 2$, which is a better inequality than Maligranda's result (1.4).

Similar results can be stated for $p<1$, however the details are not presented here.

6. Applications for power series

For power series $f(z)=\sum_{n=0}^{\infty}a_n z^n$ with complex coefficients we can naturally construct another power series which have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(z):=\sum_{n=0}^{\infty}|a_n|z^n$. It is obvious that this new power series have the same radius of convergence as the original series, and that if all coefficients $a_n \geqslant 0$, then $f_a=f$.

In the following we denote by $D(0,1):=\{z\in\mathbb{C},\ |z|<1\}$.

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned}
f(z)&=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^n=\ln\frac{1}{1+z},\ z\in D(0,1); \qquad (6.1)\\
g(z)&=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}=\cos z,\ z\in\mathbb{C};\\
h(z)&=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}=\sin z,\ z\in\mathbb{C};\\
l(z)&=\sum_{n=0}^{\infty}(-1)^n z^n=\frac{1}{1+z},\ z\in D(0,1);
\end{aligned}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(z)=\sum_{n=1}^{\infty}\frac{1}{n}z^n=\ln\frac{1}{1-z},\ z\in D(0,1); \tag{6.2}$$

$$g_a(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z,\ z\in\mathbb{C};$$

$$h_A(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ z\in\mathbb{C};$$

$$l_A(z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z},\ z\in D(0,1).$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n \qquad z\in\mathbb{C}, \tag{6.3}$$

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}, \qquad z\in D(0,1);$$

$$\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1}, \qquad z\in D(0,1);$$

$$\tanh^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}, \qquad z\in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0, \qquad z\in D(0,1);$$

where Γ is *Gamma function*.

THEOREM 4. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. If $(X;\|\cdot\|)$ is a normed linear space and $x,y\in X$ with $\|x\|,\|y\|<R$, then*

$$\begin{aligned}
&\|f(\|x\|)x-f(\|y\|)y\| \\
&\leqslant\left\|\frac{x+y}{2}\right\|\,|\|x\|-\|y\||\max\left\{f_a'(\|x\|),f_a'(\|y\|)\right\}+\|x-y\|\frac{f_a(\|x\|)+f_a(\|y\|)}{2} \\
&\leqslant\|x-y\|\left\{\left\|\frac{x+y}{2}\right\|\max\left\{f_a'(\|x\|),f_a'(\|y\|)\right\}+\frac{f_a(\|x\|)+f_a(\|y\|)}{2}\right\} \\
&\leqslant\|x-y\|\left[\max\{\|x\|,\|y\|\}\max\left\{f_a'(\|x\|),f_a'(\|y\|)\right\}+\max\{f_a(\|x\|),f_a(\|y\|)\}\right].
\end{aligned} \tag{6.4}$$

Proof. If we write the inequality (5.6) for $p=n+1$, n a natural number, then we get

$$\begin{aligned} &\left\| \|x\|^n x - \|y\|^n y \right\| \\ &\leqslant n \left[\max\left\{\|x\|, \|y\|\right\}\right]^{n-1} \left\| \frac{x+y}{2} \right\| \left| \|x\| - \|y\| \right| + \frac{\|x\|^n + \|y\|^n}{2} \|x-y\| \end{aligned} \tag{6.5}$$

for any $x, y \in X$.

Let $m \geqslant 1$. Then we have, by the generalized triangle inequality and by (6.5), that

$$\begin{aligned} &\left\| \left(\sum_{n=0}^{m} a_n \|x\|^n \right) x - \left(\sum_{n=0}^{m} a_n \|y\|^n \right) y \right\| \\ &\leqslant \sum_{n=0}^{m} |a_n| \left\| \|x\|^n x - \|y\|^n y \right\| \\ &\leqslant \left\| \frac{x+y}{2} \right\| \left| \|x\| - \|y\| \right| \sum_{n=0}^{m} n |a_n| \left[\max\left\{\|x\|, \|y\|\right\}\right]^{n-1} + \|x-y\| \sum_{n=0}^{m} |a_n| \frac{\|x\|^n + \|y\|^n}{2}. \end{aligned} \tag{6.6}$$

Since $\|x\|, \|y\| < R$, the series

$$\sum_{n=0}^{\infty} a_n \|x\|^n, \quad \sum_{n=0}^{\infty} a_n \|y\|^n, \quad \sum_{n=0}^{\infty} |a_n| \|x\|^n, \quad \sum_{n=0}^{\infty} |a_n| \|y\|^n$$

and

$$\sum_{n=0}^{\infty} n |a_n| \left[\max\left\{\|x\|, \|y\|\right\}\right]^{n-1}$$

are convergent and

$$\sum_{n=0}^{\infty} a_n \|x\|^n = f(\|x\|), \quad \sum_{n=0}^{\infty} a_n \|y\|^n = f(\|y\|),$$

$$\sum_{n=0}^{\infty} |a_n| \|x\|^n = f_a(\|x\|), \quad \sum_{n=0}^{\infty} |a_n| \|y\|^n = f_a(\|y\|)$$

while

$$\begin{aligned} \sum_{n=0}^{\infty} n |a_n| \left[\max\left\{\|x\|, \|y\|\right\}\right]^{n-1} &= f_a'\left(\max\left\{\|x\|, \|y\|\right\}\right) \\ &= \max\left\{f_a'(\|x\|), f_a'(\|y\|)\right\}. \end{aligned}$$

Taking the limit over $m \to \infty$ in (6.6) we obtain the first part of (6.4).

The second part is obvious. □

REMARK 3. If we take $f(z) := \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ then we have from (6.4) the following inequality

$$\begin{aligned} &\|\exp(\|x\|)x - \exp(\|y\|)y\| \\ &\leqslant \left\|\frac{x+y}{2}\right\| |\|x\| - \|y\|| \max\{\exp\|x\|, \exp\|y\|\} + \|x-y\| \frac{\exp\|x\| + \exp\|y\|}{2} \\ &\leqslant \|x-y\| \left\{ \left\|\frac{x+y}{2}\right\| \max\{\exp\|x\|, \exp\|y\|\} + \frac{\exp\|x\| + \exp\|y\|}{2} \right\} \\ &\leqslant \|x-y\| \max\{\exp\|x\|, \exp\|y\|\} (\max\{\|x\|, \|y\|\} + 1) \end{aligned} \tag{6.7}$$

for any $x, y \in X$.

If we apply the inequality (6.4) for the functions $f(z) := \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ and $f(z) := \frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$ then we have

$$\begin{aligned} &\left\| \frac{x}{1 \pm \|x\|} - \frac{y}{1 \pm \|y\|} \right\| \\ &\leqslant \left\|\frac{x+y}{2}\right\| |\|x\| - \|y\|| \max\left\{ (1-\|x\|)^{-2}, (1-\|y\|)^{-2} \right\} \\ &\quad + \|x-y\| \frac{(1-\|x\|)^{-1} + (1-\|y\|)^{-1}}{2} \\ &\leqslant \|x-y\| \left\{ \left\|\frac{x+y}{2}\right\| \max\left\{ (1-\|x\|)^{-2}, (1-\|y\|)^{-2} \right\} \right. \\ &\quad \left. + \frac{(1-\|x\|)^{-1} + (1-\|y\|)^{-1}}{2} \right\} \\ &\leqslant \|x-y\| \left[\max\{\|x\|, \|y\|\} \max\left\{ (1-\|x\|)^{-2}, (1-\|y\|)^{-2} \right\} \right. \\ &\quad \left. + \max\left\{ (1-\|x\|)^{-1}, (1-\|y\|)^{-1} \right\} \right] \end{aligned} \tag{6.8}$$

for any $x, y \in X$ with $\|x\|, \|y\| < 1$.

The interested reader may apply the above results for other power series as pointed out in (6.1)–(6.3). However the details are not provided here.

Acknowledgement. The author would like to thank the anonymous referee for valuable comments that have been implemented in the final version of the paper.

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(Received September 12, 2013)

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