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BERWALD TYPE INEQUALITY FOR EXTREMAL UNIVERSAL INTEGRALS BASED ON (α,m)–CONCAVE FUNCTION

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Abstract. The aim of this work is to show a Berwald type inequality for the extremal universal integrals based on (α,m) concave function. Some examples are given to illustrate the validity of these inequalities.

1. Introduction

As an important tool, the theory of fuzzy measure and fuzzy integrals introduced by Sugeno in [1], can be used for modelling problems in non deterministic environment. However, in several real situations, the Sugeno integral is restricted because of the special operations. Therefore, many authors generalized the Sugeno integral by using some other operators to replace the special operator(s) $\wedge$ and/or $\vee$ and introduced Choquet-like integral [2], Shilkret integral [3], $\perp$-integral [4], and pseudo-integral [5]. In [6] Klement provided a concept of universal integrals generalizing both the Choquet and the Sugeno integrals.

The following Berwald inequality is well known in [7]:

Let f be a nonnegative concave function on $[a,b]$. Then, for all r,s such that $0<r<s<\infty$, the following inequality hold

$$\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\int_a^b f^s(x)\,dx}{b-a}\right)^{\frac{1}{s}} \leqslant \left(\frac{\int_a^b f^r(x)\,dx}{b-a}\right)^{\frac{1}{r}}. \tag{1.1}$$

In [8], Agahi proved the Berwald type inequality for Sugeno integral.

THEOREM 1. *Let $0<r<s<\infty$ and $f:[a,b]\to[0,\infty)$ be a concave function and μ be the Lebesgue measure on* $\mathbf{R}$. *Then*

(a) if $f(a)<f(b)$, then

$$\left((S)\int_a^b f^r\,d\mu\right)^{\frac{1}{r}} \geqslant \min\left\{t,\left(b-\frac{(b-a)t+af(b)-bf(a)}{f(b)-f(a)}\right)^{\frac{1}{r}}\right\} \tag{1.2}$$

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where $t = \frac{(1+s)^{\frac{1}{s}}(b-a)^{\frac{1}{r}}}{(1+r)^{\frac{1}{r}}}\left(\frac{(S)\int_a^b f^s d\mu}{b-a}\right)^{\frac{1}{s}}$.

(b) if $f(a) = f(b)$, *then*

$$\left((S)\int_a^b f^r d\mu\right)^{\frac{1}{r}} \geqslant \min\left\{f(a), (b-a)^{\frac{1}{r}}\right\}. \tag{1.3}$$

(c) if $f(a) < f(b)$, *then*

$$\left((S)\int_a^b f^r d\mu\right)^{\frac{1}{r}} \geqslant \min\left\{t, \left(\frac{(b-a)t + af(b) - bf(a)}{f(b)-f(a)} - a\right)^{\frac{1}{r}}\right\} \tag{1.4}$$

where $t = \frac{(1+s)^{\frac{1}{s}}(b-a)^{\frac{1}{r}}}{(1+r)^{\frac{1}{r}}}\left(\frac{(S)\int_a^b f^s d\mu}{b-a}\right)^{\frac{1}{s}}$.

In [9], Agahi proved the Berwald type inequality for universal integral.

THEOREM 2. *Let* $0 < r < s < \infty$ *and* $f \in \mathscr{F}^{([a,b],\mathscr{A})}$ *be a concave function. If* $\otimes : [0,\infty]^2 \to [0,\infty]$ *is a pseudo-multiplication on* $[0,\infty]$ *with neutral element* $e \in (0,\infty]$, *then for any monotone measure* $\mu \in \mathscr{M}^{(X,\mathscr{A})}$, *we have*

(a) if $f(a) < f(b)$, *then*

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \left(\beta^r \otimes \mu\left([a,b] \cap \left\{x | x \geqslant \frac{(b-a)\beta + af(b) - bf(a)}{f(b)-f(a)}\right\}\right)\right)^{\frac{1}{r}} \tag{1.5}$$

where $\beta = \frac{(1+s)^{\frac{1}{s}}(b-a)^{\frac{1}{r}-\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\mathbf{I}_{\otimes}^{\frac{1}{s}}(\mu, f^s)$.

(b) if $f(a) = f(b)$, *then*

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r(a)). \tag{1.6}$$

(c) if $f(a) > f(b)$, *then*

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \left(\beta^r \otimes \mu\left([a,b] \cap \left\{x | x \leqslant \frac{(b-a)\beta + af(b) - bf(a)}{f(b)-f(a)}\right\}\right)\right)^{\frac{1}{r}} \tag{1.7}$$

where $\beta = \frac{(1+s)^{\frac{1}{s}}(b-a)^{\frac{1}{r}-\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\mathbf{I}_{\otimes}^{\frac{1}{s}}(\mu, f^s)$.

In this paper, we generalize Berwald inequality for smallest universal integral from concave function to (α, m)-concave function. Specially, concave function is equivalent to $(1,1)$-concave function.

2. Preliminary results

In this section, we will review some well known results of universal integrals [6] and (α, m)-concave function [10].

DEFINITION 1. A *monotone measure* μ on a measurable space $(\mathrm{X}, \mathscr{A})$ is a function $\mu : \mathscr{A} \to [0,\infty]$ satisfying

(i) $\mu(\phi) = 0$,

(ii) $\mu(\mathrm{X}) > 0$,

(iii) $\mu(A) \leqslant \mu(B)$, whenever $A \subseteq B$.

Normed monotone measures on $(\mathrm{X}, \mathscr{A})$, i.e. monotone measures satisfying $\mu(\mathrm{X}) = 1$, are also called capacities [1, 11].

For a measurable space $(\mathrm{X}, \mathscr{A})$, i.e. a non-empty set X equipped with a σ-algebra $\mathscr{A}$, recall that a function $f : X \to [0,\infty]$ is called $\mathscr{A}$-measurable if, for each $B \in \mathscr{B}([0,\infty])$, the σ-algebra of Borel subsets of $[0,\infty]$, the preimage $f^{-1}(B) \in \mathscr{A}$ is a element of $\mathscr{A}$.

DEFINITION 2. Let $(\mathrm{X}, \mathscr{A})$ be a measurable space.

(i) $\mathscr{F}^{(\mathrm{X},\mathscr{A})}$ denotes the set of all $\mathscr{A}$-measurable functions $f : X \to [0,\infty)$;

(ii) For each number $a \in (0,\infty]$, $\mathscr{M}_a^{(X,\mathscr{A})}$ denotes the set of all monotone measures satisfying $\mu(X) = a$; and we take

$$\mathscr{M}^{(X,\mathscr{A})} = \bigcup_{a\in(0,\infty]} \mathscr{M}_a^{(X,\mathscr{A})}.$$

(iii) Let S be the class of all measurable spaces, and we take

$$\mathscr{D}_{[0,\infty]} = \bigcup_{(X,\mathscr{A})} \mathscr{M}^{(X,\mathscr{A})} \times \mathscr{F}^{(X,\mathscr{A})}.$$

$$\mathscr{D}_{[0,\infty]} = \bigcup_{(X,\mathscr{A})\in S} \mathscr{M}^{(X,\mathscr{A})} \times \mathscr{F}^{(X,\mathscr{A})}.$$

DEFINITION 3. The *Sugeno* [1], *Shikret* [3] and *Choquet* [12] integrals (see also [13, 2, 14]) respectively, are given, for any measurable space $(X, \mathscr{A})$, for any measurable function $f \in \mathscr{F}^{(X,\mathscr{A})}$ and for any monotone measure $\mu \in \mathscr{M}^{(X,\mathscr{A})}$, i.e. for any $(\mu, f) \in \mathscr{D}_{[0,\infty]}$, by

$$\mathbf{Ch}(\mu, f) = \int_0^\infty \mu(\{f \geqslant t\})\, dt,$$

$$\mathbf{Su}(\mu, f) = \sup\{\min(t, \mu(\{f \geqslant t\})) \,|\, t \in (0,\infty]\},$$

$$\mathbf{Sh}(\mu, f) = \sup\{t \cdot \mu(\{f \geqslant t\}) \,|\, t \in (0,\infty]\},$$

where the convention $0 \cdot \infty = 0$ is used. All these integrals map $\mathscr{M}^{(X,\mathscr{A})} \times \mathscr{F}^{(X,\mathscr{A})}$ into $[0,\infty]$, and fixing an arbitrary $f \in \mathscr{F}^{(X,\mathscr{A})}$, they are non-decreasing functions from $\mathscr{M}^{(X,\mathscr{A})}$ into $[0,\infty]$.

DEFINITION 4. Two pairs $(\mu_1, f_1) \in \mathscr{M}^{(X_1,\mathscr{A}_1)} \times \mathscr{F}^{(X_1,\mathscr{A}_1)}$ and $(\mu_2, f_2) \in \mathscr{M}^{(X_2,\mathscr{A}_2)} \times \mathscr{F}^{(X_2,\mathscr{A}_2)}$ satisfying $\mu_1(\{f_1 \geqslant t\}) = \mu_2(\{f_2 \geqslant t\})$ for all $t \in (0,\infty]$, will be called integral equivalent in symbols $(\mu_1, f_1) \sim (\mu_2, f_2)$.

DEFINITION 5. [15, 16] A function $\otimes$ is called a *pseudo-multiplication* if it satisfies the following properties:

(i) it is non-decreasing in each component, i.e. for all $a_1, a_2, b_1, b_2 \in [0,\infty]$ with $a_1 \leqslant a_2, b_1 \leqslant b_2$ we have $a_1 \otimes b_1 \leqslant a_2 \otimes b_2$;

(ii) 0 is an annihilator of $\otimes$, i.e., for all $a \in [0,\infty]$ we have $0 \otimes a = a \otimes 0 = 0$;

(iii) $\otimes$ has a neutral element different from 0, i.e., there exists an element $e \in (0,\infty]$ such that, for all $a \in [0,\infty]$, we have $e \otimes a = a \otimes e = a$.

DEFINITION 6. For a given pseudo-multiplication on $[0,\infty]$, we suppose the existence of a pseudo-addition $\oplus : [0,\infty]^2 \to [0,\infty]$ which is continuous, associative, non-decreasing and has 0 as neutral element and which is left-distributive with respect to $\otimes$, i.e., for all $a,b,c \in [0,\infty]$, we have $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$. The pair $(\oplus, \otimes)$ is called an integral operation pair.

DEFINITION 7. A function $\mathbf{I} : \mathscr{D}_{[0,\infty]} \to [0,\infty]$ is called a *universal integral* if the following axioms hold:

(i) For any measurable space $(X, \mathscr{A})$, the restriction of the function $\mathbf{I}$ to $\mathscr{M}^{(X,\mathscr{A})} \times \mathscr{F}^{(X,\mathscr{A})}$ is non-decreasing in each coordinate;

(ii) there exists a pseudo-multiplication $\otimes : [0,\infty]^2 \to [0,\infty]$ such that for all pairs $(\mu, c \cdot 1_A) \in \mathscr{D}_{[0,\infty]}$, $\mathbf{I}(\mu, c \cdot 1_A) = c \otimes \mu(A)$;

(iii) for all integral equivalent pairs $(\mu_1, f_1), (\mu_2, f_2) \in \mathscr{D}_{[0,\infty]}$, we have $\mathbf{I}(\mu_1, f_1) = \mathbf{I}(\mu_2, f_2)$.

THEOREM 3. Let $\otimes : [0,\infty]^2 \to [0,\infty]$ be a pseudo-multiplication on $[0,\infty]$. Then the *smallest universal integral* $\mathbf{I}$ and the *greatest universal integral* $\mathbf{I}$ based on $\otimes$ are given by

$$\mathbf{I}_{\otimes}(\mu, f) = \sup\{t \otimes \mu(X \cap \{f \geqslant t\}) \,|\, t \in (0,\infty]\} \tag{2.1}$$

$$\mathbf{I}^{\otimes}(\mu, f) = \underset{\mu}{essup} f \otimes \sup\{\mu(X \cap \{f \geqslant t\}) \,|\, t \in (0,\infty]\} \tag{2.2}$$

where $essup_{\mu} f = \sup\{t \in [0,\infty] \,|\, \mu(X \cap \{f \geqslant t\})\} > 0$

REMARK 1. When pseudo-multiplication are given by $Min(a,b) = \min(a,b)$ and $\Pr od(a,b) = a \cdot b$, the smallest universal integral reduce to the Sugeno and Skilkret integral, i.e., $\mathbf{Su} = \mathbf{I}_{Min}$ and $\mathbf{Sh} = \mathbf{I}_{\Pr od}$, respectively.

REMARK 2. There is neither a smallest nor a greatest pseudo-multiplication on $[0,\infty]$. But, if we fix the neutral element e in $[0,\infty]$, then the smallest pseudo-multiplication $\otimes_e$ and the greatest pseudo-multiplication $\otimes^e$ with neutral element e are given by

$$a \otimes_e b = \begin{cases} 0 & if\,(a,b) \in [0,e)^2 \\ \max(a,b) & if\,(a,b) \in [e,\infty]^2 \\ \min(a,b) & otherwise \end{cases}$$

and

$$a\otimes^e b=\begin{cases}\min(a,b) & if\min(a,b)=0\ or\ (a,b)\in[0,e)^2\\ \infty & if\,(a,b)\in[e,\infty]^2\\ \max(a,b) & otherwise\end{cases}$$

PROPOSITION 1. There exists the extremal universal integral $\mathbf{I}_{\otimes_e}$ among all universal integrals satisfying the conditions

(i) for each $\mu\in\mathscr{M}_e^{(X,\mathscr{A})}$ and each $c\in[0,\infty]$ we have $\mathbf{I}(\mu,c\cdot 1_X)=c$;

(ii) for each $\mu\in\mathscr{M}^{(X,\mathscr{A})}$ and each $A\in\mathscr{A}$ we have $\mathbf{I}(\mu,e\cdot 1_X)=\mu(A)$ given by

$$(\mathbf{I}_{\otimes_e}(\mu,f)=\max\{\mu(\{f\geqslant e\}),essinf_{\otimes}f\}$$

where $essinf_{\otimes}f=\sup\{t\in[0,\infty]\,|\mu(f\geqslant t)=\mu(X)\}$.

REMARK 3. Restricting now to the unit interval $[0,\infty]$ we shall consider function $f\in\mathscr{F}^{(X,\mathscr{A})}$ satisfying $Ran(f)\subseteq[0,1]$ (in which case we shall write shortly $f\in\mathscr{F}_{[0,1]}^{(X,\mathscr{A})}$). Observe that, in this case, universal integrals are restricted to the class $\mathscr{D}_{[0,1]}=\bigcup_{(X,\mathscr{A})\in S}\mathscr{M}^{(X,\mathscr{A})}\times\mathscr{F}_{[0,1]}^{(X,\mathscr{A})}$.

DEFINITION 8. Assume that $\circledast:[0,1]^2\to[0,1]$ is a *semicopula* or *conjunctor* (see [17]). The smallest universal integral $\mathbf{I}_{\circledast}$ on the $[0,1]$ scale related to the semicopula $\circledast$ is given by

$$\mathbf{I}_{\circledast}(\mu,f)=\sup\{t\circledast\mu(\{f\geqslant t\})\,|t\in[0,1]\}. \tag{2.3}$$

This type integral was called *seminormed integral* in [18]. Specially, for a fixed strict t-norm T, the corresponding universal integral $\mathbf{I}_T$ is the so-called *Sugeno-Weber integral* in [19]

DEFINITION 9. [10] The function $f:[0,b]\to R$ is said to be (α,m)-concave, where $(\alpha,m)\in[0,1]^2$, if for every $x,y\in[0,b]$ and $t\in[0,1]$, satisfies

$$f(tx+m(1-t)y)\geqslant t^{\alpha}f(x)+m(1-t^{\alpha})f(y). \tag{2.4}$$

3. Main results

THEOREM 4. *Let $0<r<s<\infty$, $\alpha,m\in(0,1)$ and $f\in\mathscr{F}^{([a,b],\mathscr{A})}$ be a (α,m)-concave function. If $\otimes:[0,\infty]^2\to[0,\infty]$ is a pseudo-multiplication on $[0,\infty]$ with neutral element $e\in(0,\infty]$, then for any monotone measure $\mu\in\mathscr{M}^{(X,\mathscr{A})}$, we have*

(a) if $f(a)\leqslant f(b)$, then

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r)\geqslant\left(\beta^r\otimes\mu\left([a,b]\cap\left\{x|x\geqslant(b-ma)\left(\frac{\beta-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}+ma\right\}\right)\right)^{\frac{1}{r}} \tag{3.1}$$

where $\beta=\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^s)}{b-a}\right)^{\frac{1}{s}}$.

(b) if $f(a) > f(b)$, then

Case i: if $m \in \left(0, \frac{f(b)}{f(a)}\right)$

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \left(\beta^r \otimes \mu\left([a,b] \cap \left\{x | x \geqslant (b-ma)\left(\frac{\beta - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + ma\right\}\right)\right)^{\frac{1}{r}} \tag{3.2}$$

where $\beta = \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^s)}{b-a}\right)^{\frac{1}{s}}$.

Case ii: $m = \frac{f(b)}{f(a)}$

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r(b)) \tag{3.3}$$

Case iii: $m \in \left(\frac{f(b)}{f(a)}, 1\right)$

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \left(\beta^r \otimes \mu\left([a,b] \cap \left\{x | x \leqslant (b-ma)\left(\frac{\beta - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + ma\right\}\right)\right)^{\frac{1}{r}} \tag{3.4}$$

where $\beta = \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^s)}{b-a}\right)^{\frac{1}{s}}$.

Proof. Let $0 < r < s < \infty$ and $\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^s)}{b-a}\right)^{\frac{1}{s}} = \beta$. Since $f : [a,b] \to [0,\infty)$ is a (α,m)-concave function, for $x \in [a,b]$ we have

$$\begin{aligned} f(x) &= f\left(m\left(1 - \frac{x-ma}{b-ma}\right)a + \frac{x-ma}{b-ma}\,b\right) \\ &\geqslant m\left(1 - \left(\frac{x-ma}{b-ma}\right)^{\alpha}\right)f(a) + \left(\frac{x-ma}{b-ma}\right)^{\alpha} f(b) = g(x) \end{aligned}$$

$$\begin{aligned} &\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, g^r) \\ &= \left[\bigvee_{\gamma>0}\left(\gamma \otimes \mu\left([a,b] \cap \left\{x | \left[m\left(1-\left(\tfrac{x-ma}{b-ma}\right)^{\alpha}\right)f(a) + \left(\tfrac{x-ma}{b-ma}\right)^{\alpha} f(b)\right] \geqslant \gamma^{\frac{1}{r}}\right\}\right)\right)\right]^{\frac{1}{r}} \\ &= \left[\bigvee_{\gamma>0}\left(\gamma \otimes \mu\left([a,b] \cap \left\{x | (f(b)-mf(a))(x-ma)^{\alpha} \geqslant (b-ma)^{\alpha}\left(\gamma^{\frac{1}{r}} - mf(a)\right)\right\}\right)\right)\right]^{\frac{1}{r}} \end{aligned} \tag{3.5}$$

(a) if $f(a) \leqslant f(b)$, then by (3.5) we have

$$\begin{aligned} &\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \\ &\geqslant \left[\bigvee_{\gamma>0}\left(\gamma \otimes \mu\left([a,b] \cap \left\{x | x \geqslant (b-ma)\left(\frac{\gamma^{\frac{1}{r}} - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + ma\right\}\right)\right)\right]^{\frac{1}{r}} \\ &\geqslant \left(\beta^r \otimes \mu\left([a,b] \cap \left\{x | x \geqslant (b-ma)\left(\frac{\beta - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + ma\right\}\right)\right)^{\frac{1}{r}} \end{aligned} \tag{3.6}$$

(b) if $f(a) > f(b)$, then by (3.5) we have
Case i: if $m \in \left(0, \frac{f(b)}{f(a)}\right)$

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \left(\beta^r \otimes \mu\left([a,b] \cap \left\{x | x \geqslant (b-ma)\left(\frac{\beta - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + ma\right\}\right)\right)^{\frac{1}{r}} \tag{3.7}$$

Case ii: $m = \frac{f(b)}{f(a)}$

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r(b)) \tag{3.8}$$

Case iii: $m \in \left(\frac{f(b)}{f(a)}, 1\right)$

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \left(\beta^r \otimes \mu\left([a,b] \cap \left\{x | x \leqslant (b-ma)\left(\frac{\beta - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + ma\right\}\right)\right)^{\frac{1}{r}} \tag{3.9}$$

and the proof is completed. □

EXAMPLE 1. Consider $X = [0,1]$ and $\otimes = \min$ with neutral element 1. Let $\mu = m^2$, where m is the Lebesge measure on X. Take the function $f(x) = \sqrt{x}$, $f(x)$ is a $\left(\frac{2}{3}, \frac{1}{3}\right)$-concave function. In fact,

$$\sqrt{x} = f\left(x \cdot 1 + \frac{1}{3}(1-x) \cdot 0\right) \geqslant \sqrt[3]{x^2} + \tfrac{1}{3}\left(1 - \sqrt[3]{x^2}\right) \cdot 0 = \sqrt[3]{x^2}$$

for $x \in [0,1]$. Let $r = \frac{1}{2}, s = 2$, a straightforward calculus shows that
(i) $\mathbf{I}_{\min}(\mu, f^r) = \sup\left\{t \wedge \mu\left([0,1] \cap \left\{x \geqslant t^4\right\}\right) \middle| t \in (0,1]\right\} = 0.6588$,
(ii) $\mathbf{I}_{\min}(\mu, f^s) = \sup\{t \wedge \mu([0,1] \cap \{x \geqslant t\}) | t \in (0,1]\} = 0.3820$,
Therefore

$$0.4340 = \left(\mathbf{I}_{\min}^2\left(\mu, f^{\frac{1}{2}}\right)\right) \geqslant \left(\begin{array}{c} \left(\frac{(1+2)^{\frac{1}{2}}}{\left(1+\frac{1}{2}\right)^2}\mathbf{I}_{\min}^{\frac{1}{2}}\left(\mu, f^2\right)\right)^{\frac{1}{2}} \wedge \\ \left(1 - \left(\frac{(1+2)^{\frac{1}{2}}}{\left(1+\frac{1}{2}\right)^2}\mathbf{I}_{\min}^{\frac{1}{2}}\left(\mu, f^2\right)\right)^{\frac{3}{2}}\right)^2 \end{array}\right)^2 = 0.2037.$$

As some special cases of (α, m)-convex functions, we get the following results.

THEOREM 5. *Let $0 < r < s < \infty$ and $f \in \mathscr{F}^{([a,b],\mathscr{A})}$ be a $(0,0)$-concave function. If $\otimes : [0,\infty]^2 \to [0,\infty]$ is a pseudo-multiplication on $[0,\infty]$ with neutral element $e \in (0,\infty]$, then for any monotone measure $\mu \in \mathscr{M}^{(X,\mathscr{A})}$, we have*

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r(b)). \tag{3.10}$$

Proof. Let $r \in (0,\infty)$. Since $f:[a,b] \to [0,\infty)$ is a $(0,0)$-concave function, for $x \in [a,b]$ we have

$$\begin{aligned} f(x) =& f\left(0 \cdot \left(1 - \frac{x - 0 \cdot a}{b - 0 \cdot a}\right) a + \frac{x - 0 \cdot a}{b - 0 \cdot a} b\right) \\ \geqslant& 0 \cdot \left(1 - \left(\frac{x - 0 \cdot a}{b - 0 \cdot a}\right)^0\right) f(a) + \left(\frac{x - 0 \cdot a}{b - 0 \cdot a}\right)^0 f(b) = f(b) \end{aligned}$$

Thus,

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r(b)). \tag{3.11}$$

and the proof is completed. □

EXAMPLE 2. Consider $X = [0,1]$ and $\otimes = \min$ with neutral element 1. Let $\mu = m^2$, where m is the Lebesge measure on X. Take the function $f(x) = 1 - x$, $f(x)$ is a decreasing function. Let $r = \frac{1}{2}$, a straightforward calculus shows that

$\mathbf{I}_{\min}(\mu, f^s) = \sup\{t \wedge \mu([0,1] \cap \{x \geqslant t\}) | t \in (0,1]\} = 0.3820$,
therefore

$$0.1459 = \mathbf{I}_{\min}^2\left(\mu, f^{\frac{1}{2}}\right) \geqslant \mathbf{I}_{\min}^2\left(\mu, f^{\frac{1}{2}}(1)\right) = 0$$

THEOREM 6. *Let $0 < r < s < \infty$, $\alpha \in (0,1)$ and $f \in \mathscr{F}^{([a,b],\mathscr{A})}$ be a $(\alpha,0)$-concave function. If $\otimes : [0,\infty]^2 \to [0,\infty]$ is a pseudo-multiplication on $[0,\infty]$ with neutral element $e \in (0,\infty]$, then for any monotone measure $\mu \in \mathscr{M}^{(X,\mathscr{A})}$, we have*

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \left(\beta^r \otimes \mu\left([a,b] \cap \left\{x | x \geqslant b\left(\frac{\beta}{f(b)}\right)^{\frac{1}{\alpha}}\right\}\right)\right)^{\frac{1}{r}} \tag{3.12}$$

where $\beta = \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}} \left(\frac{\mathbf{I}_{\otimes}(\mu, f^s)}{b-a}\right)^{\frac{1}{s}}$.

Proof. Let $0 < r < s < \infty$, $\alpha \in (0,1)$. Since $f:[a,b] \to [0,\infty)$ is a $(\alpha,0)$-concave function, for $x \in [a,b]$ we have

$$\begin{aligned} f(x) =& f\left(0 \cdot \left(1 - \frac{x - 0 \cdot a}{b - 0 \cdot a}\right) a + \frac{x - 0 \cdot a}{b - 0 \cdot a} b\right) \\ \geqslant& 0 \cdot \left(1 - \left(\frac{x - 0 \cdot a}{b - 0 \cdot a}\right)^\alpha\right) f(a) + \left(\frac{x - 0 \cdot a}{b - 0 \cdot a}\right)^\alpha f(b) = \left(\frac{x}{b}\right)^\alpha f(b) \end{aligned}$$

$$\begin{aligned}\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r) \geqslant & \mathbf{I}_{\otimes}^{\frac{1}{r}}\left(\mu,\left(\frac{x}{b}\right)^{\alpha}f(b)\right) = \left[\bigvee_{\gamma>0}\left(\gamma\otimes\mu\left([a,b]\cap\left\{x|\left(\frac{x}{b}\right)^{\alpha}f(b)\geqslant\gamma^{\frac{1}{r}}\right\}\right)\right)\right]^{\frac{1}{r}}\\ &= \left[\bigvee_{\gamma>0}\left(\gamma\otimes\mu\left([a,b]\cap\left\{x|x\geqslant\frac{b\gamma^{\frac{1}{\alpha r}}}{f(b)^{\frac{1}{\alpha}}}\right\}\right)\right)\right]^{\frac{1}{r}}\\ &\geqslant\left(\beta^r\otimes\mu\left([a,b]\cap\left\{x|x\geqslant b\left(\frac{\beta}{f(b)}\right)^{\frac{1}{\alpha}}\right\}\right)\right)^{\frac{1}{r}} \qquad (3.13)\end{aligned}$$

where $\beta = \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^s)}{b-a}\right)^{\frac{1}{s}}$ and the proof is completed. □

EXAMPLE 3. Consider $X=[0,1]$ and $\otimes=\min$ with neutral element 1. Let $\mu = m^2$, where m is the Lebesge measure on X. Take the function $f(x)=\sqrt[3]{x}$, $f(x)$ is a $\left(\frac{1}{3},0\right)$-concave function. Let $r=\frac{1}{2}, s=3$, a straightforward calculus shows that

(i) $\mathbf{I}_{\min}\left(\mu,f^{\frac{1}{2}}\right)=\sup\left\{t\wedge\mu\left([0,1]\cap\left\{x\geqslant t^6\right\}\right)\middle|t\in(0,1]\right\}=0.7268$,

(ii) $\mathbf{I}_{\min}\left(\mu,f^3\right)=\sup\{t\wedge\mu([0,1]\cap\{x\geqslant t\})|t\in(0,1]\}=0.3820$.

Therefore

$$0.5282=\left(\mathbf{I}_{\min}^2\left(\mu,f^{\frac{1}{2}}\right)\right)\geqslant\left(\begin{array}{c}\left(\frac{(1+3)^{\frac{1}{3}}}{\left(1+\frac{1}{2}\right)^2}\mathbf{I}_{\min}^{\frac{1}{3}}\left(\mu,f^3\right)\right)^{\frac{1}{2}}\wedge\\ \left(1-\left(\frac{(1+3)^{\frac{1}{3}}}{\left(1+\frac{1}{2}\right)^2}\mathbf{I}_{\min}^{\frac{1}{3}}\left(\mu,f^3\right)\right)^3\right)^2\end{array}\right)^2=0.1613.$$

THEOREM 7. *Let $0<r<s<\infty$ and $f\in\mathscr{F}^{([a,b],\mathscr{A})}$ be $(1,0)$-concave function. If $\otimes:[0,\infty]^2\to[0,\infty]$ is a pseudo-multiplication on $[0,\infty]$ with neutral element $e\in(0,\infty]$, then for any monotone measure $\mu\in\mathscr{M}^{(X,\mathscr{A})}$, we have*

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r)\geqslant\left(\beta^r\otimes\mu\left([a,b]\cap\left\{x|x\geqslant\frac{b\beta}{f(b)}\right\}\right)\right)^{\frac{1}{r}} \qquad (3.14)$$

where $\beta = \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^s)}{b-a}\right)^{\frac{1}{s}}$.

Proof. Let $0<r<s<\infty$. Since $f:[a,b]\to[0,\infty)$ is a $(1,0)$-concave function, for $x\in[a,b]$ we have

$$\begin{aligned}f(x)=&f\left(0\cdot\left(1-\frac{x-0\cdot a}{b-0\cdot a}\right)a+\frac{x-0\cdot a}{b-0\cdot a}b\right)\\ &\geqslant 0\cdot\left(1-\left(\frac{x-0\cdot a}{b-0\cdot a}\right)\right)f(a)+\left(\frac{x-0\cdot a}{b-0\cdot a}\right)f(b)=\left(\frac{x}{b}\right)f(b)\end{aligned}$$

$$\begin{aligned}\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^{r}) \geqslant & \mathbf{I}_{\otimes}^{\frac{1}{r}}\left(\mu,\left(\frac{f(b)x}{b}\right)^{r}\right)=\left[\bigvee_{\gamma>0}\left(\gamma\otimes\mu\left([a,b]\cap\left\{x\middle|\,\frac{f(b)x}{b}\geqslant\gamma^{\frac{1}{r}}\right\}\right)\right)\right]^{\frac{1}{r}}\\ &=\left[\bigvee_{\gamma>0}\left(\gamma\otimes\mu\left([a,b]\cap\left\{x|x\geqslant\frac{b\gamma^{\frac{1}{r}}}{f(b)}\right\}\right)\right)\right]^{\frac{1}{r}}\\ &\geqslant\left(\beta^{r}\otimes\mu\left([a,b]\cap\left\{x|x\geqslant\tfrac{b\beta}{f(b)}\right\}\right)\right)^{\frac{1}{r}}\end{aligned} \tag{3.15}$$

where $\beta=\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^{s})}{b-a}\right)^{\frac{1}{s}}$ and the proof is completed. □

EXAMPLE 4. Consider $X=[0,1]$ and $\otimes=\min$ with neutral element 1. Let $\mu=m^{2}$, where m is the Lebesge measure on X. Take the function $f(x)=\sqrt[3]{x}$, $f(x)$ is a $(1,0)$-concave function. Let $r=\frac{1}{2}, s=3$, a straightforward calculus shows that

(i) $\mathbf{I}_{\min}\left(\mu,f^{\frac{1}{2}}\right)=\sup\left\{t\wedge\mu\left([0,1]\cap\left\{x\geqslant t^{6}\right\}\right)\middle|\,t\in(0,1]\right\}=0.7268,$

(ii) $\mathbf{I}_{\min}\left(\mu,f^{3}\right)=\sup\left\{t\wedge\mu\left([0,1]\cap\{x\geqslant t\}\right)|t\in(0,1]\right\}=0.3820.$

Therefore

$$0.5282=\left(\mathbf{I}_{\min}^{2}\left(\mu,f^{\frac{1}{2}}\right)\right)\geqslant\left(\begin{array}{c}\left(\frac{(1+3)^{\frac{1}{3}}}{(1+\frac{1}{2})^{2}}\mathbf{I}_{\min}^{\frac{1}{3}}\left(\mu,f^{3}\right)\right)^{\frac{1}{2}}\wedge\\ \left(1-\left(\frac{(1+3)^{\frac{1}{3}}}{(1+\frac{1}{2})^{2}}\mathbf{I}_{\min}^{\frac{1}{3}}\left(\mu,f^{3}\right)\right)\right)^{2}\end{array}\right)^{2}=0.0649.$$

THEOREM 8. *Let* $0<r<s<\infty$, $m\in(0,1)$ *and* $f\in\mathscr{F}^{([a,b],\mathscr{A})}$ *be a* $(1,m)$*-concave function. If* $\otimes:[0,\infty]^{2}\to[0,\infty]$ *is a pseudo-multiplication on* $[0,\infty]$ *with neutral element* $e\in(0,\infty]$*, then for any monotone measure* $\mu\in\mathscr{M}^{(X,\mathscr{A})}$*, we have*

Case i: if $f(a)\leqslant f(b)$

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^{r})\geqslant\left(\beta^{r}\otimes\mu\left([a,b]\cap\left\{x|x\geqslant\frac{(b-ma)(\beta-mf(a))}{f(b)-mf(a)}+ma\right\}\right)\right)^{\frac{1}{r}} \tag{3.16}$$

where $\beta=\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^{s})}{b-a}\right)^{\frac{1}{s}}$.

Case ii: if $f(a)>f(b)$

$(a)\, m\in\left(0,\frac{f(b)}{f(a)}\right)$

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^{r})\geqslant\left(\beta^{r}\otimes\mu\left([a,b]\cap\left\{x|x\geqslant\frac{(b-ma)(\beta-mf(a))}{f(b)-mf(a)}+ma\right\}\right)\right)^{\frac{1}{r}} \tag{3.17}$$

where $\beta=\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^{s})}{b-a}\right)^{\frac{1}{s}}$.

$(b)\, m = \frac{f(b)}{f(a)}$

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r(b)). \tag{3.18}$$

$(c)\, m \in \left(\frac{f(b)}{f(a)}, 1\right)$

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \left(\beta^r \otimes \mu\left([a,b] \cap \left\{x | x \leqslant \frac{(b-ma)(\beta - mf(a))}{f(b) - mf(a)} + ma\right\}\right)\right)^{\frac{1}{r}} \tag{3.19}$$

where $\beta = \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu, f^s)}{b-a}\right)^{\frac{1}{s}}$.

Proof. Let $0 < r < s < \infty$, $m \in (0,1)$. Since $f : [a,b] \to [0,\infty)$ is a $(1,m)$-concave function, for $x \in [a,b]$ we have

$$\begin{aligned} f(x) =& f\left(0 \cdot \left(1 - \frac{x - 0 \cdot a}{b - 0 \cdot a}\right) a + \frac{x - 0 \cdot a}{b - 0 \cdot a}\, b\right) \\ \geqslant& m\left(1 - \left(\frac{x - ma}{b - ma}\right)\right) f(a) + \left(\frac{x - ma}{b - ma}\right) f(b) = g(x) \end{aligned}$$

$$\begin{aligned} &\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, g^r) \\ =&\left[\bigvee_{\gamma>0}\left(\gamma \otimes \mu\left([a,b] \cap \left\{x | \left[m\left(1 - \left(\tfrac{x-ma}{b-ma}\right)\right) f(a) + \left(\tfrac{x-ma}{b-ma}\right) f(b)\right] \geqslant \gamma^{\frac{1}{r}}\right\}\right)\right)\right]^{\frac{1}{r}} \\ =&\left[\bigvee_{\gamma>0}\left(\gamma \otimes \mu\left([a,b] \cap \left\{x | (f(b) - mf(a))(x - ma) \geqslant (b - ma)\left(\gamma^{\frac{1}{r}} - mf(a)\right)\right\}\right)\right)\right]^{\frac{1}{r}} \end{aligned} \tag{3.20}$$

(a) if $f(a) \leqslant f(b)$, then by (3.20) we have

$$\begin{aligned} &\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \\ \geqslant& \left[\bigvee_{\gamma>0}\left(\gamma \otimes \mu\left([a,b] \cap \left\{x | x \geqslant (b - ma)\left(\frac{\gamma^{\frac{1}{r}} - mf(a)}{f(b) - mf(a)}\right) + ma\right\}\right)\right)\right]^{\frac{1}{r}} \\ \geqslant& \left(\beta^r \otimes \mu\left([a,b] \cap \left\{x | x \geqslant \frac{(b-ma)(\beta - mf(a))}{f(b) - mf(a)} + ma\right\}\right)\right)^{\frac{1}{r}} \end{aligned} \tag{3.21}$$

where $\beta = \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu, f^s)}{b-a}\right)^{\frac{1}{s}}$.

(b) if $f(a) > f(b)$, then by (3.20) we have

Case i: if $m \in \left(0, \frac{f(b)}{f(a)}\right)$

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu, f^r) \geqslant \left(\beta^r \otimes \mu\left([a,b] \cap \left\{x | x \geqslant \frac{(b-ma)(\beta - mf(a))}{f(b) - mf(a)} + ma\right\}\right)\right)^{\frac{1}{r}} \tag{3.22}$$

where $\beta = \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^s)}{b-a}\right)^{\frac{1}{s}}$.

Case ii: $m = \frac{f(b)}{f(a)}$

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r) \geqslant \mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r(b)) \tag{3.23}$$

Case iii: $m \in \left(\frac{f(b)}{f(a)},1\right)$

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r) \geqslant \left(\beta^r \otimes \mu\left([a,b] \cap \left\{x | x \leqslant \frac{(b-ma)(\beta-mf(a))}{f(b)-mf(a)} + ma\right\}\right)\right)^{\frac{1}{r}} \tag{3.24}$$

where $\beta = \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^s)}{b-a}\right)^{\frac{1}{s}}$ and the proof is completed. □

REMARK 4. Example 4 can be used to illustrate (a) of Theorem 8. In fact,

$$\sqrt[3]{x} = f(x \cdot 1 + m(1-x) \cdot 0) \geqslant f(1)x + m(1-x)f(0) = x$$

for all $x,m \in [0,1]$.

THEOREM 9. *Let $0 < r < s < \infty$, $\alpha \in (0,1), m = 1$ and $f \in \mathscr{F}^{([a,b],\mathscr{A})}$ be a $(\alpha,1)$-concave function. If $\otimes : [0,\infty]^2 \to [0,\infty]$ is a pseudo-multiplication on $[0,\infty]$ with neutral element $e \in (0,\infty]$, then for any monotone measure $\mu \in \mathscr{M}^{(X,\mathscr{A})}$, we have*

(a) if $f(a) < f(b)$, then

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r) \geqslant \left(\beta^r \otimes \mu\left([a,b] \cap \left\{x | x \geqslant (b-a)^{\alpha}\left(\frac{(\beta-f(a))}{f(b)-f(a)}\right)^{\frac{1}{\alpha}} + a\right\}\right)\right)^{\frac{1}{r}} \tag{3.25}$$

where $\beta = \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^s)}{b-a}\right)^{\frac{1}{s}}$.

(b) if $f(a) = f(b)$, then

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r) \geqslant \mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r(a)). \tag{3.26}$$

(c) if $f(a) > f(b)$, then

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r) \geqslant \left(\beta^r \otimes \mu\left([a,b] \cap \left\{x | x \leqslant (b-a)^{\alpha}\left(\frac{(\beta-f(a))}{f(b)-f(a)}\right)^{\frac{1}{\alpha}} + a\right\}\right)\right)^{\frac{1}{r}} \tag{3.27}$$

where $\beta = \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^s)}{b-a}\right)^{\frac{1}{s}}$.

Proof. Let $0<r<s<\infty$, $\alpha\in(0,1)$. Since $f:[a,b]\to[0,\infty)$ is a $(\alpha,1)$-concave function, for $x\in[a,b]$ we have

$$\begin{aligned} f(x) &= f\left(\left(1-\frac{x-a}{b-a}\right)a+\frac{x-a}{b-a}\,b\right)\\ &\geqslant \left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)f(a)+\left(\frac{x-a}{b-a}\right)^{\alpha}f(b)=g(x)\end{aligned}$$

$$\begin{aligned} &\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r)\geqslant \mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,g^r)\\ &=\left[\bigvee_{\gamma>0}\left(\gamma\otimes\mu\left([a,b]\cap\left\{x\middle|\left[\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)f(a)+\left(\frac{x-a}{b-a}\right)^{\alpha}f(b)\right]\geqslant\gamma^{\frac{1}{r}}\right\}\right)\right)\right]^{\frac{1}{r}}\\ &=\left[\bigvee_{\gamma>0}\left(\gamma\otimes\mu\left([a,b]\cap\left\{x\middle|(f(b)-f(a))(x-a)^{\alpha}\geqslant(b-a)^{\alpha}\left(\gamma^{\frac{1}{r}}-f(a)\right)\right\}\right)\right)\right]^{\frac{1}{r}} \end{aligned} \tag{3.28}$$

(a) if $f(a)<f(b)$, then by (3.28) we have

$$\begin{aligned} &\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r)\\ &\geqslant\left[\bigvee_{\gamma>0}\left(\gamma\otimes\mu\left([a,b]\cap\left\{x\middle|x\geqslant(b-a)\left(\frac{\gamma^{\frac{1}{r}}-f(a)}{f(b)-f(a)}\right)^{\frac{1}{\alpha}}+a\right\}\right)\right)\right]^{\frac{1}{r}}\\ &\geqslant\left(\beta^r\otimes\mu\left([a,b]\cap\left\{x\middle|x\leqslant(b-a)\left(\frac{(\beta-f(a))}{f(b)-f(a)}\right)^{\frac{1}{\alpha}}+a\right\}\right)\right)^{\frac{1}{r}} \end{aligned} \tag{3.29}$$

where $\beta=\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^s)}{b-a}\right)^{\frac{1}{s}}$.

(b) if $f(a)=f(b)$, then

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r)\geqslant\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r(a)). \tag{3.30}$$

(c) if $f(a)>f(b)$, then by (3.28) we have

$$\begin{aligned} &\mathbf{I}_{\otimes}^{\frac{1}{r}}(\mu,f^r)\\ &\geqslant\left[\bigvee_{\gamma>0}\left(\gamma\otimes\mu\left([a,b]\cap\left\{x\middle|x\leqslant(b-a)\left(\frac{\gamma^{\frac{1}{r}}-f(a)}{f(b)-f(a)}\right)^{\frac{1}{\alpha}}+a\right\}\right)\right)\right]^{\frac{1}{r}}\\ &\geqslant\left(\beta^r\otimes\mu\left([a,b]\cap\left\{x\middle|x\leqslant(b-a)\left(\frac{(\beta-f(a))}{f(b)-f(a)}\right)^{\frac{1}{\alpha}}+a\right\}\right)\right)^{\frac{1}{r}} \end{aligned} \tag{3.31}$$

$\beta=\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{I}_{\otimes}(\mu,f^s)}{b-a}\right)^{\frac{1}{s}}$ and the proof is completed. □

EXAMPLE 5. Consider $X=[0,1]$ and $\otimes=\min$ with neutral element 1. Let $\mu=m^2$, where m is the Lebesge measure on X. Take the function $f(x)=\sqrt[3]{x}$, $f(x)$ is a $\left(\frac{1}{2},1\right)$-concave function. Let $r=\frac{1}{2}, s=3$, a straightforward calculus shows that

(i) $\mathbf{I}_{\min}\left(\mu,f^{\frac{1}{2}}\right)=\sup\left\{t\wedge\mu\left([0,1]\cap\left\{x\geqslant t^6\right\}\right)\middle|t\in(0,1]\right\}=0.7268$,

(ii) $\mathbf{I}_{\min}\left(\mu,f^3\right)=\sup\{t\wedge\mu([0,1]\cap\{x\geqslant t\})|t\in(0,1]\}=0.3820$.

Therefore

$$0.5282=\left(\mathbf{I}_{\min}^2\left(\mu,f^{\frac{1}{2}}\right)\right)\geqslant\left(\begin{array}{c}\left(\frac{(1+3)^{\frac{1}{3}}}{(1+\frac{1}{2})^2}\mathbf{I}_{\min}^{\frac{1}{3}}\left(\mu,f^3\right)\right)^{\frac{1}{2}}\wedge\\ \left(1-\left(\frac{(1+3)^{\frac{1}{3}}}{(1+\frac{1}{2})^2}\mathbf{I}_{\min}^{\frac{1}{3}}\left(\mu,f^3\right)\right)^2\right)^2\end{array}\right)^2=0.2966.$$

REMARK 5. Let $0<r<s<\infty$, $\alpha=1, m=1$ and $f\in\mathscr{F}^{([a,b],\mathscr{A})}$ be a (α,m)-concave function. If $\otimes:[0,\infty]^2\to[0,\infty]$ is a pseudo-multiplication on $[0,\infty]$ with neutral element $e\in(0,\infty]$, then for any monotone measure $\mu\in\mathscr{M}^{(X,\mathscr{A})}$, then we obtain the Berwald type inequality for universal integral based on concave function in [9].

REMARK 6. *Case i*: If $\otimes$ is minimum in Theorem 4 and $(\alpha,m)\in[0,1]^2$, then we obtain the Berwald type inequalities for Sugeno integral based on (α,m)-concave function encompassing some special cases, specially, if we take $\alpha=1$, $m=1$ and μ is the Lebesgue measure on $\mathbb{R}$, then we have the results of [8].

Case ii: If $\otimes$ is standard product in Theorem 4 and $(\alpha,m)\in[0,1]^2$, then we obtain the Berwald type inequalities for Shilkret integral based on (α,m)-concave function including some special cases.

Case iii: We consider on $[0,1]$ in Theorem 4. If $\otimes=\circledast$ is semicopula (t-seminorm), $e=1$ and $(\alpha,m)\in[0,1]^2$, then we obtain the Berwald type inequalities for seminormed integral based on (α,m)-concave function containing some special cases.

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