ADDITIVE $\rho$–FUNCTIONAL INEQUALITIES AND EQUATIONS

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Abstract. In this paper, we investigate the additive $\rho$-functional inequalities

$$\| f \left( \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f(x_j) \| \leq \| \rho \left( k f \left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) \right) \|$$

(0.1)

and

$$\left| k f \left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) \right| \leq \| \rho \left( f \left( \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f(x_j) \right) \|,$$

(0.2)

where $\rho$ is a fixed complex number with $|\rho| < 1$.

Furthermore, we prove the Hyers-Ulam stability of the additive $\rho$-functional inequalities (0.1) and (0.2) in complex Banach spaces and prove the Hyers-Ulam stability of additive $\rho$-functional equations associated with the additive $\rho$-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and preliminaries


The functional equation

$$f(x + y) = f(x) + f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ Theorem was generalized by Aoki [1] for additive mappings and by Rassias [8] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [3] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach.

In [4], Gilányi showed that if $f$ satisfies the functional inequality

$$\| 2f(x) + 2f(y) - f(xy^{-1}) \| \leq \| f(xy) \|$$

(1.1)


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then \( f \) satisfies the Jordan-von Neumann functional equation

\[ 2f(x) + 2f(y) = f(xy) + f(xy^{-1}). \]


In Section 2, we investigate the additive \( \rho \)-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive \( \rho \)-functional inequality (0.1) in complex Banach spaces. We moreover prove the Hyers-Ulam stability of an additive \( \rho \)-functional equation associated with the additive \( \rho \)-functional inequality (0.1) in complex Banach spaces.

In Section 3, we investigate the additive \( \rho \)-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive \( \rho \)-functional inequality (0.2) in complex Banach spaces. We moreover prove the Hyers-Ulam stability of an additive \( \rho \)-functional equation associated with the additive \( \rho \)-functional inequality (0.2) in complex Banach spaces.

Throughout this paper, let \( k \) be a fixed integer with \( k \geq 2 \) and let \( G \) be a \( k \)-divisible abelian group. Assume that \( X \) is a real or complex normed space with norm \( \| \cdot \| \) and that \( Y \) is a complex Banach space with norm \( \| \cdot \| \). Assume that \( \rho \) is a fixed complex number with \( |\rho| < 1 \).

### 2. Additive \( \rho \)-functional inequality (0.1)

In this section, we investigate the additive \( \rho \)-functional inequality (0.1) in complex Banach spaces.

**Lemma 2.1.** A mapping \( f : G \rightarrow Y \) satisfies

\[
\left\| f \left( \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f(x_j) \right\| \leq \left\| \rho \left( k f \left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) \right) \right\| (2.1)
\]

for all \( x_1, x_2, \ldots, x_k \in G \) if and only if \( f : G \rightarrow Y \) is additive.

**Proof.** Assume that \( f : G \rightarrow Y \) satisfies (2.1).

Letting \( x_1 = x_2 = \cdots = x_k = 0 \) in (2.1), we get

\[ \|(k-1)f(0)\| \leq 0. \]

So \( f(0) = 0 \).

Letting \( x_1 = x_2 = \cdots = x_k = x \) in (2.1), we get

\[ \|f(kx) - kf(x)\| \leq 0 \]

and so \( f(kx) = kf(x) \) for all \( x \in G \). Thus

\[
f \left( \frac{x}{k} \right) = \frac{1}{k} f(x) \quad (2.2)
\]
for all \( x \in G \).

It follows from (2.1) and (2.2) that

\[
\| f \left( \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f(x_j) \| \leq \| \rho \left( kf \left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) \right) \| = |\rho| \left\| f \left( \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f(x_j) \right\|.
\]

and so

\[
f \left( \sum_{j=1}^{k} x_j \right) = \sum_{j=1}^{k} f(x_j)
\]

for all \( x_1, x_2, \ldots, x_k \in G \). Hence \( f : G \rightarrow Y \) is additive.

The converse is obviously true. \( \square \)

**Corollary 2.2.** A mapping \( f : G \rightarrow Y \) satisfies

\[
f \left( \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f(x_j) = \rho \left( kf \left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) \right)
\]

(2.3)

for all \( x_1, x_2, \ldots, x_k \in G \) if and only if \( f : G \rightarrow Y \) is additive.

The equation (2.3) is called an additive \( \rho \)-functional equation.

We prove the Hyers-Ulam stability of the additive \( \rho \)-functional inequality (2.1) in complex Banach spaces.

**Theorem 2.3.** Let \( r > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \rightarrow Y \) be a mapping such that

\[
\| f \left( \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f(x_j) \| \leq \| \rho \left( kf \left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) \right) \| + \theta \sum_{j=1}^{k} \| x_j \|^r
\]

(2.4)

for all \( x_1, x_2, \ldots, x_k \in X \). Then there exists a unique additive mapping \( h : X \rightarrow Y \) such that

\[
\| f(x) - h(x) \| \leq \frac{k \theta}{k^r - k} \| x \|^r
\]

(2.5)

for all \( x \in X \).

**Proof.** Letting \( x_1 = x_2 = \cdots = x_k = 0 \) in (2.4), we get

\[
\| (k - 1) f(0) \| \leq 0.
\]

So \( f(0) = 0 \).
Letting $x_1 = x_2 = \cdots = x_k = x$ in (2.4), we get
\[
\|f(kx) - kf(x)\| \leq k\theta\|x\|^r
\] (2.6)
for all $x \in X$. So
\[
\|f(x) - kf\left(\frac{x}{k}\right)\| \leq \frac{k}{k^r}\theta\|x\|^r
\]
for all $x \in X$. Hence
\[
\left\|k^l f\left(\frac{x}{k^l}\right) - k^m f\left(\frac{x}{k^m}\right)\right\| \leq \sum_{j=1}^{m-1} \left\|k^j f\left(\frac{x}{k^j}\right) - k^{j+1} f\left(\frac{x}{k^{j+1}}\right)\right\|
\]
\[
\leq \frac{k}{k^r} \sum_{j=1}^{m-1} \frac{k^j}{k^{rj}}\theta\|x\|^r
\] (2.7)
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{k^m f\left(\frac{x}{k^m}\right)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{k^m f\left(\frac{x}{k^m}\right)\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by
\[
h(x) := \lim_{n \rightarrow \infty} k^n f\left(\frac{x}{k^n}\right)
\]
for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.7), we get (2.5).

It follows from (2.4) that
\[
\left\|h\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} h(x_j)\right\|
\]
\[
= \lim_{n \rightarrow \infty} k^n \left\|f\left(\frac{\sum_{j=1}^{k} x_j}{k^n}\right) - \sum_{j=1}^{k} f\left(\frac{x_j}{k^n}\right)\right\|
\]
\[
\leq \lim_{n \rightarrow \infty} k^n \left\|\rho \left(kf\left(\frac{\sum_{j=1}^{k} x_j}{k^{n+1}}\right) - \sum_{j=1}^{k} f\left(\frac{x_j}{k^n}\right)\right)\right\| + \lim_{n \rightarrow \infty} \frac{k^n \theta}{k^{rn}} \sum_{j=1}^{k} \|x_j\|^r
\]
\[
= \left\|\rho\left(kh\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - \sum_{j=1}^{k} h(x_j)\right)\right\|
\]
for all $x_1, x_2, \cdots, x_k \in X$. So
\[
\left\|h\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} h(x_j)\right\| \leq \left\|\rho\left(kh\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - \sum_{j=1}^{k} h(x_j)\right)\right\|
\]
for all $x_1, x_2, \cdots, x_k \in X$. By Lemma 2.1, the mapping $h : X \rightarrow Y$ is additive.
Now, let $T : X \to Y$ be another additive mapping satisfying (2.5). Then we have
\[
\|h(x) - T(x)\| = kn \left\| \frac{x}{k^n} - T \left( \frac{x}{k^n} \right) \right\| \\
\leq kn \left( \left\| \frac{x}{k^n} - f \left( \frac{x}{k^n} \right) \right\| + \left\| T \left( \frac{x}{k^n} \right) - f \left( \frac{x}{k^n} \right) \right\| \right) \\
\leq \frac{2k^{n+1}}{(k^r - k)knr} \theta \|x\|^r,
\]
which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $h$. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (2.5).

**Theorem 2.4.** Let $r < 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying (2.4). Then there exists a unique additive mapping $h : X \to Y$ such that
\[
\|f(x) - h(x)\| \leq \frac{k\theta}{k - k^r} \|x\|^r
\]  
(2.8)
for all $x \in X$.

**Proof.** Letting $x_1 = x_2 = \cdots = x_k = 0$ in (2.4), we get
\[
\| (k - 1)f(0) \| \leq 0.
\]
So $f(0) = 0$.

It follows from (2.6) that
\[
\left\| \frac{f(x) - 1}{k^r} f(kx) \right\| \leq \theta \|x\|^r
\]
for all $x \in X$. Hence
\[
\left\| \frac{1}{k^l} f(k^l x) - \frac{1}{k^m} f(k^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{k^j} f(k^j x) - \frac{1}{k^{j+1}} f(k^{j+1} x) \right\| \\
\leq \sum_{j=l}^{m-1} \frac{k^r j}{kj} \theta \|x\|^r
\]  
(2.9)
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.9) that the sequence $\left\{ \frac{1}{k^n} f(k^n x) \right\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\left\{ \frac{1}{k^n} f(k^n x) \right\}$ converges. So one can define the mapping $h : X \to Y$ by
\[
h(x) := \lim_{n \to \infty} \frac{1}{k^n} f(k^n x)
\]
for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.8).
The rest of the proof is similar to the proof of Theorem 2.3. □

By the triangle inequality, we have

\[ \| f \left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) \| \leq \| f \left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \rho \left( k f \left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) \right) \| \]

As corollaries of Theorems 2.3 and 2.4, we obtain the Hyers-Ulam stability results for the additive \( \rho \)-functional equation (2.3) in complex Banach spaces.

**Corollary 2.5.** Let \( r > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \rightarrow Y \) be a mapping such that

\[ \| \rho \left( \sum_{j=1}^{k} f(x_j) \right) \| \leq \theta \frac{\| x \| \| y \|}{2} \]  

for all \( x, y \in G \). Then there exists a unique additive mapping \( h : X \rightarrow Y \) satisfying (2.5).

**Corollary 2.6.** Let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \rightarrow Y \) be a mapping satisfying (2.10). Then there exists a unique additive mapping \( h : X \rightarrow Y \) satisfying (2.8).

**Remark 2.7.** If \( \rho \) is a real number such that \(-1 < \rho < 1\) and \( Y \) is a real Banach space, then all the assertions in this section remain valid.

3. Additive \( \rho \)-functional inequality (0.2)

In this section, we investigate the additive \( \rho \)-functional inequality (0.2) in complex Banach spaces.

**Lemma 3.1.** A mapping \( f : G \rightarrow Y \) satisfies \( f(0) = 0 \) and

\[ \| k f \left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) \| \leq \rho \left( f \left( \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f(x_j) \right) \]  

for all \( x_1, x_2, \cdots, x_k \in G \) if and only if \( f : G \rightarrow Y \) is additive.

**Proof.** Assume that \( f : G \rightarrow Y \) satisfies (3.1).

Letting \( x_1 = x \) and \( x_2 = \cdots = x_k = 0 \) in (3.1), we get

\[ \| k f \left( \frac{x}{k} \right) - f(x) \| \leq 0 \]
and so
\[ f\left(\frac{x}{k}\right) = \frac{1}{k} f(x) \] (3.2)
for all \( x \in G \).

It follows from (3.1) and (3.2) that
\[
\| f\left( \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f(x_j) \| = \| k f\left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) \| \\
\leq |\rho| \| f\left( \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f(x_j) \|
\]
and so
\[
f\left( \sum_{j=1}^{k} x_j \right) = \sum_{j=1}^{k} f(x_j)
\]
for all \( x_1, x_2, \ldots, x_k \in G \). Hence \( f : G \to Y \) is additive.

The converse is obviously true. \( \square \)

**COROLLARY 3.2.** A mapping \( f : G \to Y \) satisfies \( f(0) = 0 \) and
\[
k f\left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) = \rho \left( f\left( \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f(x_j) \right)
\] (3.3)
for all \( x_1, x_2, \ldots, x_k \in G \) if and only if \( f : G \to Y \) is additive.

The equation (3.3) is called an additive \( \rho \)-functional equation.

We prove the Hyers-Ulam stability of the additive \( \rho \)-functional inequality (3.1) in complex Banach spaces.

**THEOREM 3.3.** Let \( r > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be a mapping with \( f(0) = 0 \) such that
\[
\| k f\left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) \| \leq k^r \theta \| x \|^r
\] (3.4)
for all \( x_1, x_2, \ldots, x_k \in X \). Then there exists a unique additive mapping \( h : X \to Y \) such that
\[
\| f(x) - h(x) \| \leq \frac{k^r \theta}{k^r - k} \| x \|^r
\] (3.5)
for all \( x \in X \).
Proof. Letting $x_1 = x$ and $x_2 = \cdots = x_k = 0$ in (3.4), we get
\[
\left\| k f \left( \frac{x}{k} \right) - f(x) \right\| \leq \theta \|x\|^r
\] (3.6)
for all $x \in X$. So
\[
\left\| k^l f \left( \frac{x}{k^l} \right) - k^m f \left( \frac{x}{k^m} \right) \right\| \leq \sum_{j=l}^{m-1} \left\| k^j f \left( \frac{x}{k^j} \right) - k^{j+1} f \left( \frac{x}{k^{j+1}} \right) \right\|
\leq \sum_{j=l}^{m-1} \frac{k^j}{k^r} \theta \|x\|^r
\] (3.7)
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{k^m f(x/k^m)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{k^m f(x/k^m)\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by
\[
h(x) := \lim_{n \to \infty} k^n f \left( \frac{x}{k^n} \right)
\]
for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.7), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.3. □

Theorem 3.4. Let $r < 1$ and $\theta$ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and (3.4). Then there exists a unique additive mapping $h : X \rightarrow Y$ such that
\[
\left\| f(x) - h(x) \right\| \leq \frac{k^r \theta}{k-k^r} \|x\|^r
\] (3.8)
for all $x \in X$.

Proof. It follows from (3.6) that
\[
\left\| f(x) - \frac{1}{k} f(kx) \right\| \leq \frac{k^r \theta}{k} \|x\|^r
\]
for all $x \in X$. Hence
\[
\left\| \frac{1}{k^l} f(k^l x) - \frac{1}{k^m} f(k^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{k^j} f(k^j x) - \frac{1}{k^{j+1}} f(k^{j+1} x) \right\|
\leq \frac{k^r}{k} \sum_{j=l}^{m-1} \frac{k^j}{k^r} \theta \|x\|^r
\] (3.9)
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{\frac{1}{k^r} f(k^r x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{k^r} f(k^r x)\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by
\[
h(x) := \lim_{n \to \infty} \frac{1}{k^n} f(k^n x)
\]
for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.9), we get (3.8).

The rest of the proof is similar to the proof of Theorem 2.4. □

By the triangle inequality, we have

\[
\left\| kf \left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) \right\| - \left\| \rho \left( f \left( \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f(x_j) \right) \right\| \\
\leq \left\| kf \left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) - \rho \left( f \left( \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f(x_j) \right) \right\|.
\]

As corollaries of Theorems 3.3 and 3.4, we obtain the Hyers-Ulam stability results for the additive \( \rho \)-functional equation (3.3) in complex Banach spaces.

**Corollary 3.5.** Let \( r > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be a mapping with \( f(0) = 0 \) such that

\[
\left\| kf \left( \frac{\sum_{j=1}^{k} x_j}{k} \right) - \sum_{j=1}^{k} f(x_j) - \rho \left( f \left( \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f(x_j) \right) \right\| \leq \theta \sum_{j=1}^{k} \| x_j \|^r \quad (3.10)
\]

for all \( x_1, x_2, \ldots, x_k \in X \). Then there exists a unique additive mapping \( h : X \to Y \) satisfying (3.5).

**Corollary 3.6.** Let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and (3.10). Then there exists a unique additive mapping \( h : X \to Y \) satisfying (3.8).

**Remark 3.7.** If \( \rho \) is a real number such that \(-1 < \rho < 1\) and \( Y \) is a real Banach space, then all the assertions in this section remain valid.

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**References**


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