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GENERALIZATION OF HUA'S INEQUALITIES AND AN APPLICATION

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Abstract. We generalize a new Hua's inequality and apply it to proof the boundedness of composition operator C_ϕ from p-Bloch space $\beta^p(Y_I(N,m,n;K))$ to q-Bloch space $\beta^q(Y_I(N,m,n;K))$ in this paper, where $Y_I(N,m,n;K)$ denotes the first Cartan-Hartogs domain, and $p \geqslant 0$, $q \geqslant 0$.

1. Introduction

In 1955, Hua Loo-Keng discovered and proved an inequality [1] in the study of the functions of several complex variables: If Z_1, Z_2 are $n\times n$ complex matrices, and $I-Z_1\bar{Z_1}^T$, $I-Z_2\bar{Z_2}^T$ are both Hermitian positive definite matrices, then

$$\det(I-Z_1\bar{Z_1}^T)\det(I-Z_2\bar{Z_2}^T)\leqslant|\det(I-Z_1\bar{Z_2}^T)|^2,$$

and equality holds if and only if $Z_1 = Z_2$.

In 2007, Yang Zhongpeng generalized a new Hua's inequality [2,3] from an application of a matrix identity:

$$\begin{aligned}&\det(I-Z_1\bar{Z_1}^T)\det(I-Z_2\bar{Z_2}^T)+|\det(Z_1-Z_2)|^2+(2^n-2)|\det(Z_1-Z_2)|\\&\times[\det(I-Z_1\bar{Z_1}^T)\det(I-Z_2\bar{Z_2}^T)]^{\frac{1}{2}}\leqslant|\det(I-Z_1\bar{Z_2}^T)|^2\leqslant\det(I+Z_1\bar{Z_1}^T)\\&\times\det(I+Z_2\bar{Z_2}^T)+(2^{2n-1}-2^{n+1}+1)|\det(Z_1+Z_2)|-(2^n-2)|\det(Z_1+Z_2)|\\&\times[(2^{2(n-1)}-2^n)|\det(Z_1+Z_2)|^2+\det(I+Z_1\bar{Z_1}^T)\det(I+Z_2\bar{Z_2}^T)]^{\frac{1}{2}}.\end{aligned}$$

As is well known that all irreducible bounded symmetric domains were divided into six types by E. Cartan in 1930s. The first four types of irreducible domains are called the classical bounded symmetric domains, the rest two types called exceptional domains consisting of two domains (a 16 and 27 dimensional domain). In 1998, Professor Yin Weiping and G. Roos introduced four kinds of domains corresponding with

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the classical bounded symmetric domains, called this four kinds of domains as Cartan-Hartogs domains. Mainly, we will study the first Cartan-Hartogs domain denoted by $Y_I(N,m,n;K)$ in this paper, which can be expressed in the following:

$$Y_I(N,m,n;K) := \{W \in C^N,\ Z \in R_I(m,n) : |W|^{2K} < \det(I_m - Z\bar{Z}^T)\}, \quad K > 0,$$

where $R_I(m,n)$ denotes

$$R_I(m,n) = \Big\{Z : Z \in C^{m\times n},\ I_m - Z\bar{Z}^T > 0\Big\}.$$

We will denote $Y_I(N,m,n;K)$ by Y_I if no ambiguity can arise.

Let $\phi = (\phi_{ij})_{mn+1}$ be a holomorphic self-map of Y_I. The class of all holomorphic functions on domain Y_I will be denoted by $H(Y_I)$ (for simplicity, set $N = 1$), the composition operator C_ϕ is defined by

$$(C_\phi f)(Z,W) = f(\phi(Z,W))$$

for all $(Z,W) \in Y_I$ and $f \in H(Y_I)$.

Following Timoney [6], we say that $f \in H(Y_I)$ is in the p-Bloch space $\beta^p(Y_I)$ if

$$\|f\|_{\beta^p(Y_I)} = |f(0,0)| + \sup_{(Z,W)\in Y_I} [\det(I_m - Z\bar{Z}^T) - |W|^{2K}]^p |\nabla f(Z,W)| < +\infty,$$

where

$$\nabla f(Z,W) = \Big(\frac{\partial f(Z,W)}{\partial z_{11}}, \frac{\partial f(Z,W)}{\partial z_{12}}, \cdots, \frac{\partial f(Z,W)}{\partial z_{mn}}, \frac{\partial f(Z,W)}{\partial W}\Big),$$

and

$$|\nabla f(Z,W)|^2 = \sum_{\substack{1\leqslant\alpha\leqslant m\\ 1\leqslant\beta\leqslant n}} \Big|\frac{\partial f(Z,W)}{\partial z_{\alpha\beta}}\Big|^2 + \Big|\frac{\partial f(Z,W)}{\partial W}\Big|^2.$$

It is clear that $\beta^p(Y_I)$ is a set of constant functions in Y_I when $p < 0$, so we assume that $p \geqslant 0$.

In the beginning, Madigan and Matheson $[4-5]$ studied the composition operators in the Bloch space and Lipschitz space of the unit disk D, and proved that C_ϕ is always bounded on $\beta(D)$. More recently, Zhou et al $[7-11]$ obtainted some sufficient and necessary conditions for C_ϕ which are bounded and compact on the function spaces on the unit disk D, the polydiscs and the unit ball. We will discuss the boundedness of composition operator C_ϕ from p-Bloch space $\beta^p(Y_I)$ to q-Bloch space $\beta^q(Y_I)$, where $p \geqslant 0$ and $q \geqslant 0$.

In this paper, we generalize a new Hua's inequality:

Let $Z_1, Z_2 \in C^{m\times n}$, $W_1, W_2 \in C^N$, $K > 0$. If $I_m - Z_1\bar{Z_1}^T > 0$, $I_m - Z_2\bar{Z_2}^T > 0$, $|W_1|^{2K} < \det(I_m - Z_1\bar{Z_1}^T)$ and $|W_2|^{2K} < \det(I_m - Z_2\bar{Z_2}^T)$, then

$$[\det(I_m - Z_1\bar{Z_1}^T) - |W_1|^{2K}][\det(I_m - Z_2\bar{Z_2}^T) - |W_2|^{2K}]$$

$$\leqslant |\det(I_m - Z_1\bar{Z_2}^T) - (W_1\bar{W_2}^T)^K|^2.$$

As an application,we also discuss the boundedness of composition operator C_ϕ from p-Bloch space $\beta^p(Y_I)$ to q-Bloch space $\beta^q(Y_I)$ by using this inequality, where $p \geqslant 0$, $q \geqslant 0$.

2. Some Lemmas

In order to prove the theorems, we need the following lemmas.

LEMMA 2.1. [12] *For any $A \in C^{m\times n}$, $B \in C^{n\times m}$, we have*

$$\det(I_m + AB) = \det(I_n + BA).$$

LEMMA 2.2. [12] *Let $\Lambda_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$, $(\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0)$ and $\Lambda_2 = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_m)$, $(\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m \geqslant 0)$. If λ_j, μ_k satisfy $\lambda_j \cdot \mu_k < 1$ $(j,k = 1,2,\cdots,m)$, then there exists an arrange matrix P which is $m\times m$, such that*

$$\inf_{U\bar{U}^T = I, V\bar{V}^T = I} \left|\det(I - \Lambda_1 U \Lambda_2 \bar{U}^T V)\right| = \left|\det(I - \Lambda_1 P \Lambda_2 P^T)\right|.$$

LEMMA 2.3. [13] *For any two $m\times n$ matrices A and B, we have*

$$\left|\mathrm{tr}(\bar{A}^T B)\right|^2 \leqslant \mathrm{tr}(\bar{A}^T A)\,\mathrm{tr}(\bar{B}^T B),$$

and the "$=$" holds if and only if $A = CB$, where C is a constant.

LEMMA 2.4. *If*

$$\frac{[\det(I - Z\bar{Z}^T) - |W|^{2K}]^q}{[\det(I - Z_2\bar{Z_2}^T) - |W_2|^{2K}]^p}|\Phi'(Z,W)| = O(1) \tag{2.1}$$

$$((Z,W) \in Y_I, (Z_2,W_2) \to \partial Y_I),$$

then

$$\frac{[\det(I - Z\bar{Z}^T) - |W|^{2K}]^q}{[\det(I - Z_2\bar{Z_2}^T) - |W_2|^{2K}]^p}|\Phi'(Z,W)| < +\infty$$

for all $(Z,W) \in Y_I$, where $(Z_2,W_2) = \Phi(Z,W)$ and $\Phi = (\phi_{11}, \phi_{12}, \cdots, \phi_{mn}, \phi_{mn+1})$.

Proof. From the condition (2.1), we know that there exists a constant $\delta > 0$ when $\mathrm{dist}((Z_2,W_2), \partial Y_I) < \delta$, we have

$$\frac{[\det(I - Z\bar{Z}^T) - |W|^{2K}]^q}{[\det(I - Z_2\bar{Z_2}^T) - |W_2|^{2K}]^p}|\Phi'(Z,W)| \leqslant C_1,$$

where C_1 is a positive number. When $\operatorname{dist}((Z_2,W_2),\partial Y_I) \geqslant \delta$, we set

$$E_\delta = \{(Z_2,W_2) \in Y_I : \operatorname{dist}((Z_2,W_2),\partial Y_I) \geqslant \delta\}.$$

It is easy to prove that E_δ is a compact of Y_I. Thus there exists a constant $M \in (0,1)$, such that

$$M \leqslant \det(I - Z_2\bar{Z_2}^T) - |W_2|^{2K} < 1,$$

we can get

$$\frac{1}{\det(I - Z_2\bar{Z_2}^T) - |W_2|^{2K}} \leqslant \frac{1}{M} < +\infty,$$

Set $C = \max\{C_1, \frac{1}{M}\}$, we have

$$\frac{[\det(I - Z\bar{Z}^T) - |W|^{2K}]^q}{[\det(I - Z_2\bar{Z_2}^T) - |W_2|^{2K}]^p}|\Phi'(Z,W)| \leqslant C$$

for all $(Z,W) \in Y_I$. The proof is completed. □

LEMMA 2.5. *Let $Z,U \in R_I(m,n)$, then there exists a constant $C > 0$, such that*

$$\left|\det(I_m - Z\bar{U}^T)\right|\Big\{\sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}\left|\operatorname{tr}[(I_m - Z\bar{U}^T)^{-1}I_{\alpha\beta}\bar{U}^T]\right|^2\Big\}^{\frac{1}{2}} \leqslant C,$$

where $I_{\alpha\beta}$ is an $m\times n$ matrix whose element of the αth row and the βth column is 1, and the other elements are 0.

Proof. By Lemma 2.3, we have

$$\begin{aligned}
&\left|\operatorname{tr}[(I_m - Z\bar{U}^T)^{-1}I_{\alpha\beta}\bar{U}^T]\right| \\
\leqslant &\left|\operatorname{tr}[\overline{(I_m - Z\bar{U}^T)^{-1}}^T(I_m - Z\bar{U}^T)^{-1}]\operatorname{tr}[\overline{(I_{\alpha\beta}\bar{U}^T)}^T(I_{\alpha\beta}\bar{U}^T)]\right|^{\frac{1}{2}} \\
= &\left|\operatorname{tr}[(I_m - Z\bar{U}^T)\overline{(I_m - Z\bar{U}^T)}^T]^{-1}\operatorname{tr}[(UI_{\alpha\beta}^T)(I_{\alpha\beta}\bar{U}^T)]\right|^{\frac{1}{2}}. \qquad (2.2)
\end{aligned}$$

Since

$$\operatorname{tr}[(UI_{\alpha\beta}^T)(I_{\alpha\beta}\bar{U}^T)] = \sum_{j=1}^m |u_{j\beta}|^2,$$

thus

$$\sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}\left|\operatorname{tr}[(UI_{\alpha\beta}^T)(I_{\alpha\beta}\bar{U}^T)]\right| = m \parallel U \parallel^2, \qquad (2.3)$$

where $U = (u_{ij})_{m\times n}$,

$$\parallel U \parallel^2 = \sum_{\substack{1\leqslant i\leqslant m\\1\leqslant j\leqslant n}} |u_{ij}|^2.$$

By (2.2) and (2.3), we have

$$
\begin{aligned}
&\left|\det\left(I_m - Z\overline{U}^T\right)\right|\left\{\sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}\left|\operatorname{tr}\left[\left(I_m - Z\overline{U}^T\right)^{-1} I_{\alpha\beta}\overline{U}^T\right]\right|^2\right\}^{\frac{1}{2}}\\
\leqslant&\left|\det\left(I_m - Z\overline{U}^T\right)\right|\left\{\sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}\left|\operatorname{tr}[(I_m - Z\bar{U}^T)\overline{(I_m - Z\bar{U}^T)}^T]^{-1}\operatorname{tr}[\overline{(UI_{\alpha\beta})}^T(I_{\alpha\beta}\bar{U}^T)]\right|\right\}^{\frac{1}{2}}\\
\leqslant&\left|\det\left(I_m - Z\overline{U}^T\right)\right|\left\{\sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}\left|\operatorname{tr}[(I_m - Z\bar{U}^T)\overline{(I_m - Z\bar{U}^T)}^T]^{-1}\right|\right\}^{\frac{1}{2}}\left\{m\parallel U\parallel^2\right\}^{\frac{1}{2}}.
\end{aligned}
\tag{2.4}
$$

Set

$$
I_m - Z\overline{U}^T = P\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_m)Q \tag{2.5}
$$

where P and Q are two $m\times m$ unitary matrices.

We have

$$
\begin{aligned}
(I_m - Z\bar{U}^T)\overline{(I_m - Z\bar{U}^T)}^T &= P\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_m)Q\bar{Q}^T\operatorname{diag}(\bar{\lambda_1},\bar{\lambda_2},\ldots,\bar{\lambda_m})\bar{P}^T\\
&= P\operatorname{diag}(|\lambda_1|^2,|\lambda_2|^2,\ldots,|\lambda_m|^2)\bar{P}^T\\
&= P\operatorname{diag}(|\lambda_1|^2,|\lambda_2|^2,\ldots,|\lambda_m|^2)P^{-1}
\end{aligned}
$$

thus

$$
\begin{aligned}
&|\operatorname{tr}[(I_m - Z\bar{U}^T)\overline{(I_m - Z\bar{U}^T)}^T]^{-1}|\\
=&\left|\operatorname{tr}[P^{-1}\operatorname{diag}(|\lambda_1|^2,|\lambda_2|^2,\ldots,|\lambda_m|^2)^{-1}P]\right|\\
=&\sum_{i=1}^{m}\frac{1}{|\lambda_i|^2}.
\end{aligned}
$$

Since

$$
\begin{aligned}
|\det(I_m - Z\overline{U}^T)| &= |\det P\det(\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_m))\det Q|\\
&= |\prod_{i=1}^{m}\lambda_i| = \prod_{i=1}^{m}|\lambda_i|,
\end{aligned}
$$

thus

$$
\begin{aligned}
&\left|\det\left(I_m - Z\bar{U}^T\right)\right|\left|\operatorname{tr}[(I_m - Z\bar{U}^T)\overline{(I_m - Z\bar{U}^T)}^T]^{-1}\right|^{\frac{1}{2}}\\
=&\left\{\left|\det\left(I_m - Z\bar{U}^T\right)\right|^2\left|\operatorname{tr}[(I_m - Z\bar{U}^T)\overline{(I_m - Z\bar{U}^T)}^T]^{-1}\right|\right\}^{\frac{1}{2}}
\end{aligned}
$$

$$= \left\{ \prod_{i=1}^{m} |\lambda_i|^2 \cdot \sum_{i=1}^{m} \frac{1}{|\lambda_i|^2} \right\}^{\frac{1}{2}} = \left\{ \sum_{i=1}^{m} \left(\prod_{\substack{j=1 \\ j \neq i}}^{m} |\lambda_i|^2 \right) \right\}^{\frac{1}{2}}. \tag{2.6}$$

Set

$$A = I_m - Z\overline{U}^T = (a_{ij})_{m \times m}; \quad Z = (z_{ij})_{m \times n}.$$

Since $Z, U \in R_I(m,n)$, we have

$$I_m - Z\overline{Z}^T > 0, \qquad I_m - U\overline{U}^T > 0,$$

furthermore

$$1 - \sum_{i=1}^{n} |z_{ki}|^2 > 0, \quad 1 - \sum_{i=1}^{n} |u_{ki}|^2 > 0 \quad (k = 1, 2, \cdots, m),$$

so we have

$$|z_{ki}| < 1, \quad |u_{ki}| < 1, \quad (1 \leqslant k \leqslant m, \ 1 \leqslant i \leqslant n). \tag{2.7}$$

Denote

$$a_{\alpha\beta} = \begin{cases} -\sum_{k=1}^{n} z_{\alpha k} \overline{u_{\beta k}}, & (\alpha \neq \beta) \\ 1 - \sum_{k=1}^{n} z_{\alpha k} \overline{u_{\beta k}}, & (\alpha = \beta) \end{cases}$$

So we have

$$\begin{aligned} \operatorname{tr}(A\overline{A}^T) &= \operatorname{tr}(\operatorname{diag}(|\lambda_1|^2, |\lambda_2|^2, \ldots, |\lambda_m|^2)) = \sum_{i=1}^{m} \lambda_i^2 = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} |a_{\alpha\beta}|^2 \\ &= \sum_{\alpha=\beta=1}^{m} \left| a_{\alpha\beta} \right|^2 + \sum_{\alpha \neq \beta} \left| a_{\alpha\beta} \right|^2 \\ &= \sum_{\alpha=\beta=1}^{m} \left| 1 - \sum_{k=1}^{n} z_{\alpha k} \overline{u_{\beta k}} \right|^2 + \sum_{\alpha \neq \beta} \left| - \sum_{k=1}^{n} z_{\alpha k} \overline{u_{\beta k}} \right|^2 \\ &\leqslant \sum_{\alpha=\beta=1}^{m} \left(1 + \sum_{k=1}^{n} |z_{\alpha k}||u_{\beta k}| \right)^2 + \sum_{\alpha \neq \beta} \left(\sum_{k=1}^{n} |z_{\alpha k}||u_{\beta k}| \right)^2 \\ &\leqslant (1+n)^2 m + n^2(m^2 - m) = m^2 n^2 + 2mn + m. \end{aligned} \tag{2.8}$$

From (2.8), we can get

$$\lambda_i^2 \leqslant n^2 m^2 + 2nm + m, \quad i = 1, 2, \cdots, m. \tag{2.9}$$

By (2.4), (2.6), (2.7) and (2.9) then there exists a constant $C > 0$, such that

$$\begin{aligned}
&\left|\det\left(I_m - Z\bar{U}^T\right)\right|\left\{\sum_{\substack{1\leqslant\alpha\leqslant m\\ 1\leqslant\beta\leqslant n}}\left|\operatorname{tr}\left[\left(I_m - Z\overline{U}^T\right)^{-1} I_{\alpha\beta}\overline{U}^T\right]\right|^2\right\}^{\frac{1}{2}}\\
&\leqslant \left\{\left|\det\left(I_m - Z\bar{U}^T\right)\right|^2 \left|\operatorname{tr}[(I_m - Z\bar{U}^T)\overline{(I_m - Z\bar{U}^T)}^T]^{-1}\right|\right\}^{\frac{1}{2}} \{m \parallel U \parallel^2\}^{\frac{1}{2}}\\
&\leqslant \{(m^2n^2 + 2mn + m)^{m-1}\cdot m\}^{\frac{1}{2}} mn^{\frac{1}{2}} = C.
\end{aligned}$$

The proof is completed. $\square$

LEMMA 2.6. *Let K be a compact subset of Y_I, then there exists a constant $C(K) > 0$, such that*

$$\int_0^1 \frac{|\langle Z,W\rangle|}{[\det(I - t^2 Z\bar{Z}^T) - |tW|^{2K}]^p} dt < C$$

for all $(Z,W) \in K$.

Proof. Denote

$$E_\delta := \{(Z,W) \in Y_I : \det(I - Z\overline{Z}^T) - |W|^{2K} \geqslant \delta\}, \quad \delta \in (0,1).$$

For any compact $K \subset Y_I$, there exists a constant $\delta \in (0,1)$, such that $K \subset E_\delta$. For any $(Z,W) \in K$, $t \in [0,1]$, we have

$$|tW|^{2K} \leqslant |W|^{2K} \leqslant \det(I - Z\overline{Z}^T) \leqslant \det(I - t^2 Z\overline{Z}^T).$$

Furthermore

$$\det(I - t^2 Z\bar{Z}^T) - |tW|^{2K} \geqslant \det(I - Z\bar{Z}^T) - |W|^{2K} \geqslant \delta > 0.$$

Thus

$$0 < \frac{1}{\det(I - t^2 Z\bar{Z}^T) - |tW|^{2K}} \leqslant \frac{1}{\det(I - Z\bar{Z}^T) - |W|^{2k}} \leqslant \frac{1}{\delta}.$$

So we have

$$\begin{aligned}
&\int_0^1 \frac{|\langle Z,W\rangle|}{[\det(I - t^2 Z\bar{Z}^T) - |tW|^{2K}]^p} dt\\
&\leqslant \int_0^1 \frac{|\langle Z,W\rangle|}{[\det(I - Z\bar{Z}^T) - |W|^{2K}]^p} dt\\
&\leqslant \frac{1}{\delta^p}\int_0^1 |\langle Z,W\rangle| dt.
\end{aligned}$$

Since $Z \in R_I(m,n)$, then there exist an $m\times m$ unitary matrix U and an $n\times n$ unitray matrix V, such that

$$Z = U \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m & 0 & \cdots & 0 \end{pmatrix} V$$

where $1 > \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0$, thus

$$Z\overline{Z}^T = U\,\mathrm{diag}(\lambda_1^2, \lambda_2^2, \ldots, \lambda_m^2))\overline{U}^T.$$

Since $|Z|^2 = \mathrm{tr}(Z\overline{Z}^T)$, we have

$$|Z|^2 = \sum_{i=1}^{m} \lambda_i^2 \leqslant m\lambda_1^2 \leqslant m. \tag{2.10}$$

Since $(Z,W) \in Y_I$, thus

$$|W|^{2k} < \det(I - Z\overline{Z}^T) < 1.$$

It is clear that

$$|W| < 1, \tag{2.11}$$

By (2.10) and (2.11), we have

$$|(Z,W)| < \sqrt{m+1},$$

furthermore we have

$$\begin{aligned} &\int_0^1 \frac{|\langle Z,W\rangle|}{[\det(I - t^2 Z\overline{Z}^T) - |tW|^{2K}]^p} dt \\ &< \frac{1}{\delta^p} \int_0^1 \sqrt{m+1}dt \\ &= \frac{\sqrt{m+1}}{\delta^p} = C. \end{aligned}$$

The proof is completed. □

LEMMA 2.7. *Let $f \in \beta^p(Y_I)$ and K be a compact subset of Y_I. If $(Z,W) \in Y_I$, then there exists a constant $C > 0$, such that*

$$|f(Z,W)| \leqslant C\|f\|_{\beta^p}.$$

Proof. Since

$$\begin{aligned} \frac{\partial f(tZ,tW)}{\partial t} &= \frac{\partial f(tZ,tW)}{\partial V_{11}} z_{11} + \frac{\partial f(tZ,tW)}{\partial V_{12}} z_{12} + \cdots + \frac{\partial f(tZ,tW)}{\partial V_{mn}} z_{mn} \\ &+ \frac{\partial f(tZ,tW)}{\partial V_{mn+1}} W = \langle \nabla f(tZ,tW), (\overline{Z}, \overline{W})\rangle. \end{aligned}$$

where $Z = (z_{11}, \cdots, z_{1m}, z_{21}, \cdots, z_{2m}, \cdots, z_{mn})$. By Lemma 2.6, there exists a constant $C' > 0$, such that

$$
\begin{aligned}
&|f(Z,W)| = |f(0,0) + \int_0^1 \langle \nabla f(tZ,tW), (\bar{Z},\bar{W})\rangle dt| \\
&\leqslant |f(0,0)| + |\int_0^1 \langle \nabla f(tZ,tW), (\bar{Z},\bar{W})\rangle dt| \\
&\leqslant |f(0,0)| + \int_0^1 |\nabla f(tZ,tW)||(\bar{Z},\bar{W})| dt \\
&= |f(0,0)| + \int_0^1 \frac{|(\bar{Z},\bar{W})|}{[\det(I - t^2 Z\bar{Z}^T) - |tW|^{2k}]^p}[\det(I - t^2 Z\bar{Z}^T) - |tW|^{2K}]^p |\nabla f(tZ,tW)| dt \\
&\leqslant |f(0,0)| + \|f\|_{\beta^p} \int_0^1 \frac{|(Z,W)|}{[\det(I - t^2 Z\bar{Z}^T) - |tW|^{2K}]^p} dt \\
&\leqslant \|f\|_{\beta^p} + C'\|f\|_{\beta^p} \\
&= (C'+1)\|f\|_{\beta^p} \\
&= C\|f\|_{\beta^p}.
\end{aligned}
$$

The proof is completed. □

3. Generalization of Hua's inequalities

THEOREM 1. *Let $Z_1, Z_2 \in C^{m\times n}$, $W_1, W_2 \in C^N$, $K > 0$. If $I_m - Z_1\bar{Z_1}^T > 0$, $I_m - Z_2\bar{Z_2}^T > 0$, $|W_1|^{2K} < \det(I_m - Z_1\bar{Z_1}^T)$ and $|W_2|^{2K} < \det(I_m - Z_2\bar{Z_2}^T)$, then*

$$
\begin{aligned}
&[\det(I_m - Z_1\bar{Z_1}^T) - |W_1|^{2K}][\det(I_m - Z_2\bar{Z_2}^T) - |W_2|^{2K}] \\
&\leqslant |\det(I_m - Z_1\bar{Z_2}^T) - (W_1\overline{W_2}^T)^K|^2. \qquad (3.1)
\end{aligned}
$$

Proof. It is easy to prove that $I_m - Z_1\bar{Z_1}^T$ and $I_m - Z_2\bar{Z_2}^T$ are both Hermitian matrices. By Lemma 2.1, we know

$$\det(I_m + AB) = \det(I_n + BA),$$

where $A \in C^{m\times n}$, $B \in C^{n\times m}$. So we suppose $m \leqslant n$. In fact, we need to proof the situation of $m = n$. Since when $m < n$, there exist an $n\times n$ unitary matrices U, such that

$$Z_1 = (M_1^{(m)}, 0)U, \qquad Z_2 = (N_1^{(m)}, N_2)U.$$

So

$$\det(I - Z_2\bar{Z_2}^T) = \det(I - N_1\bar{N_1}^T - N_2\bar{N_2}^T) \leqslant \det(I - N_1\bar{N_1}^T),$$

furthermore

$$\begin{aligned}
&[\det(I-Z_1\bar{Z_1}^T)-|W_1|^{2K}][\det(I-Z_2\bar{Z_2}^T)-|W_2|^{2K}]\\
\leqslant &[\det(I-M_1\bar{M_1}^T)-|W_1|^{2K}][\det(I-N_1\bar{N_1}^T)-|W_2|^{2K}]\\
\leqslant &[\det(I-M_1\bar{N_1}^T)-(W_1\overline{W_2}^T)^K]^2\\
= &[\det(I-Z_1\bar{Z_2}^T)-(W_1\overline{W_2}^T)^K]^2.
\end{aligned}$$

It is well known that every $m\times m$ matrix A may be written $A=U(\sum_{k=1}^{m}\lambda_k E_{kk})\bar{U_0}^T$, where U and U_0 is two $m\times m$ unitary matrices, $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_m\geqslant 0$. E_{kk} is an $m\times m$ matrix whose element of the kth row and kth column is 1, and the other elements are 0. So there exist $m\times m$ unitary matrices U, U_0, V and V_0, such that

$$Z_1=U\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_m))\bar{U_0}^T\triangleq U\Lambda_1\bar{U_0}^T \tag{3.2}$$

and

$$Z_2=V\operatorname{diag}(\mu_1,\mu_2,\ldots,\mu_m))\bar{V_0}^T\triangleq V\Lambda_2\bar{V_0}^T \tag{3.3}$$

where

$$\begin{aligned}
&1\geqslant\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_m\geqslant 0,\\
&1\geqslant\mu_1\geqslant\mu_2\geqslant\cdots\geqslant\mu_m\geqslant 0.
\end{aligned}$$

Since W_1,W_2 are two complex numbers, then there exist $\theta_1,\theta_2\in[0,2\pi]$, such that

$$W_1=e^{i\theta_1}\xi_1, W_2=e^{i\theta_2}\xi_2. \tag{3.4}$$

where $\xi_1=|W_1|$, $\xi_2=|W_2|$.

Combine (3.2), (3.3) and (3.4), we can get the left side of (3.1).

$$\text{left}=[(1-\lambda_1^2)(1-\lambda_2^2)\cdots(1-\lambda_m^2)-\xi_1^{2k}][(1-\mu_1^2)(1-\mu_2^2)\cdots(1-\mu_m^2)-\xi_2^{2k}].$$

Set $\xi_1=\xi_1^k$, $\xi_2=\xi_2^k$, then $(3.1)\Longleftrightarrow$

$$\begin{aligned}
&[(1-\lambda_1^2)(1-\lambda_2^2)\cdots(1-\lambda_m^2)-\xi_1^2][(1-\mu_1^2)(1-\mu_2^2)\cdots(1-\mu_m^2)-\xi_2^2]\\
\leqslant &|\det(I-Z_1\bar{Z_2}^T)-\xi_1\xi_2e^{i(\theta_1+\theta_2)}|^2\\
= &|e^{i\theta}\det(I-U\Lambda_1\bar{U}^T V\Lambda_2\bar{V}^T)-\xi_1\xi_2|^2.
\end{aligned}$$

Since

$$\begin{aligned}
&\det(I-U\Lambda_1\bar{U_0}^T V\Lambda_2\bar{V_0}^T)\\
= &\det\left(U(I-\Lambda_1\bar{U_0}^T V\Lambda_2\bar{V_0}^T U)\bar{U}^T\right)\\
= &\det(I-\Lambda_1\bar{U_0}^T V\Lambda_2\bar{V_0}^T U)\\
= &\det(I-\Lambda_1\bar{U_0}^T V\Lambda_2\bar{V}^T U_0\bar{U_0}^T V\bar{V_0}^T U)\\
= &\det(I-\Lambda_1 U_1\Lambda_2\bar{U_1}^T V_1).
\end{aligned}$$

By Lemma 2.2, there exists an $m \times m$ arrange matrix P, such that

$$\inf_{U_1\bar{U_1}^T=I, V_1\bar{V_1}^T=I} \left|\det(I-\Lambda_1 U_1 \Lambda_2 \bar{U_1}^T V_1)\right| = \left|\det(I-\Lambda_1 P \Lambda_2 P^T)\right|.$$

Since

$$\begin{aligned}
&\inf\left|e^{i\theta}\det(I-Z_1\overline{Z_2}^T)-\xi_1\xi_2\right|^2 \\
\geqslant &\inf_{U_1\bar{U_1}^T=I, V_1\bar{V_1}^T=I}\left|\det(I-\Lambda_1 U_1\Lambda_2\bar{U_1}^T V_1)-\xi_1\xi_2\right|^2 \\
= &\left[\det(I-\Lambda_1 P\Lambda_2 P^T)-\xi_1\xi_2\right]^2 \\
= &\left[\prod_{i=1}^m (1-\lambda_i\nu_i)-\xi_1\xi_2\right]^2
\end{aligned}$$

where $\nu_1,\nu_2,\ldots,\nu_m$ is the rearrangement of $\mu_1,\mu_2,\ldots,\mu_m$.

To prove the inequality (3.1), we just only prove the following inequality.

$$\left[\prod_{i=1}^m(1-\lambda_i^2)-\xi_1^2\right]\left[\prod_{i=1}^m(1-\mu_i^2)-\xi_2^2\right] \leqslant \left[\prod_{i=1}^m(1-\lambda_i\mu_i)-\xi_1\xi_2\right]^2,$$

where

$$\xi_1^2 < \prod_{i=1}^m(1-\lambda_i^2), \xi_2^2 < \prod_{i=1}^m(1-\mu_i^2),$$

further more, we just only prove the following inequality:

$$\prod_{i=1}^m(1-\lambda_i\mu_i) \geqslant \sqrt{\prod_{i=1}^m(1-\lambda_i^2)-\xi_1^2}\sqrt{\prod_{i=1}^m(1-\mu_i^2)-\xi_2^2}+\xi_1\xi_2, \tag{3.5}$$

It is easy to prove the situation when $m=1$. If the inequality of (3.5) is true when $k=m$, then when $k=m+1$, we have

$$\begin{aligned}
&\prod_{i=1}^{m+1}(1-\lambda_i\mu_i) = (1-\lambda_{m+1}\mu_{m+1})\prod_{i=1}^m(1-\lambda_i\mu_i) \\
\geqslant &(1-\lambda_{m+1}\mu_{m+1})\left[\sqrt{\prod_{i=1}^m(1-\lambda_i^2)-\xi_1^2}\sqrt{\prod_{i=1}^m(1-\mu_i^2)-\xi_2^2}+\xi_1\xi_2\right].
\end{aligned}$$

In the following, we prove this following inequality:

$$\begin{aligned}
&(1-\lambda_{m+1}\mu_{m+1})\left[\sqrt{\prod_{i=1}^m(1-\lambda_i^2)-\xi_1^2}\sqrt{\prod_{i=1}^m(1-\mu_i^2)-\xi_2^2}+\xi_1\xi_2\right] \\
\geqslant &\sqrt{\prod_{i=1}^{m+1}(1-\lambda_i^2)-\xi_1^2}\sqrt{\prod_{i=1}^{m+1}(1-\mu_i^2)-\xi_2^2}+\xi_1\xi_2
\end{aligned} \tag{3.6}$$

If $\xi_1 = 0$ or $\xi_2 = 0$, it is easy to prove the inequality (3.6).

Else $\xi_1 \neq 0$ and $\xi_2 \neq 0$, set $\lambda = \lambda_{m+1}$, $\mu = \mu_{m+1}$, $A = \dfrac{\prod_{i=1}^{m}(1-\lambda_i^2)}{\xi_1^2}$, $B = \dfrac{\prod_{i=1}^{m}(1-\mu_i^2)}{\xi_2^2}$. Then the inequality (3.6) is:

$$(1-\lambda\mu)[\sqrt{A-1}\sqrt{B-1}+1] \geqslant \sqrt{(1-\lambda^2)A-1}\sqrt{(1-\mu^2)B-1}+1.$$

$\Longleftrightarrow$

$$(1-\lambda\mu)\sqrt{A-1}\sqrt{B-1} \geqslant \lambda\mu+\sqrt{(1-\lambda^2)A-1}\sqrt{(1-\mu^2)B-1},$$

where

$$A > \frac{1}{1-\lambda^2}, \quad B > \frac{1}{1-\mu^2}, \quad \lambda,\mu \in [0,1).$$

$\Longleftrightarrow$

$$(1-\lambda\mu)\sqrt{\frac{C}{1-\lambda^2}-1}\sqrt{\frac{D}{1-\mu^2}-1} \geqslant \lambda\mu+\sqrt{C-1}\sqrt{D-1},$$

where

$$C = (1-\lambda^2)A > 1, \qquad D = (1-\mu^2)B > 1.$$

$\Longleftrightarrow$

$$(1-\lambda\mu)\sqrt{C-1+\lambda^2}\sqrt{D-1+\mu^2} \geqslant \sqrt{1-\lambda^2}\sqrt{1-\mu^2}(\sqrt{C-1}\sqrt{D-1}+\lambda\mu).$$

It is easy to prove

$$1-\lambda\mu \geqslant \sqrt{1-\lambda^2}\sqrt{1-\mu^2}.$$

In the following we will prove

$$\sqrt{C-1+\lambda^2}\sqrt{D-1+\mu^2} \geqslant (\sqrt{C-1}\sqrt{D-1}+\lambda\mu).$$

$\Longleftrightarrow$

$$\begin{aligned}&(C-1)(D-1)+\lambda^2(D-1)+\mu^2(C-1)+\lambda^2\mu^2.\\ &\geqslant (C-1)(D-1)+2\lambda\mu\sqrt{C-1}\sqrt{D-1}+\lambda^2\mu^2.\end{aligned}$$

$\Longleftrightarrow$

$$\lambda^2(D-1)+\mu^2(C-1) \geqslant 2\lambda\mu\sqrt{C-1}\sqrt{D-1}.$$

This ends the proof. □

4. An application of Hua's inequalities

THEOREM 2. (boundedness) *If*

$$\frac{[\det(I-Z\overline{Z}^T)-|W|^{2K}]^q}{[\det(I-Z_2\overline{Z_2}^T)-|W_2|^{2K}]^p}|\phi'(Z,W)|=O(1), \tag{4.1}$$

$$((Z,W)\in Y_I,(Z_2,W_2)\to\partial Y_I)$$

then $C_\phi:\beta^p(Y_I)\longrightarrow\beta^q(Y_I)$ *is bounded.*

Conversely, if $C_\phi:\beta^p(Y_I)\longrightarrow\beta^q(Y_I)$ *is bounded, then*

$$\frac{[\det(I-Z\bar{Z}^T)-|W|^{2K}]^q G(Z,W)}{[\det(I-Z_2\bar{Z_2}^T)-|W_2|^{2K}]^p}=O(1),$$

$$((Z,W)\in Y_I,(Z_2,W_2)\to\partial Y_I) \tag{4.2}$$

where $(Z_2,W_2)=\phi(Z,W)$, $(Z,W)\in Y_I$, $\phi=(\phi_{11},\phi_{12},\cdots,\phi_{mn},\phi_{mn+1})$, $K>1$, *and*

$$|\phi'(Z,W)|^2=\sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\left|\frac{\partial\phi_{kl}}{\partial z_{\alpha\beta}}\right|^2+\sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}\left|\frac{\partial\phi_{mn+1}}{\partial z_{\alpha\beta}}\right|^2$$
$$+\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\left|\frac{\partial\phi_{kl}}{\partial W}\right|^2+\left|\frac{\partial\phi_{mn+1}}{\partial W}\right|^2,$$

and

$$G(Z,W)=\Bigg\{\sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}\Bigg|\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\det(I-Z_2\bar{Z_2}^T)\operatorname{tr}[(I-Z_2\bar{Z_2}^T)^{-1}I_{kl}\bar{Z_2}^T]\frac{\partial\phi_{kl}}{\partial z_{\alpha\beta}}$$
$$+K|W_2|^{2K-2}\bar{W_2}^T\frac{\partial\phi_{mn+1}}{\partial z_{\alpha\beta}}\Bigg|^2+\Bigg|\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\det(I-Z_2\bar{Z_2}^T)\operatorname{tr}[(I-Z_2\bar{Z_2}^T)^{-1}I_{kl}\bar{Z_2}^T]\frac{\partial\phi_{kl}}{\partial W}$$
$$+K|W_2|^{2K-2}\bar{W_2}^T\frac{\partial\phi_{mn+1}}{\partial W}\Bigg|^2\Bigg\}^{\frac{1}{2}},$$

Proof. It is well known that

$$\nabla f(Z,W)=\left(\frac{\partial f}{\partial z_{11}}(Z,W),\frac{\partial f}{\partial z_{12}}(Z,W),\cdots,\frac{\partial f}{\partial z_{mn}}(Z,W),\frac{\partial f}{\partial W}(Z,W)\right),$$

and

$$\nabla(f\circ\phi)(Z,W)$$
$$=\Bigg(\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\frac{\partial f}{\partial V_{kl}}(\phi(Z,W))\frac{\partial\phi_{kl}}{\partial z_{11}}(Z,W)+\frac{\partial f}{\partial V_{mn+1}}(\phi(Z,W))\frac{\partial\phi_{mn+1}}{\partial z_{11}}(Z,W),$$

$$\cdots, \sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}} \frac{\partial f}{\partial V_{kl}}(\phi(Z,W))\frac{\partial \phi_{kl}}{\partial z_{mn}}(Z,W)+\frac{\partial f}{\partial V_{mn+1}}(\phi(Z,W))\frac{\partial \phi_{mn+1}}{\partial z_{mn}}(Z,W),$$

$$\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}} \frac{\partial f}{\partial V_{kl}}(\phi(Z,W))\frac{\partial \phi_{kl}}{\partial W}(Z,W)+\frac{\partial f}{\partial V_{mn+1}}(\phi(Z,W))\frac{\partial \phi_{mn+1}}{\partial W}(Z,W)\Big). \quad (4.3)$$

By (4.3), we have

$$\begin{aligned}
&|(\nabla(f\circ\phi)(Z,W)|^2\\
&= \sum_{\substack{1\leqslant \alpha\leqslant m\\1\leqslant \beta\leqslant n}} |\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}} \frac{\partial f}{\partial V_{kl}}(\phi(Z,W))\frac{\partial \phi_{kl}}{\partial z_{\alpha\beta}}(Z,W)+\frac{\partial f}{\partial V_{mn+1}}(\phi(Z,W))\frac{\partial \phi_{mn+1}}{\partial z_{\alpha\beta}}(Z,W)|^2\\
&\quad +|\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}} \frac{\partial f}{\partial V_{kl}}(\phi(Z,W))\frac{\partial \phi_{kl}}{\partial W}(Z,W)+\frac{\partial f}{\partial V_{mn+1}}(\phi(Z,W))\frac{\partial \phi_{mn+1}}{\partial W}(Z,W)|^2\\
&\leqslant 2\sum_{\substack{1\leqslant \alpha\leqslant m\\1\leqslant \beta\leqslant n}} |\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}} \frac{\partial f}{\partial V_{kl}}(\phi(Z,W))\frac{\partial \phi_{kl}}{\partial z_{\alpha\beta}}(Z,W)|^2\\
&\quad +2\sum_{\substack{1\leqslant \alpha\leqslant m\\1\leqslant \beta\leqslant n}} |\frac{\partial f}{\partial V_{mn+1}}(\phi(Z,W))\frac{\partial \phi_{mn+1}}{\partial z_{\alpha\beta}}(Z,W)|^2\\
&\quad +2|\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}} \frac{\partial f}{\partial V_{kl}}(\phi(Z,W))\frac{\partial \phi_{kl}}{\partial W}(Z,W)|^2+2|\frac{\partial f}{\partial V_{mn+1}}(\phi(Z,W))\frac{\partial \phi_{mn+1}}{\partial W}(Z,W)|^2\\
&\leqslant 2[\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}} |\frac{\partial f}{\partial V_{kl}}(\phi(Z,W))|^2+|\frac{\partial f}{\partial V_{mn+1}}(\phi(Z,W))|^2][\sum_{\substack{1\leqslant \alpha\leqslant m\\1\leqslant \beta\leqslant n}}\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}} |\frac{\partial \phi_{kl}}{\partial z_{\alpha\beta}}(Z,W)|^2\\
&\quad +\sum_{\substack{1\leqslant \alpha\leqslant m\\1\leqslant \beta\leqslant n}} |\frac{\partial \phi_{mn+1}}{\partial z_{\alpha\beta}}|^2+\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}} |\frac{\partial \phi_{kl}}{\partial W}(Z,W)|^2+|\frac{\partial \phi_{mn+1}}{\partial W}(Z,W)|^2]\\
&= 2|\nabla f(\phi(Z,W))|^2|\phi'(Z,W)|^2.
\end{aligned}$$

By Lemma 2.4, then there exists a constant $C>0$ if the condition (4.1) holds, we have

$$\frac{[\det(I-Z\bar{Z}^T)-|W|^{2K}]^q}{[\det(I-Z_2\bar{Z_2}^T)-|W_2|^{2K}]^p}|\phi'(Z,W)|\leqslant C$$

for all $(Z,W)\in Y_I$. Let $f\in \beta^p(Y_I)$ and $(Z,W)\in Y_I$, we have

$$\begin{aligned}
&[\det(I-Z\bar{Z}^T)-|W|^{2K}]^q|\nabla(C_\phi f)(Z,W)|\\
&= [\det(I-Z\bar{Z}^T)-|W|^{2K}]^q|\nabla(f\circ\phi)(Z,W)|\\
&\leqslant \sqrt{2}[\det(I-Z\bar{Z}^T)-|W|^{2K}]^q|\phi'(Z,W)||\nabla f(\phi(Z,W))|
\end{aligned}$$

$$= \frac{\sqrt{2}[\det(I - Z\bar{Z}^T) - |W|^{2K}]^q}{[\det(I - Z_2\bar{Z_2}^T) - |W_2|^{2K}]^p}|\phi'(Z,W)|[\det(I - Z_2\bar{Z_2}^T) - |W_2|^{2K}]^p|\nabla f(Z_2,W_2)|$$
$$\leqslant \sqrt{2}C\|f\|_{\beta^p} = C'\|f\|_{\beta^p}.$$

By Lemma 2.7, we have

$$f(\phi(0,0)) \leqslant C''|f\|_{\beta^p},$$

thus

$$\|C_\phi f\|_{\beta^p} \leqslant C\|f\|_{\beta^p}. \tag{4.4}$$

(4.4) shows that $C_\phi : \beta^p(Y_I) \longrightarrow \beta^q(Y_I)$ is bounded.

For the converse, assume $C_\phi : \beta^p(Y_I) \longrightarrow \beta^q(Y_I)$ is a bounded operator with

$$\|C_\phi f\|_{\beta^q} \leqslant C\|f\|_{\beta^p}.$$

for all f in $\beta^p(Y_I)$.

If $p \neq \frac{1}{2}$, we will make use of a family test functions $\{f_\omega : \omega \in Y_I\}$ in $\beta^p(Y_I)$ defined by

$$f_{(U,V)}(Z,W) = \frac{1}{2p-1}\left[\frac{1}{[\det(I - Z\bar{U}^T) - \langle W,V\rangle^K]^{2p-1}} - 1\right].$$

Then

$$\begin{aligned}
\frac{\partial f}{\partial z_{\alpha\beta}} &= \frac{1}{2p-1}(1-2p)[\det(I - Z\bar{U}^T) - \langle W,V\rangle^K]^{-2p}\det(I - Z\bar{U}^T) \\
&\quad \times \operatorname{tr}[(I - Z\bar{U}^T)^{-1}(-I_{\alpha\beta})\bar{U}^T] \\
&= \frac{\det(I - Z\bar{U}^T)\operatorname{tr}[(I - Z\bar{U}^T)^{-1}I_{\alpha\beta}\bar{U}^T]}{[\det(I - Z\bar{U}^T) - \langle W,V\rangle^K]^{2p}},
\end{aligned}$$

and

$$\frac{\partial f}{\partial W} = \frac{K\langle W,V\rangle^{K-1}\overline{V}^T}{[\det(I - Z\overline{U}^T) - \langle W,V\rangle^K]^{2p}}.$$

Thus

$$\begin{aligned}
&\left|\frac{\partial f_{(U,V)}(Z,W)}{\partial(Z,W)}\right|^2 \\
&= \sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}\left|\frac{\det(I - Z\bar{U}^T)\operatorname{tr}[(I - Z\bar{U}^T)^{-1}I_{\alpha\beta}\bar{U}^T]}{\det(I - Z\bar{U}^T) - \langle W,V\rangle^K]^{2p}}\right|^2 + \left|\frac{K\langle W,V\rangle^{K-1}\bar{V}^T}{\det(I - Z\bar{U}^T) - \langle W,V\rangle^K]^{2p}}\right|^2 \\
&= \frac{\sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}|\det(I - Z\bar{U}^T)\operatorname{tr}[(I - Z\bar{U}^T)^{-1}I_{\alpha\beta}\bar{U}^T]|^2 + K^2|\langle W,V\rangle|^{2K-2}|\bar{V}^T|^2}{|\det(I - Z\bar{U}^T) - \langle W,V\rangle^{2K}|^{4p}}.
\end{aligned}$$

On one hand,we have

$$
\begin{aligned}
&[\det(I-Z\bar{Z}^T)-|W|^{2K}]^p|\nabla f_{(U,V)}(Z,W)|\\
=&[\det(I-Z\bar{Z}^T)-|W|^{2K}]^p\\
&\times\frac{\{\sum\limits_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}|\det(I-Z\bar{U}^T)\operatorname{tr}[(I-Z\bar{U}^T)^{-1}I_{\alpha\beta}\bar{U}^T]|^2+K^2|\langle W,V\rangle|^{2K-2}|\bar{V}^T|^2\}^{\frac{1}{2}}}{|\det(I-Z\bar{U}^T)-\langle W,V\rangle^{2K}|^{2p}}\\
=&\frac{[\det(I-Z\bar{Z}^T)-|W|^{2K}]^p[\det(I-U\bar{U}^T)-|V|^{2K}]^p}{|\det(I-Z\bar{U}^T)-\langle W,V\rangle^{K}|^{2p}}\\
&\times\frac{\{\sum\limits_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}|\det(I-Z\bar{U}^T)\operatorname{tr}[(I-Z\bar{U}^T)^{-1}I_{\alpha\beta}\bar{U}^T]|^2+K^2|\langle W,V\rangle|^{2K-2}|\bar{V}^T|^2\}^{\frac{1}{2}}}{[\det(I-U\bar{U}^T)-|V|^{2K}]^p}\\
\leqslant&\frac{\{\sum\limits_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}|\det(I-Z\bar{U}^T)\operatorname{tr}[(I-U\bar{U}^T)^{-1}I_{\alpha\beta}\bar{U}^T]|^2+K^2|\langle W,V\rangle|^{2K-2}|\bar{V}^T|^2\}^{\frac{1}{2}}}{[\det(I-U\bar{U}^T)-|V|^{2K}]^p}\\
\leqslant&\frac{\{\sum\limits_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}|\det(I-Z\bar{U}^T)\operatorname{tr}[(I-U\bar{U}^T)^{-1}I_{\alpha\beta}\bar{U}^T]|^2\}^{\frac{1}{2}}+K|\langle W,V\rangle|^{K-1}|\bar{V}^T|}{[\det(I-U\bar{U}^T)-|V|^{2K}]^p}\\
\leqslant&\frac{C+K|\langle W,V\rangle|^{K-1}|\bar{V}^T|}{[\det(I-U\bar{U}^T)-|V|^{2K}]^p}\\
\leqslant&\frac{C'}{[\det(I-U\bar{U}^T)-|V|^{2K}]^p}.
\end{aligned}
$$

Since $f_{(U,V)}(0,0)=0$, so we have

$$
\begin{aligned}
\|f_{(U,V)}\|_{\beta^p}&=|f_{(U,V)}(0,0)|+\sup_{(Z,W)\in Y_I}[\det(I_m-Z\bar{Z}^T)-|W|^{2K}]^p|\nabla f(Z,W)|\\
&\leqslant\frac{C'}{[\det(I-U\bar{U}^T)-|V|^{2K}]^p}.
\end{aligned}
$$

On the other hand, we can get

$$
\begin{aligned}
&[\det(I-Z\bar{Z}^T)-|W|^{2K}]^q|\nabla(C_\phi f)(Z,W)|\\
=&[\det(I-Z\bar{Z}^T)-|W|^{2K}]^q|\nabla(f\circ\phi)(Z,W)|\\
=&[\det(I-Z\bar{Z}^T)-|W|^{2K}]^q\{\sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}|\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\frac{\partial f}{\partial V_{kl}}(\phi(Z,W))\frac{\partial\phi_{kl}}{\partial z_{\alpha\beta}}(Z,W)
\end{aligned}
$$

$$+\frac{\partial f}{\partial V_{mn+1}}(\phi(Z,W))\frac{\partial \phi_{mn+1}}{\partial z_{\alpha\beta}}(Z,W)|^2+|\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\frac{\partial f}{\partial V_{kl}}(\phi(Z,W))\frac{\partial \phi_{kl}}{\partial W}(Z,W)$$

$$+\frac{\partial f}{\partial V_{mn+1}}(\phi(Z,W))\frac{\partial \phi_{mn+1}}{\partial W}(Z,W)|^2\}^{\frac{1}{2}},$$

where

$$\frac{\partial f}{\partial V_{kl}}(\phi(Z,W))=\frac{\det(I-Z_2\bar{U}^T)\operatorname{tr}[(I-Z_2\bar{U}^T)^{-1}I_{kl}\bar{U}^T]}{[\det(I-Z_2\bar{U}^T)-\langle W_2,V\rangle^K]^{2p}},$$

and

$$\frac{\partial f}{\partial V_{mn+1}}(\phi(Z,W))=\frac{K\langle W_2,V\rangle^{K-1}\bar{V}^T}{[\det(I-Z_2\bar{U}^T)-\langle W_2,V\rangle^K]^{2p}}.$$

Thus

$$\begin{aligned}
&[\det(I-Z\bar{Z}^T)-|W|^{2K}]^q|\nabla(C_\phi f)(Z,W)|\\
&=[\det(I-Z\bar{Z}^T)-|W|^{2K}]^q\{\sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}|\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\frac{\det(I-Z_2\bar{U}^T)\operatorname{tr}[(I-Z_2\bar{U}^T)^{-1}I_{kl}\bar{U}^T]}{[\det(I-Z_2\bar{U}^T)-\langle W_2,V\rangle^K]^{2p}}\\
&\quad\times\frac{\partial\phi_{kl}}{\partial z_{\alpha\beta}}(Z,W)+\frac{K\langle W_2,V\rangle^{K-1}\bar{V}^T}{[\det(I-Z_2\overline{U}^T)-\langle W_2,V\rangle^K]^{2p}}\frac{\partial\phi_{mn+1}}{\partial z_{\alpha\beta}}(Z,W)|^2\\
&\quad+|\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\frac{\det(I-Z_2\bar{U}^T)\operatorname{tr}[(I-Z_2\bar{U}^T)^{-1}I_{kl}\bar{U}^T]}{[\det(I-Z_2\bar{U}^T)-\langle W_2,V\rangle^K]^{2p}}\frac{\partial\phi_{kl}}{\partial W}(Z,W)\\
&\quad+\frac{K\langle W_2,V\rangle^{K-1}\bar{V}^T}{[\det(I-Z_2\bar{U}^T)-\langle W_2,V\rangle^K]^{2p}}\frac{\partial\phi_{mn+1}}{\partial W}(Z,W)|^2\}^{\frac{1}{2}}\\
&=\frac{[\det(I-Z\bar{Z}^T)-|W|^{2K}]^q}{|\det(I-Z_2\bar{U}^T)-\langle W_2,V\rangle^K|^{2p}}\{\sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}|\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\det(I-Z_2\bar{U}^T)\operatorname{tr}[(I-Z_2\bar{U}^T)^{-1}I_{kl}\bar{U}^T]\\
&\quad\times\frac{\partial\phi_{kl}}{\partial z_{\alpha\beta}}(Z,W)+K\langle W_2,V\rangle^{K-1}\bar{V}^T\frac{\partial\phi_{mn+1}}{\partial z_{\alpha\beta}}(Z,W)|^2+|\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\det(I-Z_2\bar{U}^T)\\
&\quad\times\operatorname{tr}[(I-Z_2\bar{U}^T)^{-1}I_{kl}\bar{U}^T]\frac{\partial\phi_{kl}}{\partial W}(Z,W)+K\langle W_2,V\rangle^{K-1}\bar{V}^T\frac{\partial\phi_{mn+1}}{\partial W}(Z,W)|^2\}^{\frac{1}{2}}.
\end{aligned}$$

Set

$$(U,V)=(Z_2,W_2)=\phi(Z,W),$$

since

$$\|C_\phi f\|_{\beta^q}\leqslant C\|f\|_{\beta^p},$$

so we have

$$\begin{aligned}&\frac{[\det(I-Z\bar{Z}^T)-|W|^{2K}]^q}{[\det(I-Z_2\bar{Z_2}^T)-|W_2|^{2K}]^{2p}}\{\sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}|\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\det(I-Z_2\bar{Z_2}^T)\operatorname{tr}[(I-Z_2\bar{Z_2}^T)^{-1}I_{kl}\bar{Z_2}^T]\\&\times\frac{\partial\phi_{kl}}{\partial z_{\alpha\beta}}(Z,W)+K|W_2|^{2K-2}\bar{W_2}^T\frac{\partial\phi_{mn+1}}{\partial z_{\alpha\beta}}(Z,W)|^2+|\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\det(I-Z_2\bar{Z_2}^T)\\&\times\operatorname{tr}[(I-Z_2\bar{Z_2}^T)^{-1}I_{kl}\bar{Z_2}^T]\frac{\partial\phi_{kl}}{\partial W}(Z,W)+K|W_2|^{2K-2}\bar{W_2}^T\frac{\partial\phi_{mn+1}}{\partial W}(Z,W)|^2\}^{\frac{1}{2}}\\\leqslant{}& C\frac{C'}{[\det(I-Z_2\bar{Z_2}^T)-|W_2|^{2K}]^p}.\end{aligned}$$

Furthermore, we have

$$\frac{[\det(I-Z\bar{Z}^T)-|W|^{2K}]^q}{[\det(I-Z_2\bar{Z_2}^T)-|W_2|^{2K}]^p}G(Z,W)\leqslant C,$$

where

$$\begin{aligned}G(Z,W)=\{&\sum_{\substack{1\leqslant\alpha\leqslant m\\1\leqslant\beta\leqslant n}}|\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\det(I-Z_2\bar{Z_2}^T)\operatorname{tr}[(I-Z_2\bar{Z_2}^T)^{-1}I_{kl}\bar{Z_2}^T]\\&\times\frac{\partial\phi_{kl}}{\partial z_{\alpha\beta}}(Z,W)+K|W_2|^{2K-2}\bar{W_2}^T\frac{\partial\phi_{mn+1}}{\partial z_{\alpha\beta}}(Z,W)|^2+|\sum_{\substack{1\leqslant k\leqslant m\\1\leqslant l\leqslant n}}\det(I-Z_2\bar{Z_2}^T)\\&\times\operatorname{tr}[(I-Z_2\bar{Z_2}^T)^{-1}I_{kl}\bar{Z_2}^T]\frac{\partial\phi_{kl}}{\partial W}(Z,W)+K|W_2|^{2K-2}\bar{W_2}^T\frac{\partial\phi_{mn+1}}{\partial W}(Z,W)|^2\}^{\frac{1}{2}}.\end{aligned}$$

If $p=\frac{1}{2}$, let

$$f_{(U,V)}(Z,W)=\ln\frac{1}{\det(I-Z\bar{U}^T)-\langle W,V\rangle^K}.$$

Then

$$\frac{\partial f}{\partial z_{\alpha\beta}}=\frac{\det(I-Z\bar{U}^T)\operatorname{tr}[(I-Z\bar{U}^T)^{-1}I_{kl}\bar{U}^T]}{\det(I-Z\bar{U}^T)-\langle W,V\rangle^K},$$

and

$$\frac{\partial f}{\partial W}=\frac{K\langle W,V\rangle^{K-1}\bar{V}^T}{\det(I-Z\bar{U}^T)-\langle W,V\rangle^K}.$$

For the same reason, it can be proved that (4.2) holds, and the details are omitted here. The proof is completed. □

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